Teaching of rotations, Euler angles, and angular velocity

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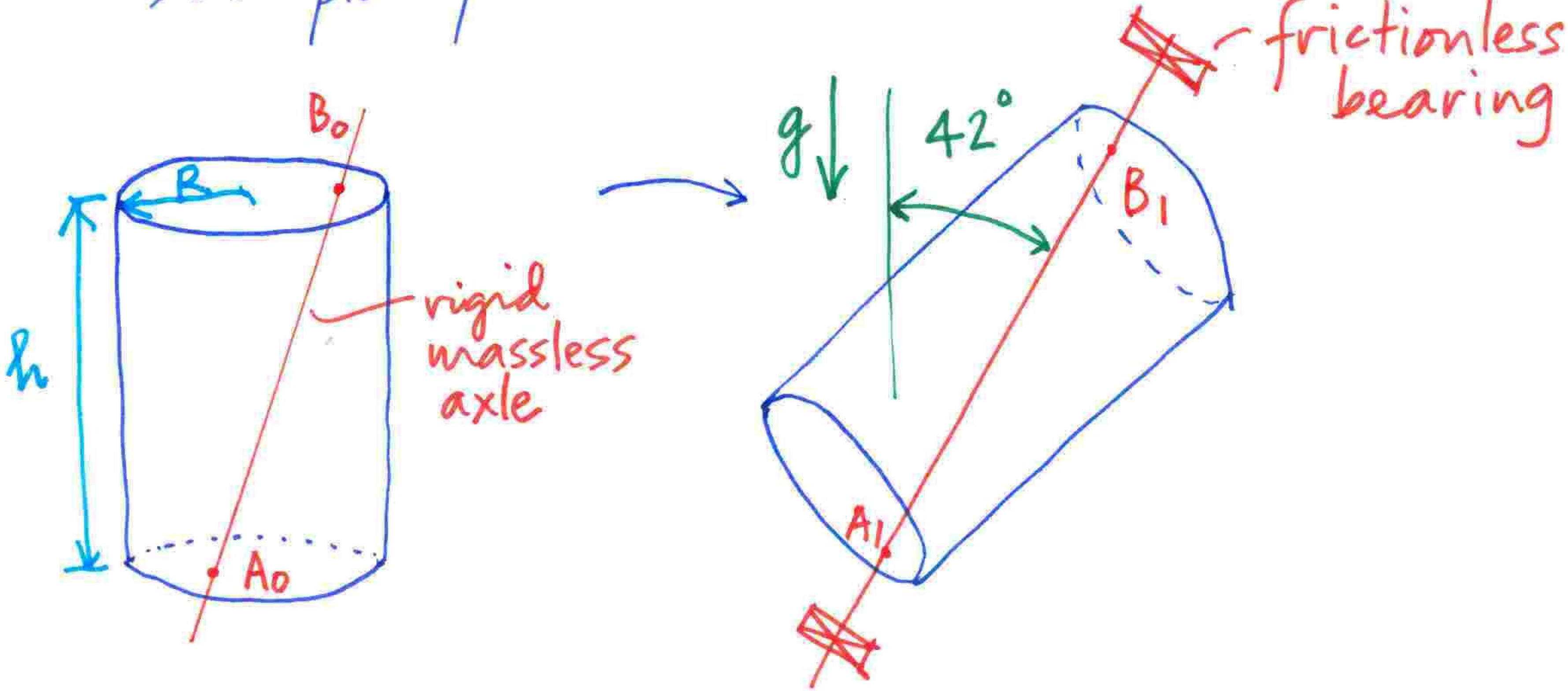
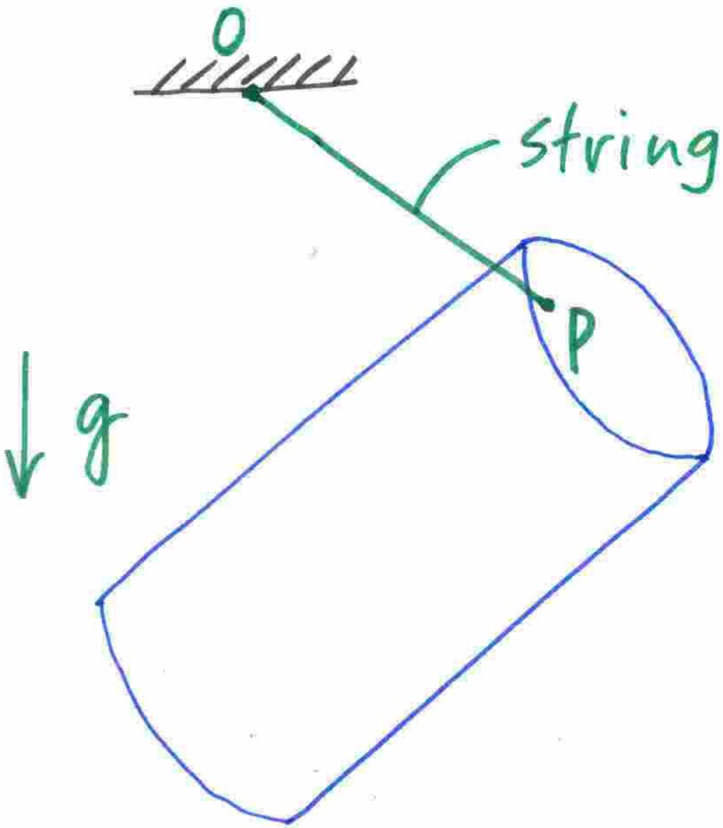
Example problem 1



