A Glimpse of Discrete Mechanics

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Basant Lal Sharma's research is centered around the mechanics of crystals and involves a theoretical study of equilibrium and motion of a defects such as dislocation, martensitic phase boundary, and crack. The crystal model, continuum as well as discrete, is typically a simplified (classical) hamiltonian system allowing application of semi-analytical methods for its analysis. The crystal dynamics also involves a study of propagation of waves and scattering in a crystal lattice. Often the research problem cannot be tackled analytically and then certain rigorous methods, that preserve the geometric structure of the model problem, are studied and analysis is done using numerical algorithms.)

"..the deeper one drives the spade the harder the digging gets; maybe it has become too hard for us unless we are given some outside help, be it even by such devilish devices as high-speed computing machines." - H. Weyl
(from an address delivered at the Princeton University Bicentennial Conference on the Problems of Mathematics, December 17-19, 1946)

INTRODUCTION

Numerical simulations play a dominant role in current science and engineering for the purpose of our understanding of physical phenomena. Typical phenomenological theories and models involve some kind of a continuum, for example, the assumption of continuous space, continuous matter and/or continuous time. Since most of the popular scientific models contain equations that do not admit closed form solutions, a strong alternative, indeed a technique, to obtain some solutions of such equations within a controllable bound on error, is available to researchers as numerical simulation. A typical such model system involves differential (ordinary or partial) equations, and the technique of numerical simulation uses a transformation, called discretization, and it is applied on the continuous model. The result is a discrete model, or rather a family of discretized models depending on discretizing parameters such as mesh size. This process involves, among other things, a shift from non-algebraic to algebraic setting and infinitesimal need to be replaced by finite, for example finite difference equations result from a discretization of differential equations [2, 20]. The final discrete model so constructed is implementable on a digital computer for numerical simulation.

Most of the focus, traditionally, in numerical simulation remains on increasing the accuracy of the numerical solution[2]. This has been found inadequate, over last few decades, due to a need for preserving the structure of the original system as well [10]. The mathematical/ geometrical structure of the discrete model, constructed for the purpose of numerical simulation, in relation to the structure of the original continuous model is not at all obvious. In this article, I shall restrict the attention to conservative mechanical systems, i.e. the systems with time independent Hamiltonian [7]. For example, consider a mechanical system, such as a simple pendulum, whose equation of motion is given in the Hamiltonian framework. If I discretize these equations then what is the structure of the discretized model system in relation to that of the original Hamiltonian system? Is there an associated discrete Hamiltonian structure? For such systems certain structure-preserving numerical algorithms have been developed by the researchers around the world which include those methods preserving some physical entities such as energy, momentum, etc, and also some structural details such as symplecticity, etc for simulating mechanical systems [3, 14, 28, 27, 10]. With the rise in research, development and application of structure-preserving algorithms in the field of numerical simulation, there are still a few fundamental aspects that require more study [10, 25].

On the other hand, from a practical point of view the assumption of a continuum should also allow a possibility to 'measure' continuously with 'arbitrary' precision. Obviously in the realm of measurements, this kind of continuity is quite out of place (more a hope) in the observable world (even if one ignores validity of quantum theoretic assumptions) and yet it is a common assumption in scientific works. Discrete mechanics[1], in contrast, may require that such measurements be at most countably possible, not necessarily continuously. Since it involves a discrete model system, it can be directly used for numerical purposes. But again similar questions may be asked. For example, is there a possibility of a discrete Hamiltonian structure so that the discrete model itself has properties

[1]There are also many other interpretations, formulations, as well as questions behind this which have been under investigation as part of the quantum theories but I shall not indulge in those.
[2]I shall not discuss any philosophical issues behind a requirement like this. Nothing at all takes place in the universe in which some rule of maximum or minimum does not appear" -Euler

analogous[2] to that of a continuous Hamiltonian model? Indeed the focus is therefore on studying discrete analogues of some fundamental continuous mathematical tools. Discrete mechanics contains the analogue of Lagrangian and Hamiltonian mechanics when continuous time is replaced by its discrete counterpart. Note that this is not the same as any arbitrary discretization of the equations of motion of a mechanical system with continuous time but rather a structure preserving discretization which also has interesting theoretical aspects analogous to continuous time Lagrangian and Hamiltonian mechanics (see, e.g.,[24, 29]). The evolution of the system occurs at discrete time instants from the outset following a discrete variational principle and a discrete analogue of the Euler-Lagrange equations, Hamilton's equations, etc. The advantage is that the discrete analogues of the concepts in continuous time such as symplecticity, the Legendre transform, momentum maps, Noether's theorem, etc, appear naturally [24]. Whereas the main topic in discrete mechanics is the development of structure-preserving algorithms for Lagrangian and Hamiltonian systems (see, e.g., [24]), the theoretical aspects of it are interesting in their own right, and furthermore provide insight into the numerical aspects as well.

