A Family of Power Allocation Schemes Achieving High Secondary User Rates in Spectrum Sharing OFDM Cognitive Radio

Mainak Chowdhury*, Anubhav Singla* and Ajit K. Chaturvedi†

***Abstract*—We propose a family of Secondary User (SU) power allocation schemes in spectrum sharing OFDM cognitive radio networks. The SU sum rates are maximized subject to a protection criterion for the Primary Users (PU) expressed through a utility function. We demonstrate that a specific choice of utility function leads to the maximization of weighted sum rate of all users, the weight being linked to a guarantee on the PU sum rate. Our formulation also allows trading off individual PU guarantees for higher SU sum rates, with a recently proposed scheme imposing individual PU Rate Loss guarantees being a limiting case.**

***Index Terms*—Cognitive radio, OFDM, Optimization methods, Distributed algorithms**

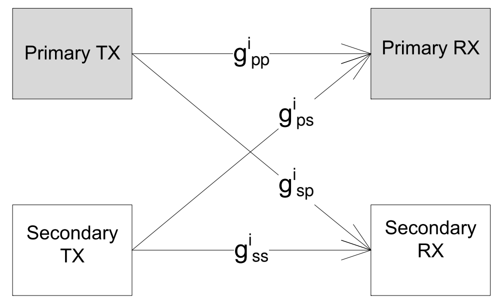

Fig. 1. The channel model for subcarrier i

I. Introduction

With the proliferation of services being offered over today's wireless links, available spectrum is becoming increasingly crowded [1]. Cognitive radio (CR) is one paradigm which has been proposed to deal with this problem. The CR deployment, in one of its typical forms (known as Spectrum Sharing [2], [3],[4],[5]), allows the transmission of multiple unlicensed users (Secondary Users or SU) on the same frequency bands as licensed users (Primary Users or PU) provided that the latter are sufficiently protected. Some of the protection criteria reported in the literature are constraints on the total interference suffered by the PU [6],[7] and on individual PU rates [8].

We explore alternative models of power allocation for SUs in the Spectrum Sharing model, with a view to increase their sum rates. We propose a family which maximizes SU sum rates subject to some guarantees on PUs expressed through the sum of utility functions. A specific member of the family have been shown to be related to an alternative formulation of the protection criteria for PU in terms of maximizing the weighted sum rates of all users (both PU and SU). We further point out a subfamily of power allocation schemes within the proposed family which trades off individual PU rate guarantees for higher SU sum rates. The subfamily also includes rate loss constraint (RLC) criterion on individual PU [8] as a limiting case.

The remaining paper is organized as follows. In Section II, we present the system model which we have considered. In Sections III and IV, we formulate the power allocation problem, describe solution strategies and derive some theoretical insights. Numerical comparisons with existing schemes have been presented in Section V.

II. System model

We consider the same system model as that in [8]. Let $\mathcal{I} = \{1, 2, \ldots, N\}$ be the set of subcarriers, $\mathcal{J} = \{1, 2, \ldots, M\}$ be the set of PUs, P_i be the SU transmitter power in i^{th} subcarrier, C_i be PU transmitter power in i^{th} subcarrier, and $\mathcal{K}_j$ be the set of all subcarriers allocated to the j^{th} PU. We assume that each subcarrier is allocated to exactly one PU, which implies that $\mathcal{K}_j$'s are disjoint sets. Let $\mathbf{P_j}$ be the vector with elements being SU transmitter powers allocated to subcarriers of the set $\mathcal{K}_j$, $\mathbf{P}$ being the full vector of SU transmitter powers allocated to all subcarriers. Also let us assume that the channel power gains between PU TxRx and SU TxRx pair are as given in Figure 1. We define the rate for the j^{th} PU as $r^p_j = \sum_{i \in \mathcal{K}_j} R^p_i$ where R^p_i is the rate achieved by the PU in the i^{th} subcarrier. In the spectrum sharing scenario,

$$R^p_i = \frac{1}{N} \log_2 \left(1 + \frac{g^i_{pp} C_i}{g^i_{ps} P_i + N_0} \right)$$

The SU rate in the i^{th} subcarrier is R^s_i where

$$R^s_i = \frac{1}{N} \log_2 \left(1 + \frac{g^i_{ss} P_i}{g^i_{sp} C_i + N_0} \right)$$

Note that PU transmission in i^{th} subcarrier suffers interference only from SU transmission in the same i^{th} subcarrier.

