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Conservation laws of linear elasticity in stress formulations

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In this paper, we present new conservation laws of linear elasticity which have been discovered. These newly discovered conservation laws are expressed solely in terms of the Cauchy stress tensor, and they are genuine, non-trivial conservation laws that are intrinsically different from the displacement conservation laws previously known. They represent the variational symmetry conditions of combined Beltrami–Michell compatibility equations and the equilibrium equations.

To derive these conservation laws, Noether's theorem is extended to partial differential equations of a tensorial field with general boundary conditions. By applying the tensorial version of Noether's theorem to Pobedrja's stress formulation of three-dimensional elasticity, a class of new conservation laws in terms of stresses has been obtained.

Keywords: elasticity; path-independent integrals; Noether's theorem

1. Introduction

The invariant integrals, or path-independent integrals, in three-dimensional (3D) elasticity are profound manifestations of intrinsic properties of elastic continua, which are not only permanent intellectual knowledge in mathematics and mechanics, but also could provide powerful analytical apparatus in applications, such as the case of Rice's J-integral (Rice 1968).

Since Eshelby's seminal work (Eshelby 1951, 1956), the subject of conservation laws in elasticity has been well studied. Landmark contributions on conservation laws of elasticity include: Eshelby (1970), Knowles & Sternberg (1972), Budiansky & Rice (1973), Fletcher (1976), Edelen (1981), Olver (1984*a*, *b*), Hill (1986), Maugin (1995) and Gurtin (1995), among others. A detailed view of the application of conservation laws in elasticity as configurational/material forces can be found in the excellent texts by Maugin (1993), Gurtin (1999) and Kienzler & Herrmann (2000).

Most of the conservation laws published so far in the literature are in terms of displacements. Recently, however, Li (2004) formulated the so-called dual conservation laws, which are in terms of the Airy stress function. The dual-conservation laws of Bui (1974) are a combination of both stresses and displacements. To the best of the

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authors' knowledge, a purely stress-based conservation law in 3D elasticity has never been obtained before.

In fact, some 30 years ago, during a discussion, C. A. Berg raised the question:

> Can one use the complementary energy density to construct a complementary energy-momentum quantity which would provide an estimate of the displacement of a defect from its equilibrium site when prescribed forces are applied to the body?

Subsequently, Eshelby (1970) made the following remark.

> The natural argument of the complementary energy is the stress. To fit the formalism I presented, the stress would have to be written as the gradient of something. I dare say that if this were done in detail something interesting might come out. But, any connection with displacement would be pretty indirect because the new energy-momentum tensor would still be a stress.

In search of a *dual energy-momentum tensor*, a systematic study on stress conservation laws in linear elasticity is carried out in this work. We first discuss stress formulations and develop an appropriate variational principle in § 2. In § 3, we present Noether's (1918) theorem for symmetric tensorial fields, and, finally, in § 4 we apply the tensorial version of Noether's theorem to derive the conservation laws of linear elasticity in terms of stresses.

2. Stress formulations

A well-known stress formulation in linear elasticity is the Beltrami–Michell stress formulation (e.g. Gurtin 1972), given by the following boundary-value problem,

$$\sigma_{ji,j} = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{2.1}$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{2.2}$$

$$\sigma_{ji} n_j = p_i, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.3}$$

where σ_{ij} is the Cauchy stress tensor, ν is Poisson's ratio, p_i is the prescribed traction vector and n_i is the normal vector field on $\partial\Omega$.

In principle, one can solve the Beltrami–Michell compatibility equations and the equilibrium equations with traction boundary conditions to find a unique stress solution (Gurtin 1972). Nevertheless, the Beltrami–Michell formulation is not mathematically well posed (e.g. nine equations, six unknowns and three boundary conditions). Moreover, its differential operator is not symmetric (not self-adjoint). On the other hand, there is a less well known but well established stress formulation in linear elasticity theory, which is the so-called Pobedrja stress formulation (Pobedrja 1978, 1980, 1994; Pobedrja & Kholmatov 1982; Kucher *et al.* 2004 (hereafter referred to as KMP)). Pobedrja's stress formulation is a set of tensorial partial differential equations, which are elliptic, self-adjoint, and well posed (Pobedrja 1980; Pobedrja & Kholmatov 1982). In the following, we study the conservation laws of linear elasticity based on Pobedrja's formulation.

(*a*) *Pobedrja's boundary-value problem*

Pobedrja's boundary-value problem (BVP) (Pobedrja 1978, 1980), which has six equations, six unknowns and six independent boundary conditions, is expressed in terms of stresses only. In the physical range of Poisson's ratio, the solution of Pobedrja's BVP satisfies both the equilibrium equations and the Beltrami–Michell compatibility equations in the domain, and satisfies traction boundary conditions along the boundary. Moreover, Pobedrja's stress formulation has the Fredholm property in the domain except at two points (KMP).

In fact, Pobedrja (1980) showed that Navier's displacement formulation of the traction boundary-value problem of an isotropic, homogeneous, linear elastic material,

$$\Delta u_i + \frac{1}{1-2\nu} u_{k,ki} = -\frac{1}{\mu} F_i, \quad \forall \boldsymbol{x} \in \Omega \tag{2.4}$$

and

$$2\mu \frac{\nu}{1-2\nu} u_{k,k} n_i + \mu(u_{i,k} + u_{k,i}) n_k = p_i, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.5}$$

is equivalent to the following boundary-value problem in terms of the Cauchy stress and vice versa:

$$\Delta\sigma_{ij} + b\sigma_{kk,ij} + c\sigma_{mn,mn}\delta_{ij} - e\Delta\sigma_{kk}\delta_{ij} + a[\sigma_{ik,kj} + \sigma_{jk,ki}] = -(a+1)\left(F_{i,j} + F_{j,i} + \frac{1-b-e}{2b-1} F_{k,k}\delta_{ij}\right), \quad \forall \boldsymbol{x} \in \Omega, \tag{2.6}$$

and

$$\sigma_{ik} n_k = p_i, \qquad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.7}$$

$$\sigma_{ik,k} = -F_i, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.8}$$

where Δ is the Laplacian operator, u_i is the displacement vector field, μ is the shear modulus, F_i is the body force vector, and parameters a, b, c, e are constants satisfying

$$b = \frac{1}{1+\nu}, \tag{2.9}$$

$$e = 1 - b + \zeta(2b-1), \tag{2.10}$$

$$c = a\frac{1-b-e}{2b-1}. \tag{2.11}$$

Note that the constants a and ζ are free parameters. The conditions (2.9)–(2.11) that, for the traction boundary-value problems, Navier's displacement formulation and Pobedrja's stress formulation are equivalent, have been revised recently by Kucher *et al*. (2004).

