Geolocation using Transmit and Receive Diversity

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***Abstract*— Geolocation using received signal strength (RSS) has large errors due to multipath fading, since fading results in high variations in RSS. We show how and when spatial diversity combined with channel knowledge at the receiver can be used to combat fading effects to increase accuracy in location estimate. We then propose a simple scheme for distance estimation, characterize the channels for which improvement in distance estimates can be thus obtained and prove that mean square error of distance estimate converges to zero with increasing diversity order. It is observed that the improvement can always be obtained for Rayleigh channel, and for Nakagami-m channel if the parameter m remains the same regardless of the distance.**

I. Introduction and Motivation

Accurate geolocation is an important and emerging field of interest for commercial, public safety and military applications. Of the popular methods like Angle of Arrival (AOA), Time of Arrival (TOA), Phase of Arrival (POA), and Received Signal Strength (RSS), RSS is the easiest and least expensive, but gives poor performance. The main reason is large variation in power estimates in fading channels. In [1], the authors use simulated data for received power and model received signal power using linear, compensated linear and multiple regression methods to investigate indoor geolocation. In [2], the authors obtain the PDF of propagation distance and use it to obtain location estimates when there is only shadowing and no multipath fading, which is not a very good assumption for wireless channels. In [3], the author proposes to look for structures and patterns in the received signal strength and then uses pattern matching to find location estimates, which can be computationally very expensive for large geographical areas.

In this paper, we take a different approach and explore the use of RSS using spatial diversity (i.e., Multiple Input Multiple Output (MIMO) systems) to give a better estimate of the distance between a transmitter and receiver. Given that MIMO systems are making their way in many of the currently proposed wireless standards, this approach has potential of important applications.

The path loss equation represents path loss (signal attenuation) as a function of distance between receiver (Rx) and transmitter (Tx). Using the empirically obtained fairly general model, the average power as a function of distance from the transmitter is given as [6]

$$E[P_r] = \frac{E[P_{r_0}]r_0^{\nu}}{r^{\nu}} \tag{1}$$

where P_r is the received power at distance r, P_{r_0} is the received power at some reference distance r_0, and ν is the path loss exponent. Typically, $\nu \geq 4$, and varies depending on environment. Throughout our analysis we assume that we have a calibrated channel and ν and $E[P_{r_0}]$ are known at the receiver. So macroscopic fading/shadowing is already taken care of and our scheme mitigates microscopic fading/multipath for a narrowband system. The distance r is bounded by the *range of transmission* – henceforth denoted by L, and we assume that the receiver always remains within this range.

As mentioned before, multipath fading makes it difficult to estimate r using P_r because the variance of P_r is large, which makes a single measurement of P_r a bad estimate for $E[P_r]$.

It is well known that use of spatial diversity eliminates deep fades ([4], pages 100-103). The criteria proposed in [4] is the *coefficient of variation* defined as the ratio of standard deviation of power to its mean. It is known that as the diversity order approaches infinity, for many channels, the coefficient of variation approaches 0. In [5] it was shown for Rayleigh and Rician channels. We show this for the case of Nakagami-m channels (see appendix). Thus one would expect that increase in diversity order should lead to increase in accuracy in geolocation. In this paper, we show that this intuition need not hold for all channels, and characterize the channels for which it does. The decay of fourth moment (with distance) of fade coefficients should satisfy a condition for the intuition to hold. A necessary and sufficient condition for this is obtained in the appendix, and a simple yet widely applicable sufficient condition is stated in **Result 1**. For example, it is shown that the condition always holds for Rayleigh channel (the most common model for fading wireless channels). We also give a simple method for estimating distance using a single measurement of power when the condition is satisfied. For these channels we formally prove that this estimate converges to the actual distance if the coefficient of variation approaches zero with increasing diversity order. In the analysis, we assume linear processing at the receiver and high SNR, since the main objective is to demonstrate the applicability of diversity techniques for increasing accuracy in geolocation.

In Section II we discuss the proposed technique and prove its usefulness. Simulation results are presented in Section III.