For the purpose of this article, discrete mechanics may be arising either out of computational discretization (usually needed to solve a typical problem in continuous mechanics) or it may be inherently involved in the behavior of a phenomenological mechanical model. Either way the subject of discrete mechanics finds a prominent place in the subject of mechanics as well as applied mathematics [13]. In this article on discrete mechanics I shall not dwell on details. For more exhaustive[3] surveys, there are many other sources available, eg. [23, 24].

1. CLASSICAL MECHANICS

Before embarking on the discrete case in the next section, it may be helpful to recall some aspects of the continuous. As it will be clear soon this necessitates some highlights from the traditional calculus of variations. The foundations of analytical mechanics can be attributed to Euler, Lagrange and Hamilton [8]. In a way, the mathematical solution of the `brachistochrone problem' [19] was the origin of the main tool involved, i.e., calculus of variations, [8, 6]. Using a discrete variation of the extremal curve between equidistant points and the discrete stationarity condition (see Fig.1), the Bernoulli brothers thus anticipated Euler's constructions for finding a general condition. Euler solved the more general extremal problem for a given a function F dependent on three variables $t, q, \dot{q}$ [4] The problem was determination of a curve q which extremizes the integral [6]

$$\int_0^a F(t, q(t), \dot{q}(t))dt. \tag{1}$$

Such an integral occurs in the brachistrochrone problem, for example. Note that for a given candidate curve, the variables q and $\dot{q}$ are assigned as: $q = q(t), \dot{q} = \frac{dq}{dt}(t)$ for a given t.

As can be seen in Fig. 1, Euler introduced equidistant discrete points t as part of the analysis. The discretized problem can be solved easily using the methods of elementary calculus. The condition for the discretized problem requires a sum to be stationary. In the limit as the distance between consecutive discrete points approaches zero, the sum approaches the original integral and the discrete stationarity condition becomes a differential equation:

$$\frac{\partial F}{\partial q}(t, q(t), \dot{q}(t)) - \frac{d}{dt}(\frac{\partial F}{\partial \dot{q}}(t, q(t), \dot{q}(t))) = 0, \quad 0 < t < a. \tag{2}$$

This equation is usually called the Euler-Lagrange differential equation and it expresses a necessary condition for an extremal of the problem [6]. For example, the famous geodesic curves (see Fig. 2) on the spherical surface in three dimensions, i.e, arcs of great circles, provide extremals for the distance function on the surface. Following Euler, one may associate a discrete set of points on a geodesic candidate and attempt to construct a discrete geodesic in such a way that the total discrete distance on the surface is minimized. In the limit when the number of points become infinite, one hopes to recover the continuous arc of a great circle.

Applying above variational formalism to mechanics requires some definitions. The action function [1] $\mathfrak{S}$ to be extremized, invoking the Hamilton's principle [1], is the integral

$$\mathfrak{S}_{q_0,t_0}(q_N, t_N) := \int_\gamma L(q(t), \dot{q}(t))dt \tag{3}$$

along a trajectory γ (in q-t plane) connecting the points (q_0, t_0) and (q_N, t_N) (see Fig. 3). Here as an interlude I describe the notation which is useful for the rest of this article. The symbol D_1, whenever it appears in front of a function of two variables, example L, refers to derivative with respect to the first entry in the expression of the function (for example L depends on two variables q and $\dot{q}$, both variables vectorial in general, so that q is respect entry and $\dot{q}$ is the second).Similarly, D_2 refers to derivative with respect to the second entry in the expression of the function. In other words,[5]

$$D_1 L = \frac{\partial L}{\partial q}, D_2 L = \frac{\partial L}{\partial \dot{q}}$$

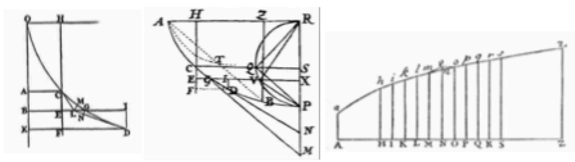


Figure 1: (left) Figure by Jacob Bernoulli providing the stationarity condition [19, 8]; (center) gure by Johann Bernoulli for his derivation of the cycloid as minimal curve [19, 8]; (right) Original Euler's figures used in his derivation [5, 8].