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III. Proposed scheme

The focus in the existing literature is on power allocation schemes which provide individual PU protection. With a view to increase SU sum rates, we explore an alternative model of ensuring protection to the PUs. We thus formulate a problem by imposing a sum utility rate guarantee on the PUs, instead of individual guarantees. In Section III-A we describe the basic formulation, and in Section III-B we describe optimization techniques to solve the problem thus formulated.

A. Formulation

We maximize the sum rate of the SU subject to a guarantee on the PUs expressed through the utility function $\mathcal{U}_j$ for the j^{th} PU. If the utility functions are concave and increasing, we ensure the convergence of an efficient decomposition-based solution for the problem.

$$\underset{\mathbf{P}\geq\mathbf{0}}{\text{maximize}} \sum_{i=1}^{N} R_i^s \tag{1a}$$

$$\text{subject to } \sum_{j=1}^{M} \mathcal{U}_j(r_j^p) \geq \delta \tag{1b}$$

$$\sum_{i=1}^{N} P_i \leq NP_a \tag{1c}$$

where P_a denotes the average (per subcarrier) power for SU(s). We are thus maximizing the sum rate of the SU(s) subject to a constraint on the total utility being provided to the PUs. Also, note that the formulation stays unchanged if the number of secondary users is changed to more than one. The case of one primary user reduces to the primary user rate constraint framework presented in [8].

B. Solving the optimization problem

This problem is a non-convex optimization problem coupled over N variables. We can de-couple the problem using primal and dual decomposition techniques ([9],Appendix A). This would lead to efficient distributed algorithms to solve the power allocation problem. The decomposition methods require only local channel power gains at the solvers, with only limited information (the subgradients) being transmitted to a central authority enforcing the scheme and hence avoids the overhead associated with transmitting all channel power information to a central authority.

Based on computational efficiency, we divide our problem into two cases: one where the number of subcarriers allocated per PU is small, and one where the number of subcarriers allocated per PU is large. A precise characterisation of 'small' and 'large' would depend on the specifics of the utility function chosen. The two conflicting issues that need to be balanced are ease of solving high dimensional problems (problem 3) and of maintaining time sharing property (Appendix D) .:

TABLE I
Dual decomposition for Problem (2)

Initialize
$(\lambda = 0, \mu = 0)$
$\mathbf{x}(\in \mathcal{R}^{2\times 1}) = (\lambda, \mu)^T$
A= $\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$
repeat
  $\lambda = \mathbf{x}(1)$ ($\mathbf{x}(i)$ refers to the i^{th} element of $\mathbf{x}$)
  $\mu = \mathbf{x}(2)$
  if (λ, μ) lies in feasible region (i.e. $\lambda \geq 0$ and $\mu \geq 0$) **then**
    for $j = 1$ to M **do**
      $\mathbf{P_j} \leftarrow \underset{\mathbf{P_j}\geq 0}{\text{argmin}} \sum_{i\in\mathcal{K}_j} -R_i^s - \lambda\mathcal{U}_j(\sum_{i\in\mathcal{K}_j} R_i^p) + \mu\sum_{i\in\mathcal{K}_j} P_i$
      (This is solvable in the most general case by an interior point solver. If problem is one dimensional then one might use simpler root finding techniques.)
    end for
    $g \leftarrow -(\delta - \sum_{j=1}^{M} \mathcal{U}_j(\sum_{i\in\mathcal{K}_j} R_i^p), \sum_{i\in\mathcal{K}_j} P_i - NP_a)^T$
  else
    $g \leftarrow \nabla h(\lambda, \mu)$ where $h(\lambda, \mu) \leq 0$ is any one feasibility condition that is violated. So $h = -\lambda$ or $h = -\mu$ according to whether $-\lambda \leq 0$ or $-\mu \leq 0$ is violated.
  end if
  $\tilde{g} \leftarrow \frac{g}{g^T A g}$
  $\mathbf{x} \leftarrow \mathbf{x} - \frac{1}{3} A\tilde{g}$
  $A = \frac{4}{3}(A - \frac{2}{3} A\tilde{g}\tilde{g}^T A)$
until $\sqrt{g^T A g} < \epsilon$
return $\mathbf{P} = (\mathbf{P_1^T}, \mathbf{P_2^T}, \ldots, \mathbf{P_M^T})^\mathbf{T}$