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II. The Proposed Scheme

The scheme is applicable to *MISO*, *SIMO* or *MIMO* systems. If the channel is known to the transmitter, it does eigen-mode transmission with water-filling on the modes (which is known to achieve capacity). The receiver in this case does a simple linear processing on the signal which is also ML optimal. In case the channel is not known to the transmitter, it uses orthogonal-space time block codes and again linear processing at the receiver gives the ML optimal performance. When there is independent fading for each path, the receiver output in either case is given by [4]

$$y_i = \sqrt{\frac{E_s}{M_T}} \sum_{n=1}^{M_R} \sum_{m=1}^{M_T} |\alpha_{imn}|^2 \; s_i + n_i \tag{2}$$

where E_s is the transmit energy per symbol, α_{imn} is the fade coefficient, M_R is the number of Rx-antennas, M_T is the number of Tx-antennas, s_i is the symbol transmitted at i^{th} instant, and n_i is the associated noise. From here on, $N = M_T M_R$ will denote the diversity order[1] of the transmit-receive system and for simplicity of notation, the summation will be performed over the index n only.

Assuming that *PSK*-constellations with unit symbol energy are used for transmission ($|s_i| = 1 \; \forall \; i$, and $E_s = 1$), and for simplification, neglecting noise in the analysis (hence results are applicable only for high SNR values), the receiver output y_i is given by

$$y_i = \frac{1}{\sqrt{M_T}} \left(\sum_{n=1}^{N} |\alpha_{in}|^2 \right) \; s_i \tag{3}$$

Thus we get the expression of receiver output power at distance r as

$$P_r^{(N)} = |y_i|^2 = \frac{1}{M_T} (\sum_{n=1}^{N} |\alpha_{in}|^2)^2 \tag{4}$$

where the superscript (N) is used to indicate the diversity order.

A. Estimation of the distance r

For estimation of r, a single observation of power is made at the receiver, and after appropriate modification to (1) for the multiple antenna case (as shown below), the modified equation is used to estimate distance. We denote the estimate thus obtained by $\widehat{r}^{(N)}$. As before, we denote the maximum value of r by L. Note that if power level at any distance is observed to be lower than the average power level at distance L, $\widehat{r}^{(N)}$ is taken to be equal to L. Thus, $\widehat{r}^{(N)} \leq L$.

The following results give the method for estimating r and establish the improvement in accuracy with increasing N. The first result gives the modification required in (1) for geolocation in a multiple antenna system. Let α_r denote the fade coefficient at distance r. We also assume that ν, $P_{r_0}^{(N)}$ and $P_L^{(N)}$ are known at the receiver.

[1]For our case we assume that diversity order is simply the product of number of Tx-antennas and number of Rx-antennas. For a more rigorous definition one may refer to [4].

Result 1: Under the condition that $E[|\alpha_r|^4] = E[|\alpha_{r_0}|^4] \frac{r_0^{2\nu}}{r^{2\nu}}$, the power loss equation (1) is modified as $E[P_r^{(N)}] = E[P_{r_0}^{(N)}] \frac{r_0^{2\nu}}{r^{2\nu}}$

Proof: If we do not use diversity, we know that power decay profile is $E[P_r] = \frac{E[P_{r_0}] \; r_0^{\nu}}{r^{\nu}}$ (1). This can be rewritten as $E[|\alpha_r|^2] = E[|\alpha_{r_0}|^2] \; \frac{r_0^{\nu}}{r^{\nu}}$. For the multiple antenna case, assuming α_n's are i.i.d. random variables distributed as α_r, we get (dropping the time index i in (4))

$$\begin{aligned} E[P_r^{(N)}] &= \frac{1}{M_T} E[(\sum_{n=1}^{N} |\alpha_n|^2)^2] \\ &= \frac{N E[|\alpha_r|^4] + N(N-1)(E[|\alpha_r|^2])^2}{M_T} \end{aligned} \tag{5}$$

If the condition stated in **Result 1** holds, then it follows that

$$\begin{aligned} E[P_r^{(N)}] &= \frac{(N \; E[|\alpha_{r_0}|^4] + N(N-1)E^2[|\alpha_{r_0}|^2]) r_0^{2\nu}}{r^{2\nu} \; M_T} \\ &= E[P_{r_0}^{(N)}] \frac{r_0^{2\nu}}{r^{2\nu}} \end{aligned} \tag{6}$$

If the $|\alpha_n|$'s are distributed as i.i.d. Nakagami-m random variables, the condition is simplified to m being a constant regardless of the distance between the transmitter and receiver[2].