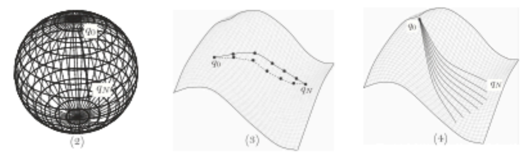


Figure 2: Paths with minimum distance between two points on a sphere together with a set of points on such a path.
Figure 3: A candidate curve γ for continuous extremal between q_0;q_N, along with a discrete set of points.
Figure 4: A family of extremals with initial point q_0 fixed but variable end point q_N used in the definition of action function $S(q_0,q_N)$

Also, a few original citations may be missing and many results or concepts could have been credited more accurately.
q and $\dot{q}$ are independent variables, possibly vector valued in the following.

Using this notation, when F in (1) is independent of t, (1) can be now rewritten as $D_1F(q(t), \dot{q}(t)) - \frac{d}{dt}(D_2F(q(t), \dot{q}(t))) = 0$.
The succinctness, as well as the second order nature, of the Euler-Lagrange differential equation becomes immediately clear after comparison with its expanded form, for suciently smooth L,
$D_{22}L(q(t), \dot{q}(t))\ddot{q}(t) + D_{12}L(q(t), \dot{q}(t))\dot{q}(t) - D_1L(q(t), \dot{q}(t)) = 0$, for γ (in q-t plane) as the unknown trajectory.

An extremal trajectory which models the evolution of the mechanical system, governed by the Hamilton's principle stated above, is a solution of the Euler-Lagrange differential equation[6]

$$\frac{d}{dt}(D_2L) - D_1L = 0 \quad (4)$$

with the boundary values

$$q(t_0) = q_0, q(t_N) = q_N. \quad (5)$$

Using a family of extremals (see Fig. 4) the action $\mathfrak{S}$ can be considered as a function $S(q_0, q_N)$, defined by

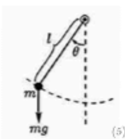
$$S(q_0, q_N) = \int_{t_0}^{t_N} L(q(t), \dot{q}(t))dt. \quad (6)$$

The solution exists uniquely locally at least if q_0, q_N are sufficiently close [1]. The partial derivatives of S with respect to q_0 and q_N, using the definition of the conjugate momenta [1], i.e., via the Legendre transform $p = D_2L$, can be found as

$$\frac{\partial S}{\partial q_0}(q_0, q_N) = -p_0, \frac{\partial S}{\partial q_N}(q_0, q_N) = p_N. \quad (7)$$

The differential of S is, therefore

$$dS = \frac{\partial S}{\partial q_N}dq_N + \frac{\partial S}{\partial q_0}dq_0 = p_N dq_N - p_0 dq_0, \quad (8)$$



(5)

Figure 5: A simple (mathematical) pendulum [1]

which is one of the basic formula for the generating functions of a symplectic transformation [1] (though derived above by the application of elementary calculus[24])

Using the definition of the conjugate momenta $p = D_2L$, the Hamiltonian [1] can be obtained as $H = p\dot{q} - L$; and one may derive the Hamilton's equations. I skip the details as they can be easily found in any classical book, for example [7, 15]. The extremals, i.e., the solutions of Euler-Lagrange equation, can be associated with the solutions of the Hamilton's equations in a one to one manner through the Legendre transform [1].