1) Small number of subcarriers per PU: In this case we solve the dual of (1):

$$\max_{\lambda\geq 0,\mu\geq 0} \sum_j h_j(\lambda, \mu) + \lambda\delta - \mu NP_a \tag{2}$$

where $h_j(\lambda, \mu)$ is defined as

$$h_j(\lambda, \mu) = \min_{\mathbf{P_j}\geq 0} - \sum_{i\in\mathcal{K}_j} R_i^s - \lambda\mathcal{U}_j\left(\sum_{i\in\mathcal{K}_j} R_i^p\right) + \mu\sum_{i\in\mathcal{K}_j} P_i \tag{3}$$

Problem as formulated in (2) is a two dimensional optimization problem and can be solved using an ellipsoid method [10], where at each iteration we solve M smaller dimensional problems (3) (details in Table I). Here M refers to the number of PUs.

2) Large number of subcarriers per PU: If we proceed as in the previous case, we find that for this case (3) remains an optimization problem in high dimensional space. So, the direct application of dual decomposition to this case would not achieve much decoupling. A more efficient approach is to perform a primal decomposition before the dual decomposition step. We note that problem 1 is equivalent to:

$$\underset{\mathbf{t},\mathbf{T}}{\text{maximize}} \sum_{j=1}^{M} \phi_j(t_j, T_j) \tag{4a}$$

$$\text{subject to } \sum_{j=1}^{M} t_j \geq \delta \tag{4b}$$

$$\sum_{j=1}^{M} T_j \leq NP_a \tag{4c}$$

$$T_j \geq 0 \tag{4d}$$

$$a_j \leq t_j \leq b_j \qquad \forall j \in \{1, \dots, M\} \tag{4e}$$

where ϕ_j is defined as follows:

$$\phi_j(t_j, T_j) = \max_{\mathbf{P_j} \geq 0} \sum_{i \in \mathcal{K}_j} R_i^s \tag{5a}$$

$$\text{subject to } r_j^p \geq \mathcal{U}_j^{-1}(t_j) \tag{5b}$$

$$\sum_{i \in \mathcal{K}_j} P_i \leq T_j \tag{5c}$$

Also note that, $\{a_j, b_j\}_{j=1}^{M}$ are constants chosen to avoid infeasibility of the subproblems to find ϕ_j. This is the primal decomposition step. The problem as formulated in (4) is an optimization in $\mathbf{t} = (t_1, t_2, \dots, t_M)$ and $\mathbf{T} = (T_1, T_2, \dots, T_M)$. The details of the solution algorithm have been presented in Table II. Justifications for the concavity of $\phi_j(\mathbf{t}, \mathbf{T})$, and the convergence of the primal decomposition step have been provided in Appendix D. To solve (5), we can use the dual decomposition technique. The dual formulation of (5) is

$$\underset{\lambda \geq 0, \mu \geq 0}{\text{maximize}} \quad \underset{\mathbf{P_j} \geq 0}{\text{minimize}} \sum_{i \in \mathcal{K}_j} -R_i^s + \lambda(-r_j^p + \mathcal{U}_j^{-1}(t_j)) + \mu(\sum_{i \in \mathcal{K}_j} P_i - T_j)$$

Plugging in the value of $r_j^p = \sum_{i \in \mathcal{K}_j} R_i^p$ and re-writing it we get the following formulation:

$$\max_{\lambda \geq 0, \mu \geq 0} \sum_{i \in \mathcal{K}_j} f_i(P_i) + \lambda \mathcal{U}_j^{-1}(t_j) - \mu T_j \tag{6}$$

$$f_i(P_i) = \min_{P_i \geq 0} -R_i^s - \lambda R_i^p + \mu P_i \tag{7}$$

Hence the problem (6) can be solved by solving one-dimensional optimization problems (7) for each subcarrier per iteration. Note that the one-dimensional problem to find $f_i(P_i)$ is a cubic polynomial root finding problem (details in appendix B). Justification of the use of dual decomposition to solve a primal problem in this case comes from the result in [11] that the duality gap tends to 0 as the number of subcarriers $\to \infty$.