If $b = c$, the boundary-value problem (2.6)–(2.8) becomes a self-adjoint BVP. The constant parameters can be then expressed in terms of Poisson's ratio ν and a free parameter ζ:

$$b = c = \frac{1}{1+\nu}, \tag{2.12}$$

$$e = \frac{1}{1+\nu}(\nu + \zeta(1-\nu)), \tag{2.13}$$

$$a = -\frac{1}{\zeta(1+\nu)}. \tag{2.14}$$

We find that, if

$$\zeta = -\frac{\nu}{1-\nu} \quad \rightarrow \quad e = 0, \tag{2.15}$$

then

$$a = \frac{1-\nu}{\nu(1+\nu)}, \tag{2.16}$$

$$b = c = \frac{1}{1+\nu}. \tag{2.17}$$

Under this condition, the Pobedrja stress formulation coincides with the Beltrami–Michell stress formulation *almost everywhere except on the boundary*. Under this condition, equation (2.6) degenerates to

$$\left(\Delta\sigma_{ij} + b\sigma_{kk,ij} + F_{i,j} + F_{j,i} + \frac{\nu}{1-\nu}F_{k,k}\delta_{ij}\right) + b\delta_{ij}(\sigma_{mn,mn} + F_{m,m}) + a[(\sigma_{ik,kj} + F_{i,j}) + (\sigma_{jk,ki} + F_{j,i})] = 0, \tag{2.18}$$

which is an identity from equations (2.1) and (2.2) (with $F_i \equiv 0$). Together with (2.7) and (2.8), (2.18) is equivalent to (2.1), (2.2) and (2.3). We refer to this case as 'the Beltrami–Michell special case'. We shall only consider the case of vanishing body force, i.e. $F_i \equiv 0$, in the rest of the paper.

(*b*) *A variational principle in terms of stress*

We now present a variational statement for Pobedrja's BVP.

Theorem 2.1 (variational principle (Pobedrja & Kholmatov 1982)). *Assume that $b = c$ and $F_i = 0$, $\forall \boldsymbol{x} \in \Omega$. Let*

$$L_\Omega := \tfrac{1}{2}\sigma_{ij,k}E_{ijk} = \{\tfrac{1}{2}\sigma_{ij,k}\sigma_{ij,k} + b\sigma_{kk,i}\sigma_{ij,j} - \tfrac{1}{2}e\sigma_{kk,\ell}\sigma_{jj,\ell} + \tfrac{1}{2}a(\sigma_{ik,k}\sigma_{ij,j} + \sigma_{jk,k}\sigma_{ji,i})\}, \tag{2.19}$$

where

$$E_{ijk}(\boldsymbol{\sigma}) := \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}, \quad \forall \boldsymbol{x} \in \Omega. \tag{2.20}$$

On the entire boundary $\boldsymbol{x} \in \partial\Omega$, we prescribe $p_i(\boldsymbol{x})$ such that

$$\delta p_i = 0, \quad \forall \boldsymbol{x} \in \partial\Omega. \tag{2.21}$$

Let

$$\chi_{ij}(\boldsymbol{x}) := E_{ijk}(\boldsymbol{\sigma}(\boldsymbol{x}))n_k(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \partial\Omega. \tag{2.22}$$

It is readily verified that the governing equations of the Pobedrja BVP may be written as

$$\frac{\mathrm{d}E_{ijk}}{\mathrm{d}x_k} = 0, \quad \forall \boldsymbol{x} \in \Omega. \tag{2.23}$$

By the divergence theorem,

$$\int_{\partial\Omega} E_{ijk} n_k \, \mathrm{d}S = 0 \quad \rightarrow \quad \int_{\partial\Omega} \delta\chi_{ij} \, \mathrm{d}S = 0, \tag{2.24}$$

which is the weak form of the so-called 'frozen' condition (see Pobedrja 1980).

The boundary-value problem (2.6)–(2.8) is then equivalent to the stationary condition of the functional

$$\begin{aligned} \Pi(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) := & \int_{\Omega} L_{\Omega}(\nabla\boldsymbol{\sigma}) \, \mathrm{d}\Omega - \int_{\partial\Omega} \chi_{ij}\sigma_{ij} \, \mathrm{d}S \\ & + \int_{\partial\Omega} [\tfrac{1}{2}(\sigma_{ij,j}\sigma_{ik,k} + \sigma_{ij}n_j\sigma_{ik}n_k) - p_i\sigma_{ij}n_j] \, \mathrm{d}S. \end{aligned} \tag{2.25}$$

Proof. Taking the first variation of (2.25),

$$\begin{aligned} \delta\Pi = & \int_{\Omega} \frac{\partial L_{\Omega}}{\partial\sigma_{ij,k}} \delta\sigma_{ij,k} \, \mathrm{d}\Omega - \int_{\partial\Omega} \chi_{ij}\delta\sigma_{ij} \, \mathrm{d}S \\ & + \int_{\partial\Omega} [\sigma_{ij,j}\delta\sigma_{ik,k} + (\sigma_{ij}n_j - p_i)\delta\sigma_{ik}n_k] \, \mathrm{d}S. \end{aligned} \tag{2.26}$$

Integration by parts and using the frozen condition $\delta\chi_{ij} = 0$, $\forall \boldsymbol{x} \in \partial\Omega$, yields

$$\begin{aligned} \delta\Pi = & -\int_{\Omega} \left(\frac{\mathrm{d}}{\mathrm{d}x_k} \frac{\partial L_{\Omega}}{\partial\sigma_{ij,k}} \right) \delta\sigma_{ij} \, \mathrm{d}\Omega + \int_{\partial\Omega} (E_{ijk}n_k - \chi_{ij})\delta\sigma_{ij} \, \mathrm{d}S. \\ & + \int_{\partial\Omega} [\sigma_{ij,j}\delta\sigma_{ik,k} + (\sigma_{ij}n_j - p_i)\delta\sigma_{ik}n_k] \, \mathrm{d}S. \end{aligned} \tag{2.27}$$

The stationary condition, $\delta\Pi = 0$, then leads to

$$\Delta\sigma_{ij} + b\sigma_{kk,ij} + c\sigma_{mn,mn}\delta_{ij} - e\Delta\sigma_{kk}\delta_{ij} + a[\sigma_{ik,kj} + \sigma_{jk,ki}] = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{2.28}$$

and boundary conditions

$$\sigma_{ij}n_j = p_i, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.29}$$

$$\sigma_{ij,j} = 0, \quad \forall \boldsymbol{x} \in \partial\Omega. \tag{2.30}$$

■

One may generalize the above variational principle into a more general form.

Proposition 2.2. *The Euler–Lagrange equations of the fundamental integral*

$$\Pi(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) = \int_{\Omega} L_{\Omega}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) \, \mathrm{d}\Omega - \int_{\partial\Omega} L_{\Gamma}(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) \, \mathrm{d}S \tag{2.31}$$

are

$$\mathcal{E}_{ij} := \frac{\partial L_\Omega}{\partial \sigma_{ij}} - \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right) = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{2.32}$$

$$\mathcal{E}_{ij}^{(B1)} := \frac{\partial L_\Omega}{\partial \sigma_{ij,k}} n_k - \frac{\partial L_\Gamma}{\partial \sigma_{ij}} = 0, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{2.33}$$

$$\mathcal{E}_{ijk}^{(B2)} := \frac{\partial L_\Gamma}{\partial \sigma_{ij,k}} = 0, \quad \forall \boldsymbol{x} \in \partial\Omega. \tag{2.34}$$