As an illustration of the Hamiltonian framework, I mention here a common example: simple pendulum (see Fig. 5). Consider the mathematical pendulum - assuming the mass $m = 1$, massless rod of length $l = 1$; gravitational acceleration $g = 1$. This is a system (see Fig. 5) with one degree of freedom($q = \theta, p = \dot{q} = \dot{\theta}$) and the Hamiltonian

$$H(q, p) = \frac{1}{2}p^2 - \cos q, \quad (9)$$

can be easily found either directly or using the Lagrangian for this system. The Hamilton's equations of motion for simple pendulum are

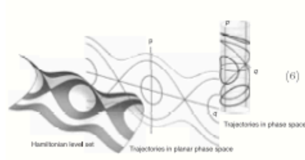
$$\dot{p} = -\sin q, \quad \dot{q} = p. \quad (10)$$

Since the left hand side is 2π-periodic in q, it is natural to consider q as a variable on the circle $\mathbb{S}^1$. Hence, the phase space (see Fig. 6) of points (p , q) becomes the cylinder $\mathbb{R} \times \mathbb{S}^1$. Figure 6 also shows some level curves of H(q, p)and the solution curves of the Hamilton's equations lie on such level curves. There exists a symplectic structure on the phase space which is preserved by the exact flow (See [1] for further details). For the case of simple pendulum, since this system has a single degree of freedom, the symplectic structure reduces to the preservation of area and Fig. 7 demonstrates the same.

In the next section I shall now discuss some discrete counterparts of above conceptual formulation.

2. DISCRETE MECHANICS

A formulation of mechanics adapted to numerical simulation may be termed discrete mechanics.[7] There have been many attempts at the development of a discrete mechanics [13] along the same lines as the traditional mechanics. In this article, for simplicity, I restrict the attention to classical mechanics. The discrete mechanics, as developed so far



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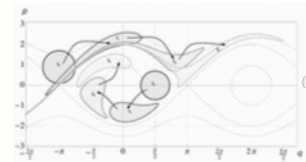
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Figure 6: The phase space (extreme right) of a simple pendulum using $q = \theta, p = \dot{q} = \dot{\theta}$, along with level curves (middle and left plots) of H(q, p). Also shown in the phase space are some characteristic trajectories of a simple pendulum [1].

Figure 7: Preservation of symplectic structure, which in this case reduces to the conservation of (signed) area, by the continuous phase flow of simple pendulum [10], darker regions evolve into lighter regions as time increases and this is shown for two sets of initial conditions bounded by circles.

[21, 22, 30, 31], starts by constructing a discrete action functional and using a discrete variational principle. Using a discrete time as dynamical variable and a given discrete Lagrangian some other developments also took place in parallel directions [16, 17, 18]. Overall this is one place where mechanics and numerical discretization and common ground and may complement each other.

A \`discrete' Lagrangian flow $\{q_k\}_{k=0}^N$ on an $n$-dimensional smooth conguration space $Q$, can be described by the following \`discrete' variational principle. Let $\mathfrak{S}_d^N$ be the following action sum of the \`discrete' Lagrangian

$L_d : Q \times Q \to \mathbb{R}$[8]

$$\mathfrak{S}_d^N(\{q_k\}_{k=0}^N) := \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}). \quad (11)$$

The discretized Hamilton's principle consists of extremizing, for given q_0 and q_N, the sum $\mathfrak{S}_d(\{q_n\}_0^N)$. Above expression for the discrete action can be considered as an approximation of the action integral $\int_0^{t_N} L(q(t), \dot{q}(t))dt$ [9] Recall the notation stated earlier concerning D_1 and D_2 So D_1 in front of L_d refers to derivative with respect to the first entry in the expression of the function L_d and so on.[9] Using the discrete variations $q_k + \varepsilon\eta_k$, for $k = 0, 1, \ldots, N, \varepsilon$ small, $\eta_0 = \eta_N = 0$, the discrete variational principle for $\delta\mathfrak{S}_d^N(\eta) = 0$ all such gives the \`discrete' Euler-Lagrange (difference) equation[10]:

$$D_2L_d(q_{k-1}, q_k) + D_1L_d(q_k, q_{k+1}) = 0. \quad (12)$$

Above is a three-term[11] difference scheme [20] for determining $q_1, \ldots, q_{N-1}$ (given q_0, q_N) Now the \`discrete' momenta can be defined via a \`discrete' Legendre transform [24], $p_n = D_2L_d(q_{n-1}, q_n)$. Also by the discrete Euler-Lagrange equation (12), $p_n = -D_1L_d(q_n, q_{n+1})$. In fact, the left and right \`discrete' Legendre transforms [26] $\mathcal{F}L_d^{\pm} : Q \times Q \to T^*Q$ can be formally defined by

$$\mathcal{F}L_d^- : (q_k, q_{k+1}) \mapsto (q_k, -D_1L_d(q_k, q_{k+1})),$$
$$\mathcal{F}L_d^+ : (q_k, q_{k+1}) \mapsto (q_{k+1}, D_2L_d(q_k, q_{k+1})), \quad (13)$$

respectively. Consequently the momenta are given by

$$p_{k,k+1}^- := -D_1L_d(q_k, q_{k+1}),$$
$$p_{k,k+1}^+ := D_2L_d(q_k, q_{k+1}). \quad (14)$$