IV. Some specific choices of the utility function

The discussion so far has assumed that $\mathcal{U}_j(x)$ is concave and monotonically increasing. In this section we propose specific utility functions ($\mathcal{U}_j(x)$ in Problem 1) and point out the nature of the optimal solution we get. However, a system can use any other utility function depending upon the application. Design of utility function for specific applications is outside the scope of this paper.

TABLE II
PRIMAL AND DUAL DECOMPOSITION FOR PROBLEM (4)

Initialize
$\mathbf{x}(\in \mathcal{R}^{2M \times 1}) = \frac{1}{M}(\delta, \dots (M - 2\,\text{times}), \delta, NP_a, \dots (M - 2\,\text{times}), NP_a)$
$A = \begin{pmatrix} \mathbf{I}_M(\delta) & 0 \\ 0 & \mathbf{I}_M(NP_a) \end{pmatrix}$
repeat
  $\mathbf{t} = \mathbf{x}(1:M)$
  ($\mathbf{x}(i:j)$ refers to the sub-vector of $\mathbf{x}$ between i^{th} and j^{th} elements)
  $\mathbf{T} = \mathbf{x}(M+1:2M)$
  if $\mathbf{x}$ lies in the feasible region **then**
    for $j = 1$ to M **do**
      $[\mathbf{P_j}, \lambda_j, \mu_j] \leftarrow \underset{\mathbf{P_j} \geq 0}{\text{argmin}}\ \phi_j(t_j, T_j)$
      (λ_j and μ_j are the dual variables for problem in (5))
      (This has been solved by dual decomposition)
    end for
    $g(\in \mathcal{R}^{2M \times 1}) \leftarrow (\frac{\lambda_1}{\mathcal{U}_{|}{}'(r_1^p)}, \dots, \frac{\lambda_M}{\mathcal{U}_{\mathcal{M}}{}'(r_M^p)}, -\mu_1, \dots, -\mu_M)^T$
  else
    $g \leftarrow \nabla h(\mathbf{x})$ where $h(\mathbf{x}) \leq 0$ is any one feasibility condition that is violated.
  end if
  $\tilde{g} \leftarrow \frac{g}{g^T A g}$
  $\mathbf{x} \leftarrow \mathbf{x} - \frac{1}{2M+1} A \tilde{g}$
  $A = \frac{4M^2}{4M^2-1}(A - \frac{2}{2M+1} A \tilde{g} \tilde{g}^T A)$
until $\sqrt{g^T A g} < \epsilon$
return $\mathbf{P} = (\mathbf{P_1^T}, \mathbf{P_2^T}, \dots, \mathbf{P_M^T})^\mathbf{T}$

A. Rate Loss Constraints (RLC)

In [8] a scheme was proposed to maximize the SU sum rate subject to rate loss guarantees (equivalent to minimum rate guarantees) to every PU.

$$\underset{\mathbf{P} \geq \mathbf{0}}{\text{maximize}} \sum_{i=1}^{N} R_i^s \tag{8a}$$

$$\text{subject to } r_j^p \geq r_j^{p0} \quad \forall j \in \mathcal{J} \tag{8b}$$

$$\sum_{i=1}^{N} P_i \leq NP_a \tag{8c}$$

For the comparison of our scheme with RLC (8) we choose δ in (1) equal to sum utility rate guarantee we get in the RLC scheme :

$$\delta = \sum_{j \in \mathcal{J}} \mathcal{U}_j(r_j^{p0}) \tag{9}$$

For this choice of δ and for any choice of utility functions $\mathcal{U}_j$ our scheme would always give SU sum rates higher than (or equal to) those of the RLC scheme (8) since we are searching over a larger search space. In other words, for the value of δ given by (9), feasible region of SU power allocation defined by RLC scheme is a subset of the feasible region of the proposed scheme. This also implies that the proposed scheme may lead to a violation of individual PU rate guarantees of the RLC scheme.