Proof. The stationary condition yields

$$\begin{aligned}\delta\Pi &= \int_\Omega \left(\frac{\partial L_\Omega}{\partial \sigma_{ij}}\delta\sigma_{ij} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\delta\sigma_{ij,k}\right)\mathrm{d}\Omega - \int_{\partial\Omega}\left(\frac{\partial L_\Gamma}{\partial \sigma_{ij}}\delta\sigma_{ij} + \frac{\partial L_\Gamma}{\partial \sigma_{ij,k}}\delta\sigma_{ij,k}\right)\mathrm{d}S \\ &= \int_\Omega \left(\frac{\partial L_\Omega}{\partial \sigma_{ij}} - \frac{\mathrm{d}}{\mathrm{d}x_k}\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\delta\sigma_{ij}\,\mathrm{d}\Omega + \int_{\partial\Omega}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}} n_k - \frac{\partial L_\Gamma}{\partial \sigma_{ij}}\right)\delta\sigma_{ij}\,\mathrm{d}S \\ &\quad - \int_{\partial\Omega}\frac{\partial L_\Gamma}{\partial \sigma_{ij,k}}\delta\sigma_{ij,k}\,\mathrm{d}S = 0.\end{aligned} \tag{2.35}$$

Therefore, the variational statement is equivalent to the Euler–Lagrange equations for a tensor field, (2.32)–(2.34). ■

3. Noether's theorem for a tensorial field

Unlike most BVP in linear elasticity, Pobedrja's BVP has two special features: it is a BVP of a tensorial field, and in order for the interior solution to satisfy the equilibrium equations, the solution on the boundary has to satisfy the equilibrium equations. Therefore, the fundamental integral (2.25) always contains a term of the boundary contribution. We start the discussion below by first assuming a transformation group, and then deriving conditions which should be satisfied for the invariance of the fundamental integral. Finally, we present a derivation of Noether's theorem as required for the present purpose.

Consider the functional

$$\Pi(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) = \Pi_\Omega(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) + \Pi_\Gamma(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}), \tag{3.1}$$

where

$$\Pi_\Omega(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) := \int_\Omega L_\Omega(x_i, \sigma_{ij}, \sigma_{ij,k})\,\mathrm{d}\Omega_{\boldsymbol{x}}, \tag{3.2}$$

$$\Pi_\Gamma(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma}) := \int_{\partial\Omega} L_\Gamma(x_i, \sigma_{ij}, \sigma_{ij,k}, n_i, p_i, \chi_{ij})\,\mathrm{d}S_{\boldsymbol{x}}. \tag{3.3}$$

Assume that we are given an r-parameter family of transformations on coordinate variable x_i, Cartesian tensor field σ_{ij}, normal vector field n_i, the traction vector field p_i and the frozen tensorial field χ_{ij}. In addition, we assume that the r-parameter family of invertible transformations with identity form an r-parameter Lie group of transformations (e.g. Olver 1986; Bluman & Kumei 1989; Ibragimov 1985). The transformations are given as

$$\bar{x}_i = \bar{x}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \tag{3.4}$$

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \tag{3.5}$$

$$\bar{n}_i = \bar{n}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \tag{3.6}$$

$$\bar{p}_i = \bar{p}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \tag{3.7}$$

$$\bar{\chi}_{ij} = \bar{\chi}_{ij}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \tag{3.8}$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$ such that

$$\begin{aligned}
\bar{x}_i|_{\boldsymbol{\varepsilon}=0} &= \bar{x}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{0}) = x_i, \\
\bar{\sigma}_{ij}|_{\boldsymbol{\varepsilon}=0} &= \bar{\sigma}_{ij}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{0}) = \sigma_{ij}, \\
\bar{n}_i|_{\boldsymbol{\varepsilon}=0} &= \bar{n}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{0}) = n_i, \\
\bar{p}_i|_{\boldsymbol{\varepsilon}=0} &= \bar{p}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{0}) = p_i, \\
\bar{\chi}_{ij}|_{\boldsymbol{\varepsilon}=0} &= \bar{\chi}_{ij}(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{0}) = \chi_{ij},
\end{aligned}$$

where $\sigma_{ij}, \chi_{ij}, \bar{\sigma}_{ij}, \bar{\chi}_{ij} \in \mathrm{Sym}$, where Sym is the linear space of all symmetric second-order Cartesian tensors.

Definition 3.1. The fundamental integral (3.1) is invariant under the r-parameter family of transformations (3.4)–(3.8) if

$$\int_{\bar{\Omega}} L_{\Omega}(\bar{x}_i, \bar{\sigma}_{ij}, \bar{\sigma}_{ij,k}) \,\mathrm{d}\Omega_{\bar{\boldsymbol{x}}} - \int_{\Omega} L_{\Omega}(x_i, \sigma_{ij}, \sigma_{ij,k}) \,\mathrm{d}\Omega_{\boldsymbol{x}} = o(\boldsymbol{\varepsilon}), \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}^r \tag{3.9}$$

and

$$\int_{\partial\bar{\Omega}} L_{\Gamma}(\bar{x}_i, \bar{\sigma}_{ij}, \bar{\sigma}_{ij,k}, \bar{n}_i, \bar{p}_i, \bar{\chi}_{ij}) \,\mathrm{d}S_{\bar{\boldsymbol{x}}} - \int_{\partial\Omega} L_{\Gamma}(x_i, \sigma_{ij}, \sigma_{ij,k}, n_i, p_i, \chi_{ij}) \,\mathrm{d}S_{\boldsymbol{x}} = o(\boldsymbol{\varepsilon}), \quad \forall \boldsymbol{\varepsilon} \in \mathbb{R}^r, \tag{3.10}$$

where

$$\bar{\Omega} := \{\bar{\boldsymbol{x}} \in \mathbb{R}^3 \mid \bar{x}_i = \bar{x}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \boldsymbol{x} \in \Omega\}$$

and

$$\partial\bar{\Omega} := \{\bar{\boldsymbol{x}} \in \mathbb{R}^3 \mid \bar{x}_i = \bar{x}_i(\boldsymbol{x}, \boldsymbol{\sigma}, \boldsymbol{n}, \boldsymbol{p}, \boldsymbol{\chi}, \boldsymbol{\varepsilon}), \boldsymbol{x} \in \partial\Omega\}.$$

Note that x_i, n_i, p_i and χ_{ij} are not independent variational arguments.

The strong form (local form) of global conditions (3.9) and (3.10) is given as follows

$$L_{\Omega}(\bar{x}_i, \bar{\sigma}_{ij}, \bar{\sigma}_{ij,k}) \det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right) - L_{\Omega}(x_i, \sigma_{ij}, \sigma_{ij,k}) = o(\boldsymbol{\varepsilon}), \tag{3.11}$$

$$L_{\Gamma}(\bar{x}_i, \bar{\sigma}_{ij}, \bar{\sigma}_{ij,k}, \bar{n}_i, \bar{p}_i, \bar{\chi}_{ij}) \det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)\left[\boldsymbol{n} \cdot \left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)^{-1} \cdot \bar{\boldsymbol{n}}\right] - L_{\Gamma}(x_i, \sigma_{ij}, \sigma_{ij,k}, n_i, p_i, \chi_{ij}) = o(\boldsymbol{\varepsilon}). \tag{3.12}$$

Note that in (3.12), Nanson's formula (e.g. Malvern 1969), i.e.

$$\bar{\boldsymbol{n}}\,\mathrm{d}\bar{S} = J\boldsymbol{n}\cdot\boldsymbol{F}^{-1}\,\mathrm{d}S \quad \text{or} \quad \mathrm{d}\bar{S} = \det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\frac{\partial x_i}{\partial\bar{x}_j}n_i\bar{n}_j\,\mathrm{d}S, \tag{3.13}$$

is used.