The discrete Euler-Lagrange equation implies that $p_{k-1,k}^+ = p_{k,k+1}^-$ In view of this, by defining

$$p_k := p_{k-1,k}^+ = p_{k,k+1}^-, \quad (15)$$

one can rewrite the discrete Euler-Lagrange equation as follows:

$$p_k = -D_1L_d(q_k, q_{k+1}),$$
$$p_{k+1} = D_2L_d(q_k, q_{k+1}). \quad (16)$$

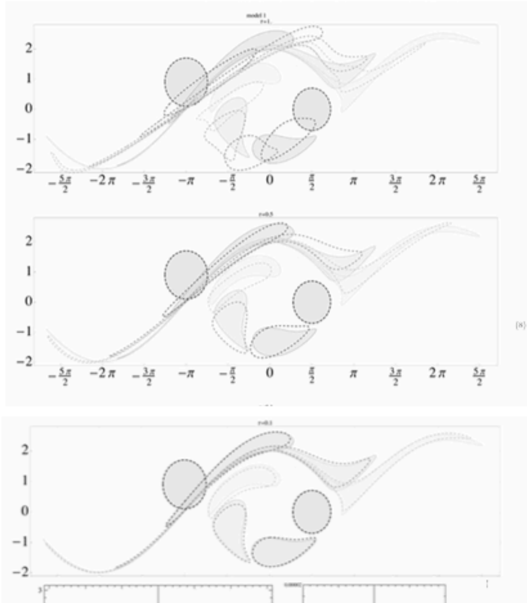
[7] This appears to be due to the name of scheme for numerical solution of classical equations of motion by LaBudde and Greenspan [9, 13] who termed it \`discrete mechanics.'

[8] The discrete Lagrangian L_d can be viewed [23] as an approximation $L_d(q_n, q_{n+1}) \approx \int_{t_n}^{t_{n+1}} L(q(t), \dot{q}(t))dt$.

[9] In particular, $D_1L_d(q_k, q_{k+1}) = \frac{\partial L_d}{\partial q_k}(q_k, q_{k+1}) = \frac{\partial L_d}{\partial x}(x, y)\big|_{x=q_k, y=q_{k+1}}$. $D_2L_d(q_k, q_{k+1}) = \frac{\partial L_d}{\partial q_{k+1}}(q_k, q_{k+1}) = \frac{\partial L_d}{\partial y}(x, y)\big|_{x=q_k, y=q_{k+1}}$.

[10] This is same as the requirement for $\frac{\partial \mathfrak{S}_d}{\partial q_n}(\{q_n\}_0^N) = 0$ an extremum.

[11] This depends on the postulated discrete Lagrangian.



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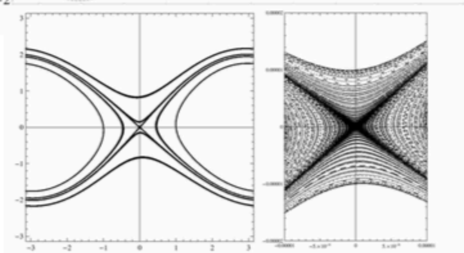

[9]

Figure 8: The discrete phase flow of one proposed discretized model of simple pendulum for parameter $\tau = 1; 0.5; 0.1$, darker regions evolve into lighter regions as time index increases and this is compared with the continuous flow shown in Fig 6 for two sets of initial conditions bounded by circles and same scaled instants. Note the preservation of symplectic structure.

Figure 9: Some trajectories for the discrete pendulum model [11, 12] in p-q plane.