Find the natural frequency of small oscillations about the stable equilibrium position.

Example problem 2

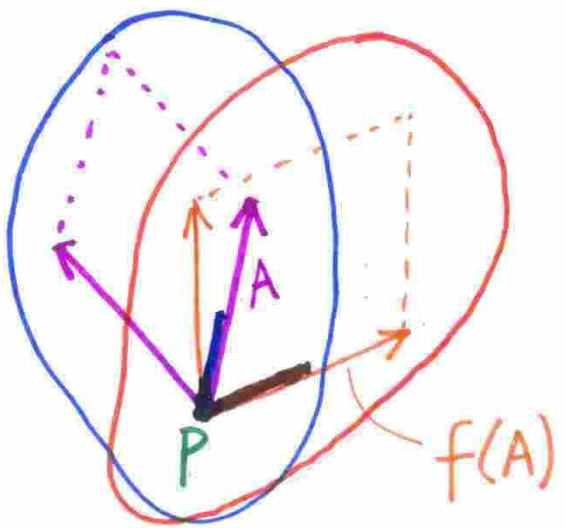


Cylinder suspended by string, released from rest.

Find the initial angular acceleration & tension in string.

Euler's Theorem

The most general displacement of a rigid body with one point fixed is a rotation about some axis passing through that point.



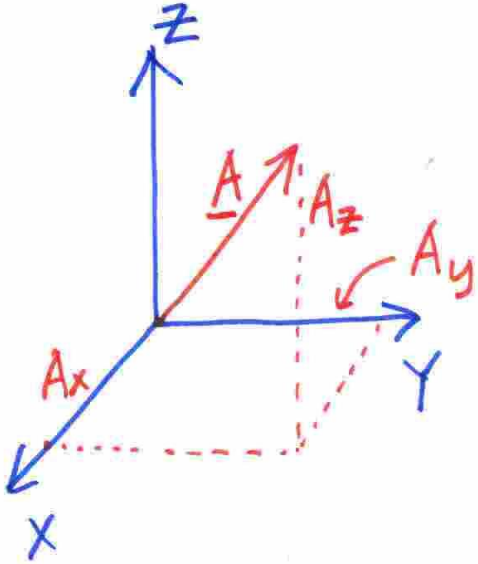
$$\underline{A} \rightarrow \underline{f}(\underline{A})$$

$$\underline{f}(\alpha \underline{A}) = \alpha \underline{f}(\underline{A})$$

$$\underline{f}(\underline{A}+\underline{B}) = \underline{f}(\underline{A}) + \underline{f}(\underline{B})$$

$\underline{f}$ is linear

Choose a Right Handed Orthonormal (RHON) coordinate system (basis) and hold it fixed.



$$\underline{A} \equiv \begin{Bmatrix} A_x \\ A_y \\ A_z \end{Bmatrix} , \quad \underline{a} \equiv \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = a$$

(equivalent, not equal)

$$\underline{f}(\div) \equiv f(\cdot)$$

$a \rightarrow Ra$ for some 3X3 matrix R.

$a' = Ra$, $\quad a'^T a' = a^T R^T R a = a^T a$

$\Rightarrow R^T R = I$ by arbitrariness of a.

$$R^T R = I \Rightarrow \det(R^T)\det(R) = 1$$

$$[\det(R)]^2 = 1, \quad \det(R) = \pm 1,$$

$$\therefore \det(R) = 1 \text{ by continuity.}$$

R is 3x3. 3 is odd. R has at least ONE real eigenvalue.

Let λ be a real eigenvalue of R, with corresponding eigenvector u.

$$Ru = \lambda u \Rightarrow (Ru)^T(Ru) = \lambda^2 u^T u$$

$$u^T R^T R u = u^T u = \lambda^2 u^T u \Rightarrow \lambda^2 = 1, \lambda = \pm 1.$$

If all eigenvalues are real, we can have 1, 1, 1 and 1, -1, -1.

If 2 eigenvalues are complex, σ and $\bar{\sigma}$, then

$$\det(R) = \lambda\sigma\bar{\sigma} = \lambda|\sigma|^2 = 1, \therefore \lambda > 0 \Rightarrow \lambda = 1.$$

$\therefore$ $\lambda = 1$ is an eigenvalue of R.