Since the Nakagami-m distribution spans via the m parameter the widest range of fading among all the multipath distributions used in practice ($m = 1/2$ - one-sided Gaussian distribution, $m = 1$ - Rayleigh distribution and $m = \frac{(1+K)^2}{1+2K}$ - Rician distribution where K is the Rician K factor [6]), this analysis is fairly general. The detection of which model (value of m) is to be used at the time of geolocation is a hypothesis testing problem discussed abundantly in literature.

Thus, for channels for which the condition in **Result 1** holds, in order to find $\widehat{r}^{(N)}$, a measurement of $P_r^{(N)}$ is performed at the receiver, and $\widehat{r}^{(N)}$ is calculated as below:

$$\widehat{r}^{(N)} = \begin{cases} r_0 \left(\frac{E[P_{r_0}^{(N)}]}{P_r^{(N)}} \right)^{\frac{1}{2\nu}} & \text{if } \; P_r^{(N)} > E[P_L^{(N)}] \\ L & \text{if } \; P_r^{(N)} \leq E[P_L^{(N)}] \end{cases} \tag{7}$$

It is worth noting here that $\widehat{r}^{(N)}$ is a continuous and bounded function of $P_r^{(N)}$ (continuity is to be checked only at $P_r^{(N)} = E[P_L^{(N)}]$, which can be verified from (6)).

For further results, we take the condition in **Result 1** to hold because of its wide applicability and simplicity. However, note that this is not a necessary condition for deriving a decay relationship. A necessary and sufficient condition has been derived in Appendix I.

In what follows, we prove the increase in accuracy by this method of estimation of r for channels for which the condition in **Result 1** is true.

Result 2:

$$\frac{P_r^{(N)}}{E[P_r^{(N)}]} \xrightarrow{p} 1 \text{ as } N \to \infty$$

[2]The statement is proved towards the end of Appendix I.

where p is used to denote convergence in probability.

Proof: Note that $E\left[\frac{P_r^{(N)}}{E[P_r^{(N)}]}\right] = 1$. Thus, by Chebyshev's inequality,

$$Pr\left(\left|\frac{P_r^{(N)}}{E[P_r^{(N)}]} - 1\right| > \epsilon\right) < \frac{Var\left(\frac{P_r^{(N)}}{E[P_r^{(N)}]}\right)}{\epsilon^2} \quad (8)$$

Now,

$$Var\left(\frac{P_r^{(N)}}{E[P_r^{(N)}]}\right) = \frac{Var(P_r^{(N)})}{E^2[P_r^{(N)}]}$$

which is square of the parameter *coefficient of variation* [4]. For many channels (in [5] it was proved for Rayleigh and Rician channels, and in appendix, we prove it for Nakagami-m channels), coefficient of variation $\to 0$, as $N \to \infty$. And thus $Var\left(\frac{P_r^{(N)}}{E[P_r^{(N)}]}\right) \to 0$, whence $\frac{P_r^{(N)}}{E[P_r^{(N)}]} \xrightarrow{p} 1$ as $N \to \infty$.