Our formulation also provides a method to trade off individual PU guarantees for higher SU sum rates, and at the same time maintaining the net (total) utility guarantees at the same level as in the original formulations in the literature. Our formulation may in fact be shown to contain the RLC formulation as a limiting case. To see this, define a subfamily of utility functions, parametrized by α as follows:

$$\mathcal{U}_j^\alpha(x) = \frac{\left(\frac{x}{r_j^{p0}}\right)^{1-\alpha}}{1-\alpha} \quad \text{if } \alpha \neq 1 \qquad (10)$$
$$= \log\left(\frac{x}{r_j^{p0}}\right) \quad \text{otherwise}$$

By letting $\alpha \to \infty$ in the problem as formulated in (1), we see that the feasible region would converge towards that of the RLC scheme (Please see Appendix C). Thus by choosing a reasonable α one can trade off higher SU sum rates with individual PU guarantees.

B. The maximization of the weighted sum rate

In this subsection we demonstrate how the solution to the problem as formulated in (1) would maximize a weighted sum of the primary and secondary user rates for a suitable choice of utility functions. We consider following problem.

$$\underset{\mathbf{P}\geq 0}{\text{maximize}} \sum_{i\in\mathcal{I}} R_i^s + \omega_p \sum_{i\in\mathcal{I}} R_i^p \text{ subject to } \sum_{i=1}^{N} P_i \leq NP_a \qquad (11a)$$

where ω_p represents the weight given to the PU sum rate with respect to unit weight of SU sum rate, which is same as rate of every PU being equally weighted by ω_p and unit weight for rate of every SU.We now consider the problem dual to the problem formulated in 1 with $\delta = \delta_0$ and the utility function chosen as $\mathcal{U}_j(x) = x$. Then the dual problem is

$$\underset{\lambda\geq 0,\mu\geq 0}{\text{maximize}} \quad \underset{\mathbf{P}\geq 0}{\text{minimize}} \sum_{i\in\mathcal{I}} -R_i^s + \lambda(-\sum_{j\in\mathcal{J}}(r_j^p) + \delta_0) + \mu(\sum_{i\in\mathcal{I}} P_i - NP_a) \qquad (12)$$

Let the optimal value of this problem be achieved at some power allocation with dual variables equal to (μ_0, λ_0). The same power allocation would also solve the dual of the following problem (using the definition of r_j^p):

$$\underset{\mathbf{P}\geq 0}{\text{maximize}} \sum_{i\in\mathcal{I}} R_i^s + \lambda_0 \sum_{i\in\mathcal{I}} R_i^p \text{ subject to } \sum_{i=1}^{N} P_i \leq NP_a \qquad (13)$$

On comparing 11 and 13 we conclude that the weight ω_p is nothing but λ_0. If we assume that duality gap is zero, then we have shown that the proposed problem (with $\mathcal{U}_j(x) = x$ and $\delta = \delta_0$) also solves the weighted sum rate maximization problem (11) with weight $\omega_p = \lambda_0$. The assumption about the duality gap being zero may be justified by a result, shown in [11], that the duality gap for our problem $\to 0$ as the number of subcarriers $\to \infty$. Thus by choosing an appropriate δ and the utility function as $\mathcal{U}_j(x) = x$, one can maximize weighted sum rate for any weight. Weighted sum rate maximization problem is relevant, for example, in determining the revenue maximizing power allocation, given that for every ω_p dollars per unit rate the PUs pay, the SUs pay 1 dollar.

V. Numerical Results and Discussions

In this section we present numerical results to study the performance of the proposed power allocation scheme (1). We have chosen the utility functions to be members of the subfamily $\mathcal{U}_j^\alpha$ as defined in (10). We consider the case in Section III-B2 since the case in Section III-B1 would lead to larger and possibly unrepresentative gains with respect to existing schemes (e.g. in [8]). We thus consider a total of 128 subcarriers ($N = 128$) and 2 PUs ($M = 2$) with equal number of subcarries allocated to both the PUs ($K_1 = \{1, 2, ..., 64\}, K_2 = \{65, 66, ..., 128\}$). Per subcarrier PU transmit power budget has been assumed to be 10 dB ($C_i = 10$). The channel gains have been assumed to be Rayleigh distributed, thus the g parameters (power gains) are exponentially distributed. $E(g_{pp}) = E(g_{ss}) = 1$ and $E(g_{ps}) = E(g_{sp}) = 0.1$ where $E(x)$ refers to the expectation of x. The noise term N_0 has been taken to be 1. Every plot has been averaged over 150 independent channel realizations.