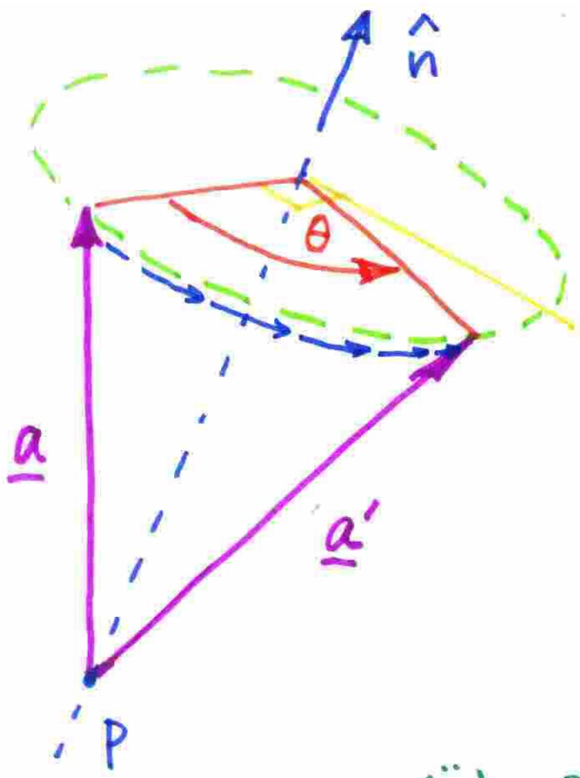
$Ru = u$ for some u.

There is a vector that is fixed in the body.

$\therefore$ The most general motion, etc. etc.

Consider a rotation about $\hat{n}$ through θ.

What is $R(n, \theta)$?



Work out R

(i) $\underline{a} \times \underline{b} \equiv S(a)\, b = -S(b)\, a$

$$S(a) = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}$$

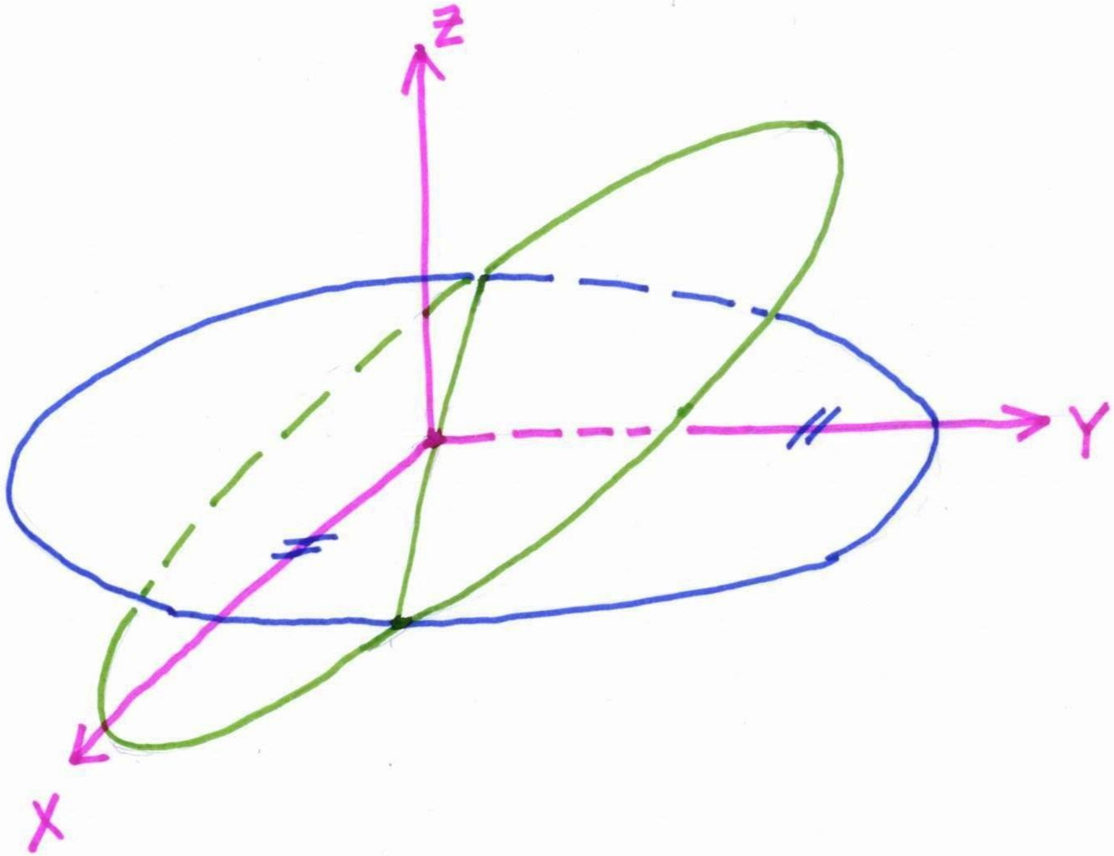
(ii) $R(n,\theta) = (1-\cos\theta)\, n n^T$