Using Taylor's theorem, one can expand the r-parameter transformations, (3.4)–(3.8) in terms of a small vector variable $\{\varepsilon\}_\alpha$, $\alpha = 1, 2, \ldots, r$,

$$\bar{x}_i = x_i + \varphi_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi})\varepsilon_\alpha + o(\boldsymbol{\varepsilon}), \tag{3.14}$$

$$\bar{\sigma}_{ij} = \sigma_{ij} + \xi_{ij\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi})\varepsilon_\alpha + o(\boldsymbol{\varepsilon}), \tag{3.15}$$

$$\bar{n}_i = n_i + \nu_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi})\varepsilon_\alpha + o(\boldsymbol{\varepsilon}), \tag{3.16}$$

$$\bar{p}_i = p_i + \gamma_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi})\varepsilon_\alpha + o(\boldsymbol{\varepsilon}), \tag{3.17}$$

$$\bar{\chi}_{ij} = \chi_{ij} + \lambda_{ij\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi})\varepsilon_\alpha + o(\boldsymbol{\varepsilon}), \tag{3.18}$$

where

$$\varphi_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi}) := \left.\frac{\partial\bar{x}_i}{\partial\varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=\boldsymbol{0}}, \tag{3.19}$$

$$\xi_{ij\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi}) := \left.\frac{\partial\bar{\sigma}_{ij}}{\partial\varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=\boldsymbol{0}}, \tag{3.20}$$

$$\nu_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi}) := \left.\frac{\partial\bar{n}_i}{\partial\varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=\boldsymbol{0}} \tag{3.21}$$

$$\gamma_{i\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi}) := \left.\frac{\partial\bar{p}_i}{\partial\varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=\boldsymbol{0}}, \tag{3.22}$$

$$\lambda_{ij\alpha}(\boldsymbol{x},\boldsymbol{\sigma},\boldsymbol{n},\boldsymbol{p},\boldsymbol{\chi}) := \left.\frac{\partial\bar{\chi}_{ij}}{\partial\varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=\boldsymbol{0}}. \tag{3.23}$$

Theorem 3.2. *The fundamental integral (3.1) is invariant (in the sense of definition 3.1.), if the following conditions hold:*

$$\left\{\frac{\partial L_\Omega}{\partial x_i}\varphi_{i\alpha} + \frac{\partial L_\Omega}{\partial\sigma_{ij}}\xi_{ij\alpha} + \frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\left(\frac{\mathrm{d}\xi_{ij\alpha}}{\mathrm{d}x_k} - \sigma_{ij,\ell}\frac{\mathrm{d}\varphi_{\ell\alpha}}{\mathrm{d}x_k}\right)\right\} + L_\Omega\frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} = 0, \quad \forall\boldsymbol{x}\in\Omega,\ \alpha = 1, 2, \ldots, r. \tag{3.24}$$

$$\left\{\frac{\partial L_\Gamma}{\partial x_i}\varphi_{i\alpha} + \chi_{ij}\xi_{ij\alpha} + \frac{\partial L_\Gamma}{\partial n_i}\nu_{i\alpha} + \frac{\partial L_\Gamma}{\partial p_i}\gamma_{i\alpha} + \frac{\partial L_\Gamma}{\partial\chi_{ij}}\lambda_{ij\alpha}\right\} + L_\Gamma\left\{\frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} - \frac{\partial\varphi_{i\alpha}}{\partial x_j}n_i n_j + n_i\nu_{i\alpha}\right\} = 0, \quad \forall\boldsymbol{x}\in\partial\Omega,\ \alpha = 1, 2, \ldots, r. \tag{3.25}$$

Proof. We first prove (3.24). Following Logan (1977) and using the short-hand notation, $(\cdot)_0 \equiv (\cdot)_{\boldsymbol{\varepsilon}=\mathbf{0}}$, we start by noting the following:

$$\left(\frac{\partial \bar{\sigma}_{ij}}{\partial x_k}\right)_0 = \left(\frac{\partial \bar{x}_i}{\partial \sigma_{jk}}\right)_0 = 0; \tag{3.26}$$

$$\left(\frac{\partial \bar{\sigma}_{ij}}{\partial \sigma_{k\ell}}\right)_0 = \tfrac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \quad \bar{\sigma}_{ij}, \sigma_{ij} \in \mathrm{Sym}; \tag{3.27}$$

$$\left(\frac{\partial \bar{x}_i}{\partial x_j}\right)_0 = \delta_{ij}; \tag{3.28}$$

$$\left(\frac{\partial^2 \bar{\sigma}_{ij}}{\partial \varepsilon_\alpha \partial x_k}\right)_0 = \frac{\partial \xi_{ij\alpha}}{\partial x_k}; \tag{3.29}$$

$$\left(\frac{\partial^2 \bar{\sigma}_{ij}}{\partial \varepsilon_\alpha \partial \sigma_{k\ell}}\right)_0 = \frac{\partial \xi_{ij\alpha}}{\partial \sigma_{k\ell}}; \tag{3.30}$$

$$\left(\frac{\partial^2 \bar{x}_i}{\partial \varepsilon_\alpha \partial x_j}\right)_0 = \frac{\partial \varphi_{i\alpha}}{\partial x_j}; \tag{3.31}$$

$$\left(\frac{\partial^2 \bar{x}_i}{\partial \varepsilon_\alpha \partial \sigma_{jk}}\right)_0 = \frac{\partial \varphi_{i\alpha}}{\partial \sigma_{jk}}. \tag{3.32}$$

Differentiating (3.11) and then evaluating at $\boldsymbol{\varepsilon} = 0$, we have

$$\left\{\frac{\partial L_\Omega}{\partial x_i}\varphi_{i\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij}}\xi_{ij\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\left(\frac{\partial \bar{\sigma}_{ij,k}}{\partial \varepsilon_\alpha}\right)_0\right\}\left[\det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)\right]_0 + L_\Omega\left[\frac{\partial}{\partial \varepsilon_\alpha}\det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)\right]_0 = 0. \tag{3.33}$$

To evaluate (3.33), we start by evaluating

$$\det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)\bigg|_0 = 1, \tag{3.34}$$

and

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon_\alpha}\left[\det\left(\frac{\partial \bar{\boldsymbol{x}}}{\partial \boldsymbol{x}}\right)\right]\bigg|_0 &= \frac{\partial}{\partial \varepsilon_\alpha}\left(\frac{\partial \bar{x}_i}{\partial x_j}\right)A_{ij}\bigg|_0 \\
&= \left(\frac{\partial^2 \bar{x}_i}{\partial \varepsilon_\alpha \partial x_j} + \frac{\partial^2 \bar{x}_i}{\partial \varepsilon_\alpha \partial \sigma_{k\ell}}\sigma_{k\ell,j}\right)A_{ij}\bigg|_0 \\
&= \left(\frac{\partial \varphi_{i\alpha}}{\partial x_j} + \frac{\partial \varphi_{i\alpha}}{\partial \sigma_{k\ell}}\sigma_{k\ell,j}\right)\delta_{ij} = \frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i},
\end{aligned} \tag{3.35}$$

where A_{ij} is the co-factor of $(\partial \bar{x}_i/\partial x_j)$, and $(A_{ij})_0 = \delta_{ij}$. To evaluate $(\partial \bar{\sigma}_{ij,k}/\partial \varepsilon_\alpha)_0$, differentiate $\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(\bar{\boldsymbol{x}})$ to obtain