We have also compared the proposed scheme with the RLC scheme [8]. For RLC scheme the maximum allowable rate losses have been chosen to be 5% for PU 1 and 25% for PU 2. This would give us value of r_j^{p0} for RLC scheme (8) as well as value of δ (9) and U_j^α (10).

Figure 2 shows the variation of SU sum rates with SU average per subcarrier power budget. Three of the curves correspond to proposed schemes with different values of $\mathcal{U}_j^\alpha$ with $\alpha = 0, 2, 8$. The fourth curve corresponds to RLC scheme [8]. Here we note that the SU sum rates achieved with proposed schemes are higher (Section IV-A) than those of the RLC scheme. In general the more strict the rate constraints on some user(s), the more pronounced will be the differences between the rate curves from the different schemes. Thus, for example Rate Loss guarantees of 15% to each user would correspond to less pronounced differences in the Figure 2.

In Figure 3, we plot the rates of individual PU for the same four schemes (as described above) as a function of SU average per subcarrier power budget. We observe that the proposed schemes do not provide individual PU rate guarantee provided in the RLC scheme. Thus higher SU rates in the proposed scheme come at the cost of individual PU guarantees of RLC scheme (as discussed in Section IV-A). However, the sum of utility of rates guarantee of the proposed scheme remain same as that of the RLC scheme (as given by eqn. (9)).

Another interesting observation which can be made from Figure 2 and Figure 3 is that the value of α can be used to trade-off higher SU sum rates with guarantees on individual PU rates. We observe that as α increases individual PU rate approaches r_j^{p0} (Section IV-A) but SU sum rates decrease. In other words, α can be used as a tuning parameter. For $\alpha = 0$, the PU guarantees are not strong, but SU sum rates are higher,

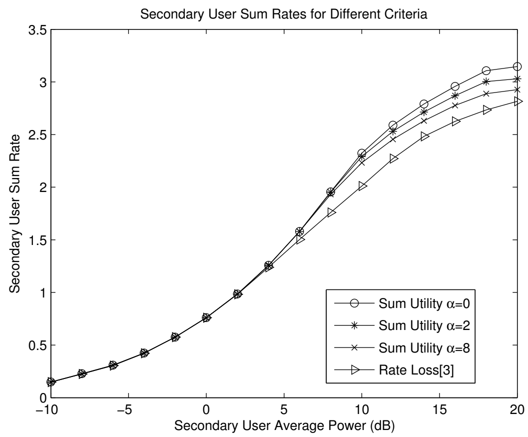

Fig. 2. SU sum rates for proposed schemes(Problem as formulated in (1) with different utility functions) compared with those for the RLC scheme (Problem as formulated in (8))

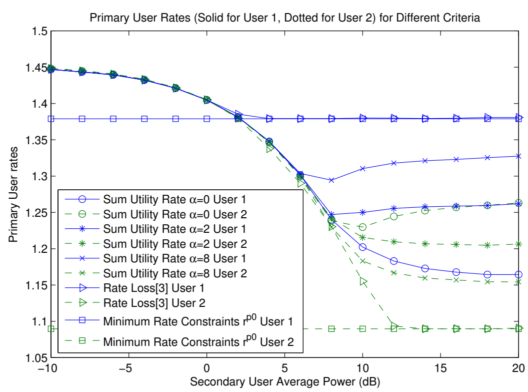

Fig. 3. Guarantees for individual Primary Users with different utility functions in Problem as formulated in (1)

where as for $\alpha = 8$, SU sum rates are lower and individual PU rates are closer to r_j^{p0}.