$+ \cos\theta \cdot I$

$+ \sin\theta \cdot S(n)$

(iii) $a' = R(n,\theta)\, a$

Euler Angles (3-1-3)



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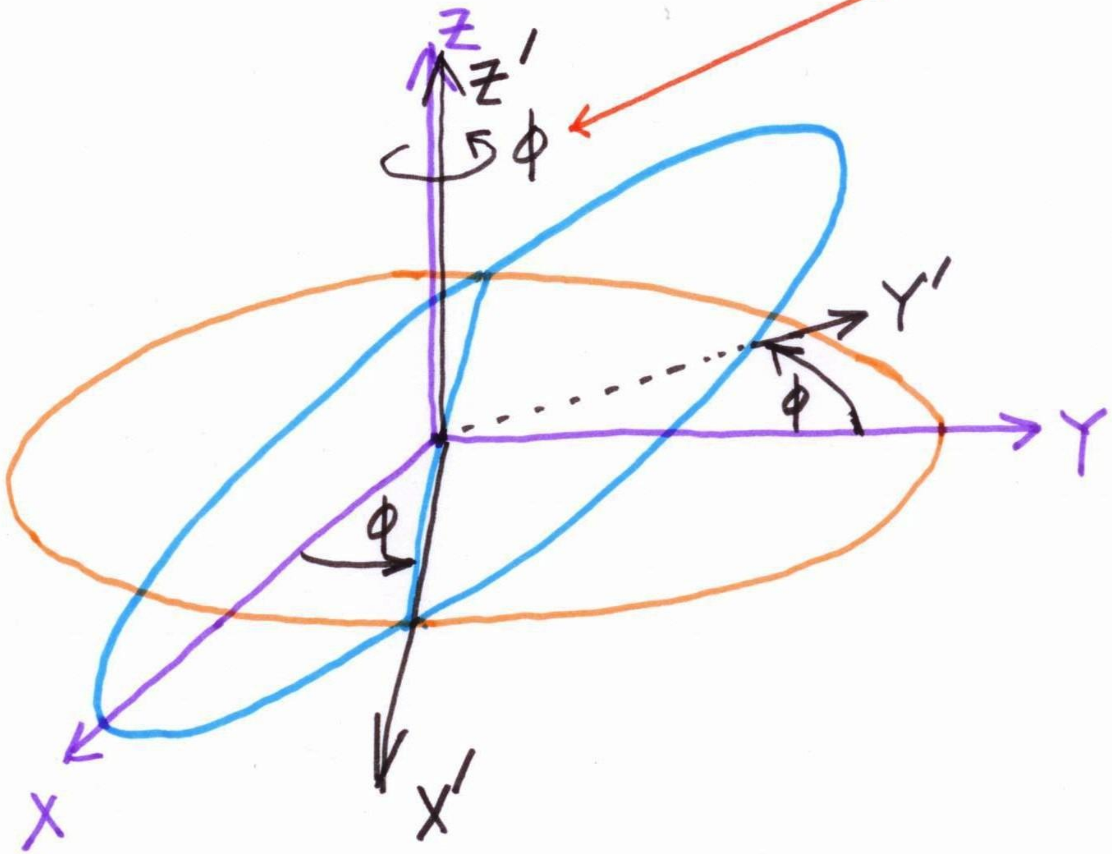
Euler Angles (3-1-3)