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_k} + \frac{\partial \bar{\sigma}_{ij}}{\partial \sigma_{\ell m}}\sigma_{\ell m,k} = \frac{\partial \bar{\sigma}_{ij}}{\partial \bar{x}_\ell}\left(\frac{\partial \bar{x}_\ell}{\partial x_k} + \frac{\partial \bar{x}_\ell}{\partial \sigma_{mn}}\sigma_{mn,k}\right). \tag{3.36}$$

By virtue of (3.26), (3.27) and (3.29) we also have

$$\left(\frac{\partial\bar{\sigma}_{ij}}{\partial\bar{x}_k}\right)_0 = \sigma_{ij,k}. \tag{3.37}$$

Finally, using (3.36) we obtain

$$\left(\frac{\partial\bar{\sigma}_{ij,k}}{\partial\varepsilon_\alpha}\right)_0 = \frac{\mathrm{d}\xi_{ij\alpha}}{\mathrm{d}x_k} - \sigma_{ij,\ell}\frac{\mathrm{d}\varphi_{\ell\alpha}}{\mathrm{d}x_k}. \tag{3.38}$$

Substitution of equations (3.34)–(3.38) into (3.33) yields the invariant condition (3.24).

We now prove (3.25). Differentiating (3.12) with respect to ε_α and evaluating the expression at $\boldsymbol{\varepsilon}=0$, we obtain

$$\begin{aligned}\left\{\frac{\partial L_\Gamma}{\partial x_i}\varphi_{i\alpha} + \frac{\partial L_\Gamma}{\partial\sigma_{ij}}\xi_{ij\alpha} + \frac{\partial L_\Gamma}{\partial\sigma_{ij,k}}\left(\frac{\partial\bar{\sigma}_{ij,k}}{\partial\varepsilon_\alpha}\right)_0 + \frac{\partial L_\Gamma}{\partial n_i}\nu_{i\alpha} + \frac{\partial L_\Gamma}{\partial p_i}\gamma_{i\alpha} + \frac{\partial L_\Gamma}{\partial\chi_{ij}}\lambda_{ij\alpha}\right\}&\\ \times\left[\det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\left(\frac{\partial x_i}{\partial\bar{x}_j}\right)n_i\bar{n}_j\right]_0 + L_\Gamma\left\{\frac{\partial}{\partial\varepsilon_\alpha}\left[\det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\left(\frac{\partial x_i}{\partial\bar{x}_j}\right)n_i\bar{n}_j\right]\right\}_0 &= 0.\end{aligned} \tag{3.39}$$

It is readily shown that

$$\left[\det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\left(\frac{\partial x_i}{\partial\bar{x}_j}\right)n_i\bar{n}_j\right]_0 = 1. \tag{3.40}$$

The last term of (3.39) can be expanded as

$$\begin{aligned}&\frac{\partial}{\partial\varepsilon_\alpha}\left\{\det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\left(\frac{\partial x_i}{\partial\bar{x}_j}\right)n_i\bar{n}_j\right\}\Bigg|_0\\ &\quad= \frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} + \det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\frac{\partial}{\partial\varepsilon_\alpha}\left(\frac{\partial\bar{x}_j}{\partial x_i}\right)^{-1}n_i\bar{n}_j\Bigg|_0 + \det\left(\frac{\partial\bar{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right)\left(\frac{\partial\bar{x}_j}{\partial x_i}\right)^{-1}n_i\frac{\partial\bar{n}_j}{\partial\varepsilon_\alpha}\Bigg|_0.\end{aligned} \tag{3.41}$$

Considering

$$\begin{aligned}\frac{\partial}{\partial\varepsilon_\alpha}\left(\frac{\partial\bar{x}_j}{\partial x_i}\right)^{-1}\Bigg|_0 &= -\left(\frac{\partial\bar{x}_j}{\partial x_k}\right)^{-1}\left(\frac{\partial^2\bar{x}_\ell}{\partial\varepsilon_\alpha\partial x_k}\right)\left(\frac{\partial\bar{x}_\ell}{\partial x_i}\right)^{-1}\Bigg|_0 = -\frac{\partial\varphi_{j\alpha}}{\partial x_i},\\ \frac{\partial\bar{n}_j}{\partial\varepsilon_\alpha}\Bigg|_0 &= \nu_{j\alpha},\end{aligned} \tag{3.42}$$

we then have

$$\begin{aligned}\left\{\frac{\partial L_\Gamma}{\partial x_i}\varphi_{i\alpha} + \frac{\partial L_\Gamma}{\partial\sigma_{ij}}\xi_{ij\alpha} + \frac{\partial L_\Gamma}{\partial\sigma_{ij,k}}\left(\frac{\mathrm{d}\xi_{ij\alpha}}{\mathrm{d}x_k} - \sigma_{ij,\ell}\frac{\mathrm{d}\varphi_{\ell\alpha}}{\mathrm{d}x_k}\right) + \frac{\partial L_\Gamma}{\partial n_i}\nu_{i\alpha} + \frac{\partial L_\Gamma}{\partial p_i}\gamma_{i\alpha} + \frac{\partial L_\Gamma}{\partial\chi_{ij}}\lambda_{ij\alpha}\right\}&\\ + L_\Gamma\left\{\frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} - \frac{\partial\varphi_{i\alpha}}{\partial x_j}n_in_j + n_i\nu_{i\alpha}\right\} &= 0.\end{aligned} \tag{3.43}$$

Using equations (2.33) and (2.34), we finally obtain the invariant conditions on the boundary as

$$\left\{\frac{\partial L_\Gamma}{\partial x_i}\varphi_{i\alpha} + \chi_{ij}\xi_{ij\alpha} + \frac{\partial L_\Gamma}{\partial n_i}\nu_{i\alpha} + \frac{\partial L_\Gamma}{\partial p_i}\gamma_{i\alpha} + \frac{\partial L_\Gamma}{\partial \chi_{ij}}\lambda_{ij\alpha}\right\} + L_\Gamma\left\{\frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} - \frac{\partial \varphi_{i\alpha}}{\partial x_j}n_i n_j + n_i\nu_{i\alpha}\right\} = 0, \quad \forall \boldsymbol{x} \in \partial\Omega,\ \alpha = 1, 2, \ldots, r. \tag{3.44}$$

■

Theorem 3.3 (Noether 1918). *If the fundamental integral, $\Pi(\boldsymbol{\sigma}, \nabla\boldsymbol{\sigma})$, is invariant (i.e. it satisfies equations (3.9) and (3.10)) under the r-parameter transformations (3.4)–(3.8), then the following conservation laws hold:*

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left[\left(L_\Omega\delta_{k\ell} - \sigma_{ij,\ell}\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\varphi_{\ell\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\xi_{ij\alpha}\right] = 0, \quad \alpha = 1, 2, \ldots, r. \tag{3.45}$$

