Department of Mechanical Engineering, Indian Institute of Technology Kanpur
Ph.D Comprehensive Examination (Mathmatics)- Written part: November 2016

Time: 2 hours **Maximum Marks: 100**

Problem 1 (Linear Algebra): (10 + 10 Marks)

(i) For what values of ω will the following system of equations posses a non-trivial solution? Why?

$$\omega x_1 + x_2 = 0$$
$$x_1 + \omega x_2 - x_3 = 0$$
$$-x_2 + \omega x_3 = 0.$$

(ii) Determine the eigenvalues and the associated eigenspaces for the following matrices: (a) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and (b) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Also determine the geometric and the algebraic multiplicity in each case.

Problem 2 (Vector Analysis): (8+4+4+4 Marks)

(i) Let $\mathbf{v}$ be a vector field such that $\text{curl}\,\mathbf{v} = \mathbf{0}$ in a three-dimensional domain Ω. Use Stokes' theorem to show that the evaluation of line integral

$$\int_{\mathbf{X}_0}^{\mathbf{Y}} \mathbf{v} \cdot d\mathbf{X}$$

is independent of the path between two points $\mathbf{X}_0$ and $\mathbf{Y}$ in Ω. In other words, it depends only on the initial and the final point. Use this result to infer that there exists a scalar field ϕ such that $\mathbf{v} = \nabla\phi$. Here ∇ indicates the gradient.

(ii) Let $\mathbf{e}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|$. Calculate $\nabla\mathbf{e}$.

(iii) Let $\mathbf{u}(\mathbf{x}) = \mathbf{x}/|\mathbf{x}|^3$. Show that $\mathbf{u}$ is harmonic (that is, its Laplacian is zero).

(iv) Let $\mathbf{w}$ and $\mathbf{v}$ be vector fields. Show that $\text{div}(\mathbf{w} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{w} - \mathbf{w} \cdot \text{curl}\,\mathbf{v}$.

Problem 3 (ODE): (7+6+7 Marks)

(i) (a) Find the solution of the following first order ODE:

$$y' = x\left(\frac{1}{y} - y\right), \quad (y \geq 2).$$

(b) Evaluate the constant of integration from the initial condition $y(0) = 2$.

(ii) (a) Find the *integrating factor* of the following first order ODE:

$$(2\cos y + 4x^2)dx + (-x\sin y)dy = 0, \quad (x > 0).$$

(b) Multiply the above ODE by the integrating factor found in part (a), and show that the resulting ODE is an *exact* differential equation.

(iii) Find the general solution of the following second order *homogeneous* ODE with constant coefficients:

$$9y'' - 30y' + 25y = 0.$$

In case of a double root of the characteristic equation, use the method of *reduction of order* to find the second independent solution.

Problem 4 (PDE): (20 Marks)

A square plate of side 12 units is held at 0^oC temperature at three boundaries: (i) $x = 0$, $x = 12$, and $y = 0$. The temperature at the boundary $y = 12$ varies with x: $T = \sin(\pi x/4)$. Thus, the temperature is governed by the following boundary value problem consisting of an *elliptic* PDE and boundary conditions:

$$\text{PDE:} \quad \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

$$\text{BC:} \quad (i)\ T(0, y) = 0, \quad (ii)\ T(12, y) = 0, \quad \text{for } 0 \leq y \leq 12;$$

$$(iii)\ T(x, 0) = 0, \quad (iv)\ T(x, 12) = \sin(\pi x/4), \quad \text{for } 0 \leq x \leq 12.$$

Obtain the solution of the problem, i.e., the expression for $T(x, y)$.

Problem 5 (Numerical Methods): (10 Marks)

Retain at least 5 significant digits in your calculation.

(a) Carry out 5 steps, using the step size of $h = 0.1$, of the *Euler method* for the following initial value problem:

$$\text{ODE:} \quad y' = -0.1y,$$

$$\text{IC: } y(0) = 2.$$

(b) The exact solution of the above problem is $y = 2e^{-0.1x}$. Find the *error* at each step.

Problem 6 (Statistics): (5+5 Marks)

(i) Find the mean and the variance of a random variable with probability density function $f(x) = 2e^{-2x}$, $(x \geq 0)$.

(ii) The breakage strength of a certain type of plastic block is normally distributed with a mean of 1250 Kg and a standard deviation of 55 Kg. What is the maximum load such that we can expect no more than 5% of the blocks to break?