In Figure 4 we plot weighted sum rates (Section IV-B) achieved as a function of SU Power Budget for two schemes: one with direct maximization of weighted sum rate (11) and the other with the proposed scheme (1) with $\mathcal{U}(x) = x$. To compare the two schemes, ω_p was fixed at 1.5 and $\sum_{i\in\mathcal{I}} R_i^p (= \delta)$ was evaluated by using the solution to the problem as formulated in (11). Using the obtained δ and $\mathcal{U}(x) = x$, the proposed problem (1) was solved. We note that the curves are indistinguishable, suggesting that the duality gap is small for the problem parameters considered. The plots thus bear out the claims made in Section IV-B, about the relation between the solutions of the problems as formulated in (1) and (11).

VI. Conclusion

A family of power allocation schemes for spectrum sharing OFDM cognitive radio networks has been presented. Through numerical and analytical investigations, it has been established that the family allows a tradeoff between higher SU sum rates and individual PU rate loss guarantees. One recent power allocation problem formulation (Rate Loss Constraints) has been shown to be a limiting case of the family described here. A specific member has also been shown to maximize the weighted sum rate of all users.

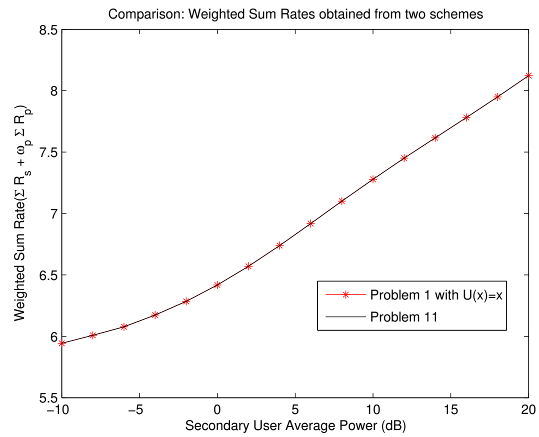

Fig. 4. Comparison of weighted sum rates obtained from Problem as formulated in (11) (direct maximization) with those from Problem 1

Appendix

A. Primal and Dual decomposition

We use decomposition techniques in conjunction with the ellipsoid algorithm [10]. The ellipsoid algorithm works by localizing the solution point in an ellipsoid. Assuming a convex objective function, the algorithm locates a hyperplane separating the feasible region from non-feasible region, and passing through the center of the ellipsoid. This hyperplane can be computed by finding out the subgradient to the objective function at the center. It then finds out the minimum volume ellipsoid containing the feasible region. The algorithm proceeds this way until the volume of the localizing ellipsoid reduces below a threshold.

1) Subgradient expression for primal decomposition methods: Here we document subgradients of functions like the following

$$\begin{aligned} \phi(a) = &\underset{\mathbf{x}}{\text{minimize}} && f(x) \\ &\text{subject to} && g(x) \le a \end{aligned} \tag{14}$$

We claim that the subgradient is $-\lambda$ where λ is the dual variable corresponding to the optimal point. The proof, which may be found in standard references (e.g. [9]), is omitted due to space constraints.

2) Subgradient expressions for dual decomposition methods: Consider the dual function for the problem above

$$h(\lambda) = \underset{\mathbf{x}}{\text{minimize}}\ f(x) + \lambda(g(x) - a)$$

We claim that the subgradient to the negative of the function (which is convex) is $g(x_1) - a$, where x_1 is the point where the dual function $h(\lambda)$ is minimized. The proof is again omitted due to space constraints.

B. One-Dimensional Problem Specification

We present here the form of the one dimensional subproblem referred to in the problem as formulated in (7). This can be rewritten as

$$\begin{aligned} \underset{\mathbf{P}\geq\mathbf{0}}{\text{minimize}} \quad & -\log\left(1+\frac{g_{ss}^i P_i}{g_{sp}^i C_i + N_0}\right) \\ & -\lambda\log\left(1+\frac{g_{pp}^i C_i}{g_{ps}^i P_i + N_0}\right) + \mu P_i \end{aligned} \tag{15}$$

Since this is a one dimensional problem the solution is either at $P = 0$ or at an unconstrained extremum. The necessary first order condition that the latter needs to satisfy is given by

$$\frac{g_{ss}^i}{g_{sp}^i C_i + N_0 + g_{ss}^i P_i} - \frac{\lambda g_{pp}^i g_{ps}^i C_i}{(g_{ps}^i P_i + N_0)^2 + (g_{ps}^i P_i + N_0) g_{pp}^i C_i} = \mu N \log 2 \tag{16}$$

This may be rewritten as a cubic equation for which analytical closed form solutions exist.