Proof. Consider the following identities:

$$\frac{\partial L_\Omega}{\partial x_i}\varphi_{i\alpha} = \frac{\mathrm{d}L_\Omega}{\mathrm{d}x_i}\varphi_{i\alpha} - \frac{\partial L_\Omega}{\partial \sigma_{jk}}\sigma_{jk,i}\varphi_{i\alpha} - \frac{\partial L_\Omega}{\partial \sigma_{jk,\ell}}\sigma_{jk,\ell i}\varphi_{i\alpha}, \tag{3.46}$$

$$\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\frac{\mathrm{d}\xi_{ij\alpha}}{\mathrm{d}x_k} = \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\xi_{ij\alpha}\right) - \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\xi_{ij\alpha}, \tag{3.47}$$

$$\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\sigma_{ij,\ell}\frac{\mathrm{d}\varphi_{\ell\alpha}}{\mathrm{d}x_k} + \frac{\partial L_\Omega}{\partial \sigma_{jk,\ell}}\sigma_{jk,\ell i}\varphi_{i\alpha} = \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\sigma_{ij,\ell}\varphi_{\ell\alpha}\right) - \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\sigma_{ij,\ell}\varphi_{\ell\alpha}. \tag{3.48}$$

Substituting (3.46)–(3.48) into (3.24) yields

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left(L_\Omega\varphi_{k\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\xi_{ij\alpha} - \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\sigma_{ij,\ell}\varphi_{\ell\alpha}\right) + \left(\frac{\partial L_\Omega}{\partial \sigma_{ij}} - \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\right)(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha}) = 0. \tag{3.49}$$

Finally, using (2.32) in (3.49), we obtain equation (3.45). ■

4. Conservation laws in stress space

Theorem 4.1. *Consider a simply connected region, $\Omega \in \mathbb{R}^3$, with Lipschitz continuous boundary $\partial\Omega$. Assume that a second-order symmetric Cartesian tensor, $\sigma_{ij}(\boldsymbol{x}) \in$ Sym, is the solution of the Beltrami–Michell boundary-value problem (Gurtin 1972)*

$$\sigma_{ji,j} = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{4.1}$$

$$\sigma_{ij,kk} + \frac{1}{1+\nu}\sigma_{kk,ij} = 0, \quad \forall \boldsymbol{x} \in \Omega, \tag{4.2}$$

$$\sigma_{ji}n_j = p_i, \quad \forall \boldsymbol{x} \in \partial\Omega, \tag{4.3}$$

where $\sigma_{ij}(\boldsymbol{x}) \in [C^2(\Omega)]^6$. The following quantities are then conserved (they are divergence free with respect to index k):

$$\text{(C1)}\quad \tilde{X}_{k\alpha} = \tfrac{1}{2}\sigma_{\ell m,n}\sigma_{\ell m,n}\delta_{k\alpha} - \sigma_{ij,\alpha}\sigma_{ij,k} - b\sigma_{ik,\alpha}\sigma_{qq,i}; \tag{4.4}$$

$$\text{(C2)}\quad \tilde{R}_{k\alpha} = \tfrac{1}{2}\sigma_{\ell m,n}\sigma_{\ell m,n}\epsilon_{pk\alpha}x_p + (\varepsilon_{\ell i\alpha}\sigma_{\ell j} + \varepsilon_{\ell j\alpha}\sigma_{i\ell} - \sigma_{ij,\ell}\varepsilon_{m\ell\alpha}x_m) \times (\sigma_{ij,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})); \tag{4.5}$$

$$\text{(C3)}\quad \tilde{S}_k = -\sigma_{\ell m,n}\sigma_{\ell m,n}x_k + (2\sigma_{ij,\ell}x_\ell + \sigma_{ij}) \times (\sigma_{ij,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})); \tag{4.6}$$

$$\text{(C4 a)}\quad \tilde{P}_k = (\sigma_{ij,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j}))c_{ij}; \tag{4.7}$$

$$\text{(C4 b)}\quad \tilde{P}_{k\alpha} = 2[\sigma_{\alpha j,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,\alpha} + \delta_{\alpha k}\sigma_{qq,j})]c_j; \tag{4.8}$$

$$\text{(C5 a)}\quad \tilde{G}_{ik} = \sigma_{ik}; \tag{4.9}$$

$$\text{(C5 b)}\quad \tilde{G}_{ijk} = (\sigma_{ij,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})); \tag{4.10}$$

$$\text{(C5 c)}\quad \tilde{E}_{ijk} = \sigma_{ij,k} + \tfrac{1}{2}b(\delta_{jk}\sigma_{pp,i} + \delta_{ik}\sigma_{pp,j} + 2\delta_{ij}\sigma_{kp,p}) + a(\delta_{jk}\sigma_{ip,p} + \delta_{ik}\sigma_{jp,p}); \tag{4.11}$$

$$\text{(C6)}\quad \tilde{B}_k = \tilde{E}_{ijk}(\boldsymbol{\sigma})\tau_{ij} - \sigma_{ij}\tilde{E}_{ijk}(\boldsymbol{\tau}); \tag{4.12}$$

where δ_{ij} is the Kronecker delta symbol, ε_{ijk} is the permutation symbol,

$$a = \frac{1-\nu}{\nu(1+\nu)}, \qquad b = \frac{1}{1+\nu},$$

c_i is an arbitrary constant vector, c_{ij} is an arbitrary constant symmetric tensor, and $\tau_{ij}(\boldsymbol{x})$ is an arbitrary solution of the Beltrami–Michell BVP. (It does not satisfy the required boundary condition (4.3).)

Proof. According to Noether's theorem, derived in the previous section, the variational-symmetric conservation laws have the form

$$\frac{\mathrm{d}C_{k\alpha}}{\mathrm{d}x_k} = 0, \quad \alpha = 1, 2, \ldots, r, \tag{4.13}$$

where the conserved quantities are

$$C_{k\alpha} = \left(L_\Omega\delta_{k\ell} - \sigma_{ij,\ell}\frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\right)\varphi_{\ell\alpha} + \frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\xi_{ij\alpha}, \quad k = 1, 2, 3, \tag{4.14}$$

and

$$L_\Omega = \tfrac{1}{2}\{\sigma_{ij,k}\sigma_{ij,k} + 2b\sigma_{kk,i}\sigma_{ij,j} - e\sigma_{ii,k}\sigma_{jj,k} + a(\sigma_{ik,k}\sigma_{im,m} + \sigma_{jk,k}\sigma_{jm,m})\}. \tag{4.15}$$

We denote

$$L_\Gamma = \chi_{ij}\sigma_{ij} + [p_i\sigma_{ij}n_j - \tfrac{1}{2}(\sigma_{ij,j}\sigma_{ik,k} + \sigma_{ij}n_j\sigma_{ik}n_k)]. \tag{4.16}$$

(*a*) *Coordinate translation*

Let

$$\bar{x}_i = x_i + \varepsilon_i, \tag{4.17}$$
$$\bar{\sigma}_{ij} = \sigma_{ij}, \tag{4.18}$$
$$\bar{n}_i = n_i, \tag{4.19}$$
$$\bar{p}_i = p_i, \tag{4.20}$$
$$\bar{\chi}_{ij} = \chi_{ij}. \tag{4.21}$$

The corresponding generators of infinitesimal transformations are

$$\varphi_{i\alpha} = \left.\frac{\partial \bar{x}_i}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} = \delta_{i\alpha}, \quad \alpha = 1, 2, 3, \tag{4.22}$$
$$\xi_{ij\alpha} = 0, \tag{4.23}$$
$$\nu_{i\alpha} = 0, \tag{4.24}$$
$$\gamma_{i\alpha} = 0, \tag{4.25}$$
$$\lambda_{ij\alpha} = 0. \tag{4.26}$$