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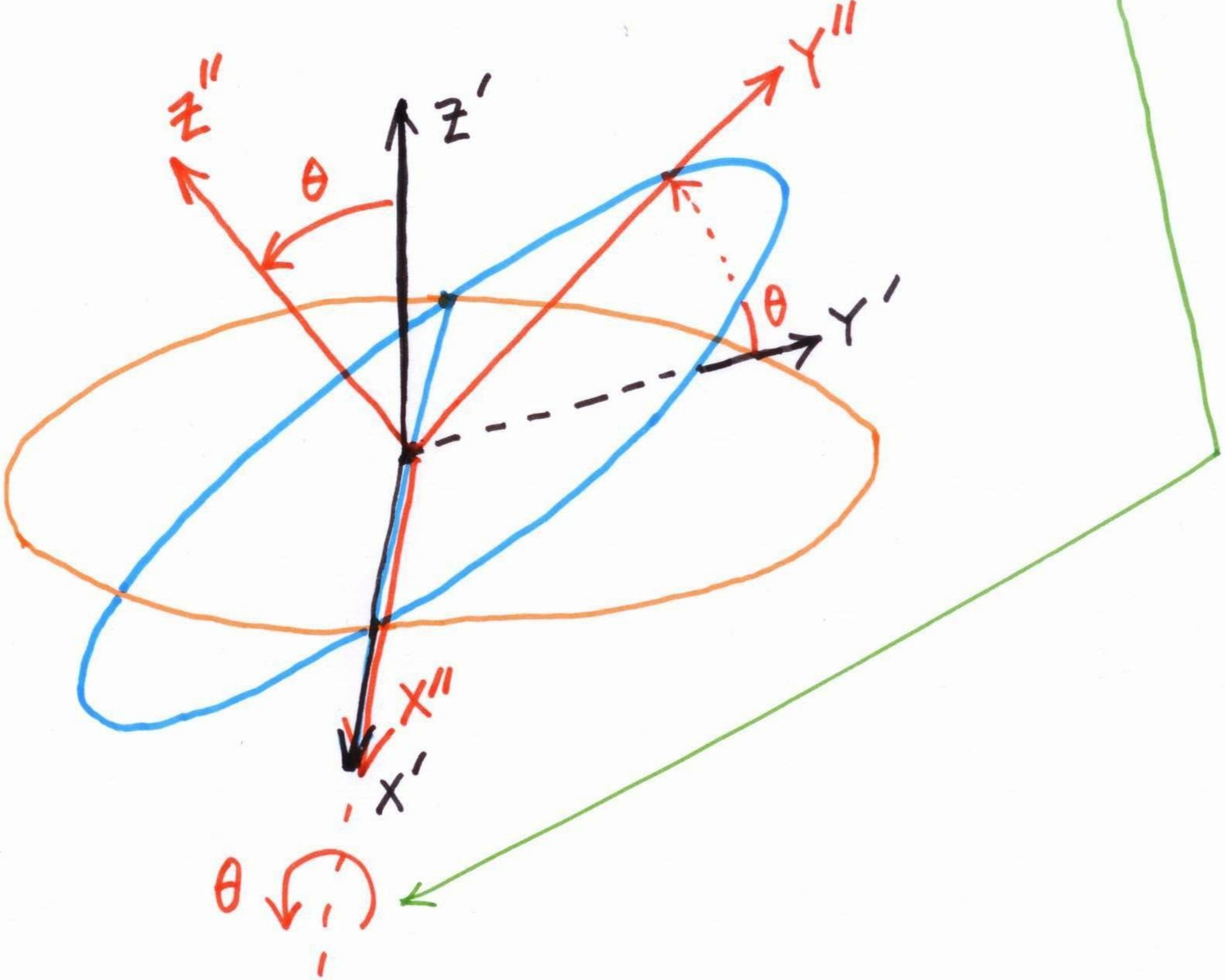
Euler Angles (3-1-3)



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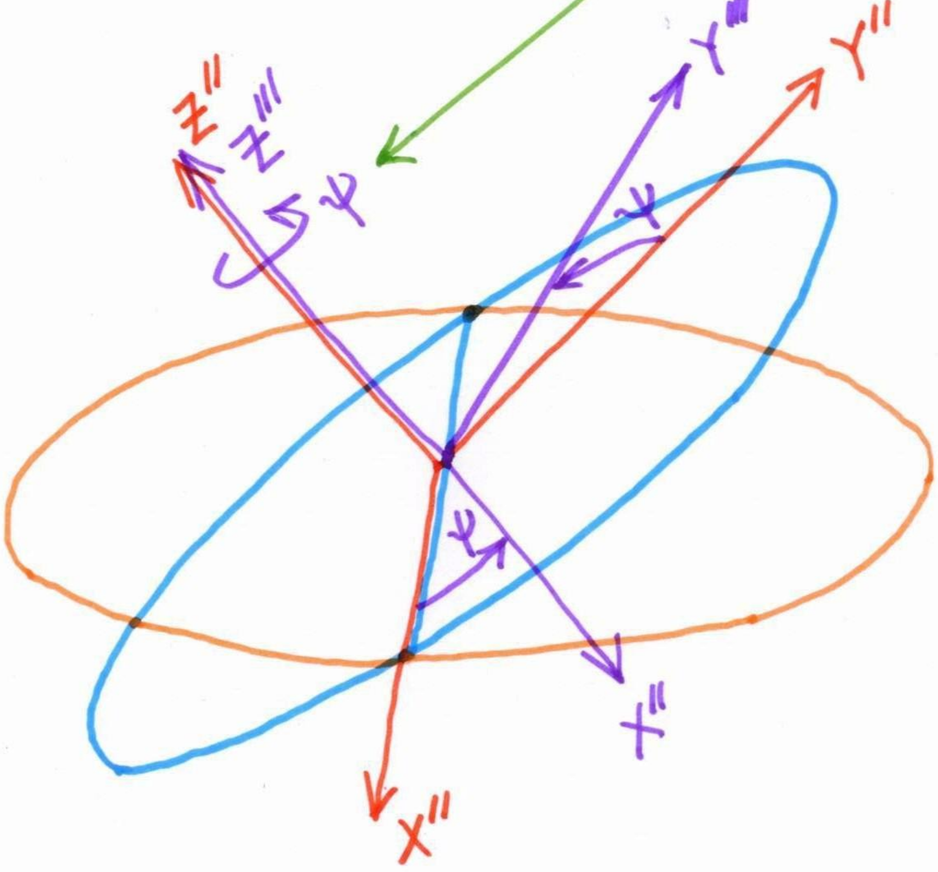
Euler Angles (3-1-3)