C. Convergence to rate loss guarantees

Consider any point $\mathbf{R} = (r_1^p, r_2^p, \ldots, r_M^p)$ on the boundary of the feasible region

$$\left(\sum_{i=1}^{M}\left(\frac{r_i^p}{r_i^{p0}}\right)^{1-\alpha} \leq c^2\right)$$

Here r_i^{p0} are constants (with respect to α) for all i (and $c^2 \geq 1$). Since all the terms in the LHS are positive, we have

$$\left(\min_i \frac{r_i^p}{r_i^{p0}}\right)^{1-\alpha} \leq c^2$$

Since $\alpha > 1$ we get $\left(\min_i \frac{r_i^p}{r_i^{p0}}\right) \geq c^{\frac{2}{1-\alpha}}$ As $\alpha \to \infty$, we get

$$\left(\min_i \frac{r_i^p}{r_i^{p0}}\right) \geq 1$$

But this is nothing but the RLC constraint. Here r_i^{p0} may be interpreted as the rate guaranteed to the i^{th} primary user.

D. Convergence guarantee for primal decomposition step

The primal decomposition step (4) used to solve problem formulated in (1) is guaranteed to converge if $\phi_j(\mathbf{t}, \mathbf{T})$, as defined in (5), is concave in $\mathbf{t}, \mathbf{T}$. We show how time-sharing property for OFDM systems together with the concavity of the utility functions would guarantee the concavity of ϕ. Let us assume that the power allocation $\mathbf{P}^A$ gives the value of $\phi_j(x^A, y^A)$ and $\mathbf{P}^B$ of $\phi_j(x^B, y^B)$. Also define $\mathbf{P}^\lambda$ as the vector obtained by "frequency sharing [11]" λ fraction of $\mathbf{P}^A$ and $(1-\lambda)$ fraction of $\mathbf{P}^B$. If we assume that time sharing property holds, $\mathbf{P}^\lambda$ satisfies the constraints as defined in (4) for $\phi_j(\lambda x^A + (1-\lambda)x^B, \lambda y^A + (1-\lambda)y^B)$:

$$\sum_{i\in\mathcal{K}_j} P_i^\lambda = \lambda \sum_{i\in\mathcal{K}_j} P_i^A + (1-\lambda)\sum_{i\in\mathcal{K}_j} P_i^B \leq \lambda y^A + (1-\lambda) y^B$$

Again using time sharing property and concavity of $\mathcal{U}_j$ (or convexity of $\mathcal{U}_j^{-1}$) we get:

$$\begin{aligned} r_j^p(\mathbf{P}^\lambda) &= \lambda r_j^p(\mathbf{P}^A) + (1-\lambda) r_j^p(\mathbf{P}^B) \\ &\geq \lambda \mathcal{U}_j^{-1}(x^A) + (1-\lambda)\mathcal{U}_j^{-1}(x^B) \\ &\geq \mathcal{U}_j^{-1}(\lambda x^A + (1-\lambda)x^B) \end{aligned}$$

Also note that

$$\begin{aligned} &\phi_j(\lambda x^A + (1-\lambda)x^B, \lambda y^A + (1-\lambda)y^B) \\ &\geq \sum_{i\in\mathcal{K}_j} R_i^s(P_i^\lambda) \\ &= \lambda \sum_{i\in\mathcal{K}_j} R_i^s(P_i^A) + (1-\lambda)\sum_{i\in\mathcal{K}_j} R_i^s(P_i^B) \\ &= \lambda\phi_j(x^A, y^A) + (1-\lambda)\phi_j(x^B, y^B) \end{aligned}$$

which gives us the required concavity:

$$\begin{aligned} &\phi_j(\lambda x^A + (1-\lambda)x^B, \lambda y^A + (1-\lambda)y^B) \\ &\geq \lambda\phi_j(x^A, y^A) + (1-\lambda)\phi_j(x^B, y^B) \end{aligned}$$

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