One may verify that invariant conditions (3.24) and (3.25) are satisfied trivially. We can then obtain the following conserved quantities due to coordinate translation:

$$X_{k\alpha} = L_\Omega \delta_{k\alpha} - (\sigma_{ij,\alpha}\sigma_{ij,k} + b\sigma_{ii,\alpha}\sigma_{kq,q} + b\sigma_{ik,\alpha}\sigma_{qq,i} - e\sigma_{ii,\alpha}\sigma_{qq,k} + 2a\sigma_{ik,\alpha}\sigma_{iq,q}), \quad k, \alpha = 1, 2, 3. \tag{4.27}$$

Considering the Beltrami–Michell special case, i.e. $\zeta = -\nu/(1-\nu)$, or $e = 0$, and using the equilibrium relations over the domain ($\sigma_{ij,j} = 0$), we obtain (C1),

$$\tilde{X}_{k\alpha} = \tfrac{1}{2}\sigma_{\ell m,n}\sigma_{\ell m,n}\delta_{k\alpha} - \sigma_{ij,\alpha}\sigma_{ij,k} - b\sigma_{ik,\alpha}\sigma_{qq,i}, \tag{4.28}$$

where the tilde denotes the special case.

(*b*) *Coordinate rotation*

Let

$$\bar{x}_i = Q_{ji}(\boldsymbol{\varepsilon}) x_j, \tag{4.29}$$
$$\bar{\sigma}_{ij} = Q_{ki}(\boldsymbol{\varepsilon}) \sigma_{k\ell} Q_{\ell j}(\boldsymbol{\varepsilon}), \tag{4.30}$$
$$\bar{n}_i = Q_{ji}(\boldsymbol{\varepsilon}) n_j, \tag{4.31}$$
$$\bar{p}_i = Q_{ji}(\boldsymbol{\varepsilon}) p_j, \tag{4.32}$$
$$\bar{\chi}_{ij} = Q_{ki}(\boldsymbol{\varepsilon}) \chi_{k\ell} Q_{\ell j}, \tag{4.33}$$

where the rotation matrix $\{Q_{ij}(\boldsymbol{\varepsilon})\} \in SO(3)$ and $\{Q_{ij}(0)\} = \{\delta_{ij}\}$. For infinitesimal rotation,

$$Q_{ij}(\boldsymbol{\varepsilon}) = \delta_{ij} + \varepsilon_{ijk}\varepsilon_k + o(\boldsymbol{\varepsilon}), \quad k = 1, 2, 3, \tag{4.34}$$

and

$$\overset{\triangledown}{\boldsymbol{\varepsilon}} := \{\varepsilon_{ijk}\varepsilon_k\} = \begin{bmatrix} 0 & \varepsilon_3 & \varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ -\varepsilon_2 & -\varepsilon_1 & 0 \end{bmatrix} \in so(3). \tag{4.35}$$

The infinitesimal generators are

$$\varphi_{i\alpha} = \left.\frac{\partial \bar{x}_i}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} = \left.\frac{\partial Q_{ji}}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} x_j = \varepsilon_{ji\alpha} x_j, \tag{4.36}$$

$$\begin{aligned}\xi_{ij\alpha} &= \left.\frac{\partial \bar{\sigma}_{ij}}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} = \left.\frac{\partial Q_{ki}}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} \sigma_{k\ell} \left. Q_{\ell j}\right|_{\boldsymbol{\varepsilon}=0} + \left. Q_{ki}\right|_{\boldsymbol{\varepsilon}=0} \sigma_{k\ell} \left.\frac{\partial Q_{\ell j}}{\partial \varepsilon_\alpha}\right|_{\boldsymbol{\varepsilon}=0} \\ &= \varepsilon_{ki\alpha}\sigma_{kj} + \varepsilon_{\ell j\alpha}\sigma_{i\ell},\end{aligned} \tag{4.37}$$

$$\nu_{i\alpha} = \varepsilon_{ji\alpha} n_j, \tag{4.38}$$

$$\gamma_{i\alpha} = \varepsilon_{ji\alpha} p_j, \tag{4.39}$$

$$\lambda_{ij\alpha} = \varepsilon_{ki\alpha}\sigma_{kj} + \varepsilon_{\ell j\alpha}\sigma_{i\ell}, \tag{4.40}$$

where $\alpha = 1, 2, 3$.

It can be easily shown that transformations (4.29)–(4.33) satisfy invariant conditions (3.24) and (3.25).

The conserved quantities are

$$\begin{aligned}R_{k\alpha} &= L_\Omega \varepsilon_{mk\alpha} x_m + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}(\varepsilon_{\ell i\alpha}\sigma_{\ell j} + \varepsilon_{\ell j\alpha}\sigma_{i\ell} - \sigma_{ij,\ell}\varepsilon_{m\ell\alpha} x_m) \\ &= L_\Omega \varepsilon_{mk\alpha} x_m + (\varepsilon_{\ell i\alpha}\sigma_{\ell j} + \varepsilon_{\ell j\alpha}\sigma_{i\ell} - \sigma_{ij,\ell}\varepsilon_{m\ell\alpha} x_m) \\ &\quad \times (\sigma_{ij,k} + b\delta_{ij}\sigma_{kq,q} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j}) \\ &\qquad - e\delta_{ij}\sigma_{qq,k} + a(\delta_{jk}\sigma_{iq,q} + \delta_{ik}\sigma_{jq,q})).\end{aligned} \tag{4.41}$$

Considering the Beltrami–Michell special case, i.e $e = 0$ and using the equilibrium equations, we obtain (C2).

Remark 4.2.

(i) The conservation laws resulting from the transformations based on the principle of material frame-indifference (Truesdell & Noll 1965) are linear combinations of conserved quantities obtained by coordinate translation and coordinate rotation.

(ii) The Lagrangian density L_Ω may be viewed as a pseudo-energy density and, by definition,

$$E_{ijk} = \frac{\partial L_\Omega}{\partial \sigma_{ij,k}} \quad \text{and} \quad \sigma_{ij,k} = \frac{\partial L_\Omega}{\partial E_{ijk}}. \tag{4.42}$$

However, one may verify that the sixth-order tensor,

$$D_{ijk\ell mn} := \frac{\partial^2 L_\Omega}{\partial \sigma_{ij,k}\partial \sigma_{\ell m,n}}, \tag{4.43}$$

is not an isotropic tensor. Therefore, the rigid-body rotation is not an invariant transformation.