$\hat{i} = \hat{e}_1 \equiv \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$ — e_1 , $\hat{j} = \hat{e}_2 \equiv \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$ — e_2

$\hat{k} = \hat{e}_3 \equiv \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$ — e_3

$$R_1 = R(e_3, \phi)$$

$$R_2 = R(R_1 e_1, \theta)$$

$$R_3 = R(R_2 R_1 e_3, \psi)$$

$$R_{net} = R_3 R_2 R_1$$

- It can be shown that

$$R_{net} = R(e_3, \phi)\, R(e_1, \theta)\, R(e_3, \psi) \quad *$$

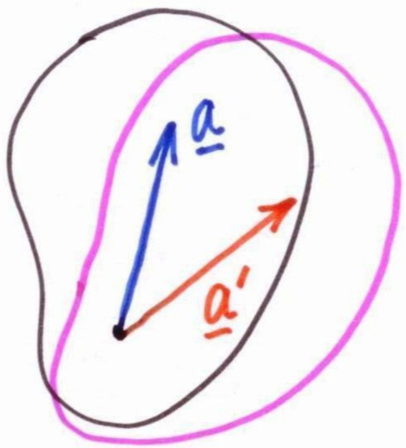
(simpler in computations) (by hand!)

- Typical textbooks hold the vector fixed and rotate coordinate systems, which leads to *, but which I find confusing

- Present approach is good for computation by computer.

  Matlab OR Maple

Note : $a' = Ra \longleftrightarrow a = R^T a'$



$$RR^T = I$$
$$\dot{R}R^T + R\dot{R}^T = 0$$
$$\dot{R}R^T + (\dot{R}R^T)^T = 0$$
$\dot{R}R^T$ is skew symmetric

$= S(\omega)$, say, for some $\underline{\underline{\omega}}$ (vector!)

$$a' = Ra$$
$$\dot{a}' = \dot{R}a = \dot{R}R^T a' = S(\omega)a'$$
$$\frac{d}{dt}\underline{a}' = \underline{\omega} \times \underline{a}'$$

Definition 1 of angular velocity.

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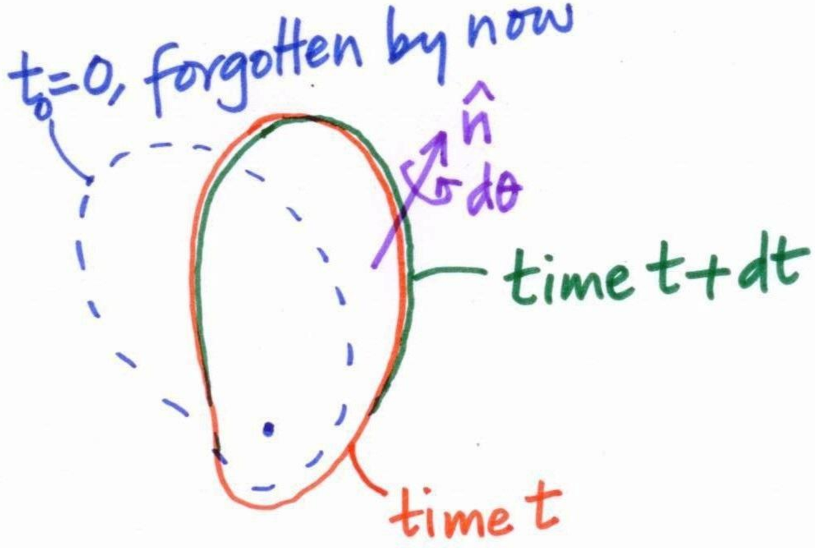
By Euler's theorem,
in infinitesimal time interval dt,
we have a rotation about some $\hat{n}$
through some angle $d\theta$.

Then we may expect $\underline{\omega} = \frac{d\theta}{dt}\hat{n}$.