Result 3: $\widehat{r}^{(N)} \xrightarrow{p} r$ as $N \to \infty$

Proof: Define $X_N = \frac{P_r^{(N)}}{E[P_r^{(N)}]}$. Then from **Result 1** it follows that

$$\begin{aligned} \widehat{r}^{(N)} &\triangleq r_0 \left[\frac{E[P_{r_0}^{(N)}]}{P_r^{(N)}}\right]^{\frac{1}{2\nu}} \\ &= \begin{cases} r\left[\frac{E[P_r^{(N)}]}{P_r^{(N)}}\right]^{\frac{1}{2\nu}} & \text{if } \frac{P_r^{(N)}}{E[P_r^{(N)}]} \geq \frac{E[P_L^{(N)}]}{E[P_r^{(N)}]} \\ L & \text{if } \frac{P_r^{(N)}}{E[P_r^{(N)}]} < \frac{E[P_L^{(N)}]}{E[P_r^{(N)}]} \end{cases} \end{aligned} \quad (9)$$

We note that $\widehat{r}^{(N)}$ is a continuous function of $X_N = \frac{P_r^{(N)}}{E[P_r^{(N)}]}$, for fixed r. It is of the form $f(x) = \frac{r}{x^{\frac{1}{2\nu}}}$ for $x \geq \frac{E[P_L^{(N)}]}{E[P_r^{(N)}]}$ and $f(x) = L$ otherwise. For verifying the continuity of $f(x)$ at the boundary i.e. at $x = \frac{E[P_L^{(N)}]}{E[P_r^{(N)}]}$, notice that when $P_r^{(N)} = E[P_L^{(N)}]$, then $\widehat{r}^{(N)} = L$. We know that if a sequence of random variables $\{X_n\}$ converges in probability to a random variable X, then for any continuous function $f(.)$, the sequence $\{f(X_n)\}$ converges in probability to $f(X)$ [7]. Since $X_N \xrightarrow{p} 1$ (by **Result 2**), $f(X_N) \xrightarrow{p} f(1) = r$. That is, $\widehat{r}^{(N)} \xrightarrow{p} r$.

Corollary: From **Result 3** we have $\widehat{r}^{(N)} \xrightarrow{p} r$. Also, $|\widehat{r}^{(N)}| \leq L$, where L is a constant. That is, $\widehat{r}^{(N)}$ is dominated by a constant, L. Therefore, by dominated convergence result [7], $\widehat{r}^{(N)} \xrightarrow{L^p} r \quad \forall \quad p > 1$, where L^p denotes convergence in L^p norm. In particular, the case $p = 2$ implies $E[|\widehat{r}^{(N)} - r|^2] \to 0$, i.e., the mean square error in distance estimation converges to zero as $N \to \infty$.

Remark: In addition, we also see that $E[\widehat{r}^{(N)}] > r$. Proof goes as follows. Since $\widehat{r}^{(N)} = r_0\left(\frac{E[P_{r_0}^{(N)}]}{P_r^{(N)}}\right)^{2\nu}$, $E[\widehat{r}^{(N)}] = r_0 E\left[\left(\frac{E[P_{r_0}^{(N)}]}{P_r^{(N)}}\right)^{2\nu}\right]$. Now $g(x) = \frac{1}{x^p}$, is a strictly convex function $\forall\, p > 0$. It follows from the Jensen's Inequality that $E[g(X)] \geq \frac{1}{(E[X])^p}$. Noting that $r = r_0\left(\frac{E[P_{r_0}^{(N)}]}{E[P_r^{(N)}]}\right)^{2\nu}$, and since $E[P_{r_0}^{(N)}]$ and r_0 are constants, the inequality $E[\widehat{r}^{(N)}] > r$ follows. Thus $\widehat{r}^{(N)}$ is a biased estimator of r. As we see in the simulations, this bias decreases with increasing N, and, in fact, the corollary to **Result 3** shows that the estimator $\widehat{r}^{(N)}$ is asymptotically unbiased.

III. Simulation Results

From Result 3, we can infer that the error in geolocation approaches zero as $N \to \infty$. For finding out how $E[(\widehat{r}^{(N)} - r)^2]$ decreases with N, we resort to simulations. Simulations have been done for Rayleigh fading (i.e. $m = 1$), $r = 4$, $\nu = 4$, $r_0 = 1$ and $E[|\alpha_0|^2] = 1$. Fig. 1 shows $E[\widehat{r}^{(N)}]$ calculated over 5000 iterations. We note that for small values of N, $E[\widehat{r}^{(N)}] > r$.