(c) Scaling

Choose

$$\bar{x}_i = (1 + c_1\varepsilon)x_i \quad \rightarrow \quad \varphi_{i\alpha} = c_1 x_i, \tag{4.44}$$
$$\bar{\sigma}_{ij} = (1 + c_2\varepsilon)\sigma_{ij} \quad \rightarrow \quad \xi_{ij\alpha} = c_2\sigma_{ij}, \tag{4.45}$$
$$\bar{n}_i = (1 + c_3\varepsilon)n_i \quad \rightarrow \quad \nu_{i\alpha} = c_3 n_i, \tag{4.46}$$
$$\bar{p}_i = p_i \quad \rightarrow \quad \gamma_{i\alpha} = 0, \tag{4.47}$$
$$\bar{\chi}_{ij} = (1 + c_4\varepsilon)\chi_{ij} \quad \rightarrow \quad \lambda_{ij\alpha} = c_4\chi_{ij}, \tag{4.48}$$

where c_1, c_2, c_3 and c_4 are real constants. Substituting (4.44)–(4.48) into the invariant conditions (3.24), (3.25), we obtain

$$(2c_2 + c_1)L_\Omega = 0, \tag{4.49}$$

and

$$(c_2 + c_3)\chi_{ij}\sigma_{ij} + L_\Gamma(2c_1 + c_4) = 0. \tag{4.50}$$

Let

$$c_1 = 1, \qquad c_2 = -\tfrac{1}{2}, \qquad c_3 = \tfrac{1}{2}, \qquad c_4 = -2. \tag{4.51}$$

The conserved quantities are then given as

$$S_k = -2L_\Omega x_k + (2\sigma_{ij,\ell}x_\ell + \sigma_{ij}) \times (\sigma_{ij,k} + b\delta_{ij}\sigma_{kq,q} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j}) - e\delta_{ij}\sigma_{qq,k} + a(\delta_{jk}\sigma_{iq,q} + \delta_{ik}\sigma_{jq,q})). \tag{4.52}$$

Considering the Beltrami–Michell special case, i.e. $e = 0$ and using the equilibrium equations, we obtain (C3).

(d) Constant pre-stress

Let

$$\bar{x}_i = x_i, \tag{4.53}$$
$$\bar{\sigma}_{ij} = \sigma_{ij} + \varepsilon c_{ij}, \quad c_{ij} \in \text{Sym}, \tag{4.54}$$
$$\bar{n}_i = n_i, \tag{4.55}$$
$$\bar{p}_i = p_i, \tag{4.56}$$
$$\bar{\chi}_{ij} = \chi_{ij}. \tag{4.57}$$

The non-zero infinitesimal generators are

$$\xi_{ij\alpha} = c_{ij}, \quad \alpha = 1. \tag{4.58}$$

Both invariant conditions (3.24) and (3.25) are satisfied. To verify condition (3.25), we note that, by the frozen condition,

$$\oint_\Gamma \chi_{ij}\,\mathrm{d}S = 0 \quad \rightarrow \quad c_{ij}\oint_\Gamma \chi_{ij}\,\mathrm{d}S = 0, \quad \forall c_{ij} \in \text{Sym}, \tag{4.59}$$

where Γ is an arbitrary closed contour.

The following quantities are conserved:

$$P_k = \frac{\partial L_\Omega}{\partial \sigma_{ij,k}} c_{ij}$$
$$= [\sigma_{ij,k} + b\delta_{ij}\sigma_{kq,q} + \tfrac{1}{2}b(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})$$
$$- e\delta_{ij}\sigma_{qq,k} + a(\delta_{jk}\sigma_{iq,q} + \delta_{ik}\sigma_{jq,q})]c_{ij}. \quad (4.60)$$

Considering the Beltrami–Michell special case, i.e. $e = 0$ and using the equilibrium equations, we obtain (C4 a).

An alternative construction is to let

$$\bar{\sigma}_{ij} = \sigma_{ij} + (\varepsilon_i c_j + c_i \varepsilon_j), \quad i, j = 1, 2, 3, \quad (4.61)$$

where $\{c_i\}$ is an arbitrary constant vector.

The corresponding non-zero infinitesimal generators are

$$\xi_{ij\alpha} = \delta_{i\alpha} c_j + c_i \delta_{j\alpha}, \quad \alpha = 1, 2, 3. \quad (4.62)$$

We obtain the following conserved quantities:

$$P_{k\alpha} = \frac{\partial L_\Omega}{\partial \sigma_{\alpha j,k}} c_j + \frac{\partial L_\Omega}{\partial \sigma_{i\alpha,k}} c_i = 2\frac{\partial L_\Omega}{\partial \sigma_{\alpha j,k}} c_j, \quad k, \alpha = 1, 2, 3. \quad (4.63)$$

We can then obtain (C4 b) as a special case.

(e) Others

For the present BVP, equation (2.32) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right) = 0, \quad \forall \boldsymbol{x} \in \Omega. \quad (4.64)$$

Noting (2.20), we conclude that E_{ijk} is a conserved quantity. We obtain (C5 c) for the Beltrami–Michell special case. We further obtain (C5 a) and (C5 b) by using the equilibrium relations.

By using the self-adjoint property of the Pobedrja formulation, a reciprocal theorem can be derived, which leads to the following conserved quantities:

$$B_k = E_{ijk}(\boldsymbol{\sigma})\tau_{ij} - \sigma_{ij} E_{ijk}(\boldsymbol{\tau}), \quad (4.65)$$

where $\boldsymbol{\tau}$ is an arbitrary solution of the BVP (it does not satisfy the required boundary conditions). We can prove that the above quantity is divergence free by first noting that

$$\frac{\mathrm{d}E_{ijk}(\boldsymbol{\sigma})}{\mathrm{d}x_k} = 0 \quad \text{and} \quad \frac{\mathrm{d}E_{ijk}(\boldsymbol{\tau})}{\mathrm{d}x_k} = 0, \quad (4.66)$$

and observing that

$$E_{ijk}(\boldsymbol{\sigma})\tau_{ij,k} = \sigma_{ij,k} E_{ijk}(\boldsymbol{\tau}). \quad (4.67)$$

We then obtain the conservation law (C6) for the Beltrami–Michell special case. Here $\boldsymbol{\tau}$ is an arbitrary solution of the Beltrami–Michell BVP. (It does not satisfy the required boundary condition (4.3).)

Equation (4.12) also provides us with a systematic way of generating higher-order stress conservation laws. For example, take $\tau_{ij} = \sigma_{ij,kk}$. It can be easily verified that τ_{ij} satisfies the Beltrami–Michell BVP (with an additional requirement that $\sigma_{ij}(\boldsymbol{x}) \in [C^4(\Omega)]^6$). We then obtain new higher-order conserved quantities (using (4.12)),

$$\tilde{A}_k = \tilde{E}_{ijk}(\boldsymbol{\sigma})\sigma_{ij,qq} - \sigma_{ij}\tilde{E}_{ijk}(\Delta\boldsymbol{\sigma}). \tag{4.68}$$

Remark 4.3. It should be noted that the above transformations may not be the only transformations under which we can obtain conservation laws.

■

5. Closure

The conservation laws in terms of the stress had been anticipated by Eshelby (1970), and it was hinted that they may be useful in studying defects in solids.

In this paper, by using Pobedrja's stress formulation of linear elasticity, we have discovered, we believe for the first time, a set of conservation laws in terms of stresses. The stress conservation laws represent the intrinsic properties of elastic materials, and they may provide physical insights to study material behaviours. It should be noted that the stress conservation laws discovered in this paper represent an intrinsic mathematical structure of 3D elasticity, and their existence does not depend on Pobedrja's stress formulation.

The laws derived here are new, genuine and non-trivial conservation laws of linear elasticity, which represent the symmetry properties of the compatibility equations and the equilibrium equations. Even though these conservation laws are expressed solely in terms of stresses, they may be also viewed as higher-order displacement conservation laws if the stress components are converted into strain components, and it may be possible to derive them as the higher-order conservation laws of Navier's equations.

cokNote that here we have not exhausted all the possibilities of conservation laws in stress formulations. We have only documented several conservation laws that have physical implications.

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