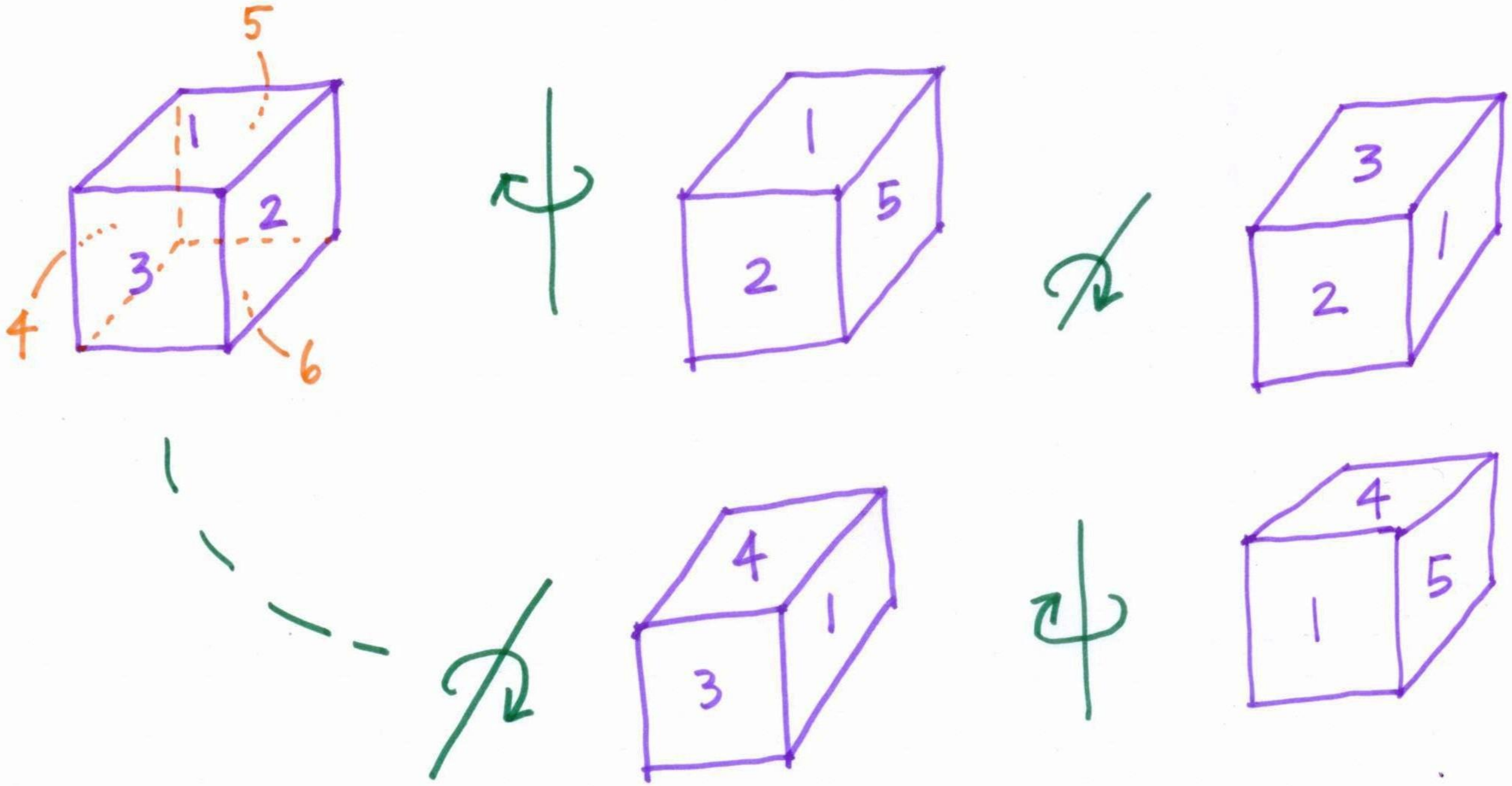
$R(t) = I$. reference position

$$R(t+dt) = (1-\cos(d\theta))\, nn^T + \cos(d\theta)\cdot I + \sin(d\theta)\cdot S(n)$$

$$\dot{R}R^T = \dot{R} = \frac{d\theta}{dt} S(n) = S(\omega)$$

for $\underline{\omega} = \frac{d\theta}{dt}\hat{n}$. DEFN. 2

Infinitesimal rotations add up like vectors.
Finite rotations do not.



$R_1 R_2 \neq R_2 R_1$ – matrix mult. not commutative.

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$R(n, \theta)$ for infinitesimal rotation θ

$= I + \theta S(n)$ to first order.