Fig. 2 shows $E[(\widehat{r}^{(N)} - r)^2]$, again, calculated by averaging over 5000 iterations for each N. We observe that there is a sharp decrease in $E[(\widehat{r}^{(N)} - r)^2]$ with increase in N, which corroborates the derived theoretical results (viz **Result 3** and its corrollary).

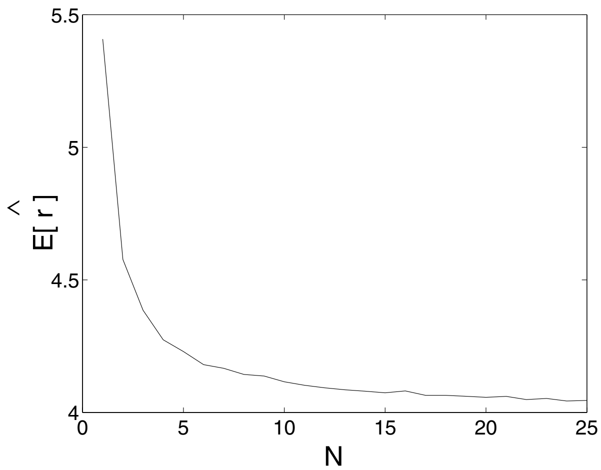

Fig. 1. $E[\widehat{r}^{(N)}]$ Vs N for $r = 4$, $\nu = 4$, $r_0 = 1$ and $E[|\alpha_0|^2] = 1$

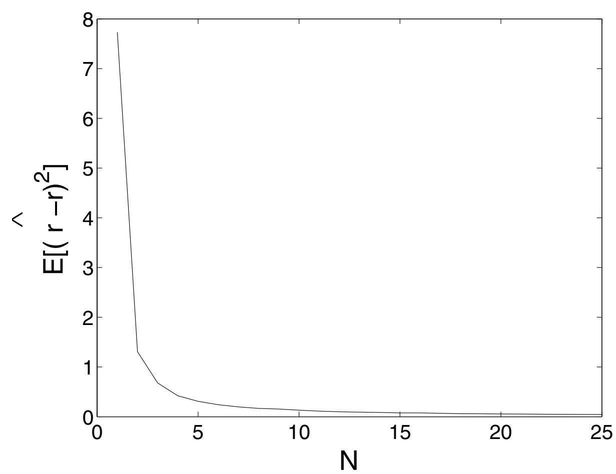

Fig. 2. $E[(\widehat{r}^{(N)} - r)^2]$ Vs N for $r = 4$, $\nu = 4$, $r_0 = 1$ and $E[|\alpha_0|^2] = 1$

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APPENDIX I
COEFFICIENT OF VARIATION FOR NAKAGAMI-M FADING CHANNEL

The Nakagami-m distribution has PDF given by [6]

$$p(|\alpha|) = \frac{2m^m|\alpha|^{2m-1}}{\Omega^m\Gamma(m)} exp(\frac{-m|\alpha|^2}{\Omega}) \tag{10}$$

where α is the fade coefficient, and $\Omega = E[|\alpha|^2]$ is the average power.
In this case, MGF of instantaneous power ($|\alpha|^2$) is given by:

$$\phi(t) = E[e^{t|\alpha|^2}] = \left(1 - \frac{\Omega \times t}{m}\right)^{-m} \tag{11}$$

From the MGF of $|\alpha|^2$, we calculate MGF of $\sum_{n=1}^{N} |\alpha_n|^2$ to be

$$\psi(t) = \big(1 - \frac{\Omega \times t}{m}\big)^{-mN} \tag{12}$$

assuming that α_n's, $n = 1, \ldots, N$ are i.i.d. fade coefficients of a Nakagami-m channel. Calculating second and fourth moments of $\sum_{n=1}^{N} |\alpha_n|^2$ using ψ, we can calculate the first and second moments of $P_r^{(N)} = \frac{1}{M_T}(\sum_{i=1}^{N} |\alpha_i|^2)^2$. So we can calculate the variance of $P_r^{(N)}$, which is

$$\begin{aligned} Var[P_r^{(N)}] &= E[(P_r^{(N)})^2] - E^2[P_r^{(N)}] \\ &= \frac{2N(mN+1)(2mN+3)\Omega^4}{m^3 \times M_T^2} \end{aligned} \tag{13}$$

and $E[P_r^{(N)}] = \frac{N(mN+1)\Omega^2}{m \times M_T}$. Hence the expression for coefficient of variation (in this case, its square) is

$$\frac{Var[P_r^{(N)}]}{E^2[P_r^{(N)}]} = \frac{2(2mN+3)}{mN(mN+1)} \to 0 \text{ as } N \to \infty \tag{14}$$