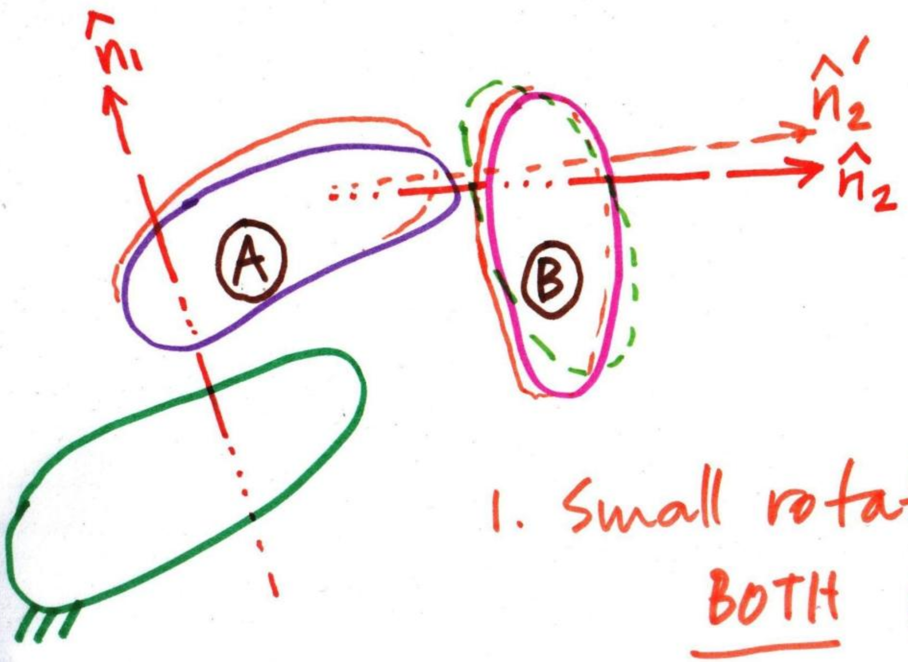
$R(n_1, \theta_1) \cdot R(n_2, \theta_2) = (I + \theta_1 S(n_1)) \cdot (I + \theta_2 S(n_2))$

$= I + \theta_1 S(n_1) + \theta_2 S(n_2)$ to first order

$= R(n_2, \theta_2) \cdot R(n_1, \theta_1)$ to first order

Infinitesimal-Rotation Matrices commute.

Two successive infinitesimal rotations



1. Small rotation about $\hat{n}_1$ of **BOTH** A and B as one unit.

2. Subsequent small rotation of B about $\hat{n}_2' \approx \hat{n}_2$ (to first order)

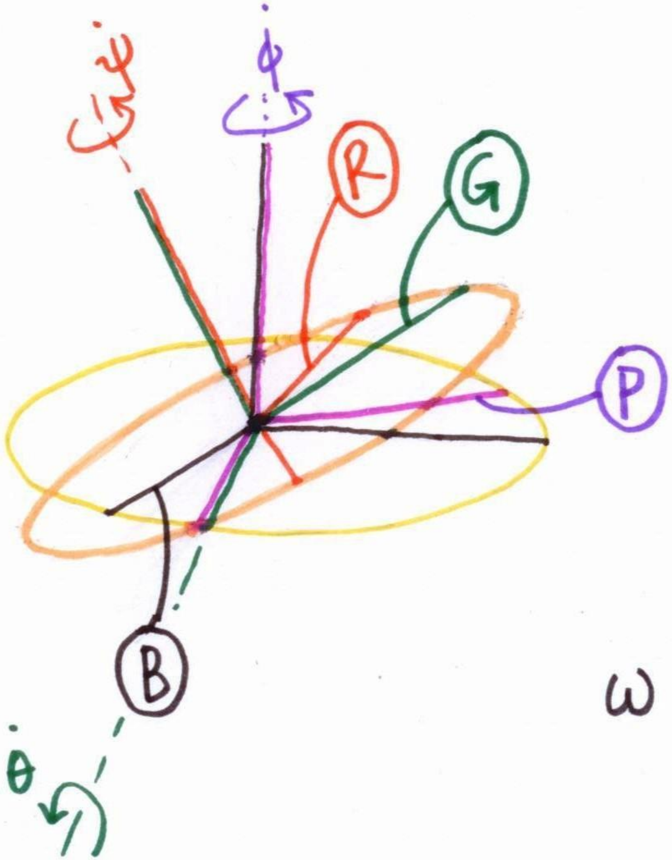
NET effect on B is two infinitesimal rotations

$R(n_1, d\theta_1)$ and $R(n_2, d\theta_2)$

can drop the prime to first order.

Relative Angular Velocities

$$\underline{\omega}_{B/Ground} = \underline{\omega}_{B/A} + \underline{\omega}_{A/Ground}$$



$$\underline{\omega}_{R/B} = \underline{\omega}_{R/G} + \underline{\omega}_{G/P} + \underline{\omega}_{P/B}$$

$$\omega = \dot{\phi}\, e_3 + \dot{\theta}\, R_1 e_1 + \dot{\psi}\, R_2 R_1 e_3$$

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Closing comments

- Rotations in 3D are difficult to understand intuitively
  - 6 dimensional (6 dof)
  - while velocity is derivative of position angular velocity is not derivative of some orientation variable.
- Euler's theorem & axis-angle formula are easy to understand
- Subsequent study of Rotation is an exercise in Notation