Now we show that for Nakagami-m channel, (6) is valid if the parameter m is the same regardless of the distance between the transmitter and receiver.

If $|\alpha_n|$ has a Nakagami-m distribution then $\gamma = |\alpha_n|^2$ has a Gamma distribution [6] given as

$$p_\gamma(\gamma) = \frac{m^m \gamma^{m-1}}{\bar{\gamma}^m \Gamma(m)} e^{-\frac{m\gamma}{\bar{\gamma}}}; \gamma \geq 0 \tag{15}$$

Hence $E[|\alpha_{r_0}|^4] = \frac{\bar{\gamma}_{r_0}^2}{m_{r_0}} + \bar{\gamma}_{r_0}^2$ and $E[|\alpha_r|^4] = \frac{\bar{\gamma}_r^2}{m_r} + \bar{\gamma}_r^2$. Now $E[|\alpha_r|^2] = E[|\alpha_{r_0}|^2]\ \frac{r_0^\nu}{r^\nu}$ or equivalently $\bar{\gamma}_r^2 = \bar{\gamma}_{r_0}^2\ \frac{r_0^{2\nu}}{r^{2\nu}}$. Hence if $m_{r_0} = m_r$, (6) follows.

APPENDIX II
NECESSARY AND SUFFICIENT CONDITION ON DECAY OF $E[|\alpha_r|^4]$

Result 4: Let the decay of $E[|\alpha_r|^4]$ be denoted by $w(r)$, that is $E[|\alpha_r|^4] = w(r) \times E[|\alpha_{r_0}|^4]$. Necessary and sufficient condition for estimating r using (7) by a single power measurement after linear processing at the receiver is that the equation:

$$\gamma w(r) + \delta \frac{r_0^{2\nu}}{r^{2\nu}} = P_r^{(N)} \tag{16}$$

has a unique solution in $[0, L]$ for all $P_r^{(N)} > E[P_L^{(N)}]$, where,

$$\gamma = \frac{N}{M_T} E[|\alpha_{r_0}|^4]$$

and

$$\delta = \frac{N(N-1)}{M_T} E^2[|\alpha_{r_0}|^2]$$

are constants.

Proof: If we do not use diversity, we know that power decay profile is $E[P_r] = \frac{E[P_{r_0}]\ r_0^\nu}{r^\nu}$ (1). This can be rewritten as $E[|\alpha_r|^2] = E[|\alpha_{r_0}|^2]\ \frac{r_0^\nu}{r^\nu}$. For the multiple antenna case, assuming α_n's are i.i.d. random variables distributed as α_r, we get

$$\begin{aligned} E[P_r^{(N)}] &= \frac{1}{M_T} E[(\sum_{n=1}^{N} |\alpha_n|^2)^2] \\ &= \frac{NE[|\alpha_r|^4] + N(N-1)E^2[|\alpha_r|^2]}{M_T} \\ &= \gamma w(r) + \delta \frac{r_0^{2\nu}}{r^{2\nu}} \end{aligned} \tag{17}$$

Since we want to estimate r from P_r, we replace $E[P_r]$ by P_r in (17). If the condition stated in **Result 4** is true, then r can be estimated using (17) uniquely (after replacing $E[P_r]$ by P_r) for all $P_r^{(N)} > E[P_L^{(N)}]$. For $P_r^{(N)} \leq E[P_L^{(N)}]$, we estimate r as $\hat{r}^{(N)} = L$.

If the condition is not true, then we will not get a unique estimate of r, and the method for estimation would fail. Thus the condition in **Result 4** is a necessary condition.

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