Recall now the continuous case in the interim and analogous to it, using the definition of S_d

$$S_d(q_0, q_N) = \sum_{n \in \mathbb{Z}_1^N} L_d(q_n, q_{n+1}), \tag{17}$$

where $\{q_n\}_{k=0}^{N}$ is a solution of the discrete Euler-Lagrange equation above with the boundary values q_0 and q_N, it can be shown that

$$\begin{aligned} dS_d(q_0, q_N) &= -p_0 dq_0 + p_N dq_N, \\ \frac{\partial S_d}{\partial q_0} &= -p_0, \frac{\partial S_d}{\partial q_N} = p_N. \end{aligned} \tag{18}$$

Suppose the discrete flow, due to the discrete Euler-Lagrange equation, is denoted by $\Phi_{L_d} : Q \times Q \to Q \times Q$, i.e.,

$$\Phi_{L_d} : (q_{k-1}, q_k) \mapsto (q_k, q_{k+1}), \tag{19}$$

i.e., Φ_{L_d} is a representation of the general solution of the discrete Euler-Lagrange equation. Then, in the presence of a well defined discrete Legendre transform between p_n and q_{n+1} for given q_n, the discrete Φ_{L_d} is symplectic [1].

For example, in the case of simple pendulum, since $L = p\dot{q} - H = T - V = \frac{1}{2}\dot{q}^2 + \cos q$,

$$L_d(q_k, q_{k+1}) = \frac{1}{2}(q_{k+1} - q_k)^2 + \tau \cos q_k, \tag{20}$$

where τ is a time constant. Using this, a discrete model is proposed:

$$\begin{aligned} q_{k+1} &= \tau(p_k - \tau \sin q_k) + q_k, \\ p_{k+1} &= p_k - \tau \sin q_k. \end{aligned} \tag{21}$$

Such discrete models lead to a symplectic flow[12]. In contrast to the original continuous integrable system, it has been established that these discrete flow maps, in general, do not correspond to a integrable[13] behavior [11,12]. For example, see the right side of Fig. 9 which has been obtained by zooming in the trajectories near the figure eight trajectory in cylindrical phase space of Fig. 6, i.e., near a homoclinic orbit. In this manner the familiar KAM (Kolmogorov-Arnold-Moser theorem [1]) features appear in the discrete pendulum model [11, 12], which are absent in the continuous case. Note that the discrete Hamilton's equations are also not quite analogous to the continuous case as presented above, though the flow is on the phase space, i.e. the cotangent bundle T*Q. With the right discrete Legendre transform

$$p_{k+1} = \mathcal{F}L_d^+(q_k, q_{k+1}) = D_2 L_d(q_k, q_{k+1}), \tag{22}$$

the following right discrete Hamiltonian can be defined [26]:

$$H_d^+(q_k, p_{k+1}) = p_{k+1}q_{k+1} - L_d(q_k, q_{k+1}). \tag{23}$$

Then, the discrete Hamiltonian map is denied by the right discrete Hamilton's equations[14] [26]

$$\begin{aligned} q_{k+1} &= D_2 H_d^+(q_k, p_{k+1}), \\ p_k &= D_1 H_d^+(q_k, p_{k+1}). \end{aligned} \tag{24}$$

Similarly, with the left discrete Legendre transform, a set of left discrete Hamilton's equations result [26]. Further as an extension of the analogy between discrete and continuous mechanics, a discrete Hamilton-Jacobi equation, i.e., Hamilton-Jacobi differential-difference equation, can be derived [4]. A list of some analogies between ingredients in continuous and discrete theories can be found in [26]. I am including here a slightly expanded list as Table 1

A word of caution with discrete approach may conclude this article appropriately. That in the hunt for quantitative prediction using discrete mechanics there is a risk of easy slide into the tumultuous river of computational mechanics. A risk avoidable to some while desirable to some.

[12] See [1] for further details concerning the definition of symplecticity.
[13] See [1] or [7] for the definition of integrability.
[14] Recall the notation stated earlier that D_1 in front of H_d^+ refers to derivative with respect to the first entry in the expression of the function H_d^+ and so on. In particular, $D_1 H_d^+(q_k, p_{k+1}) = \frac{\partial H_d^+}{\partial q_k}(q_k, p_{k+1}) = \frac{\partial H_d^+}{\partial x}(x, y)\big|_{x=q_k, y=p_{k+1}}$, $D_2 H_d^+(q_k, p_{k+1}) = \frac{\partial H_d^+}{\partial p_{k+1}}(q_k, p_{k+1}) = \frac{\partial H_d^+}{\partial y}(x, y)\big|_{x=q_k, y=p_{k+1}}$. The same notation requires attention in Table 1.

Continuous	Discrete
$(q,t) \in Q \times \mathbb{R}_0^+$	$(q,t) \in Q \times \mathbb{Z}_0^+$
$L(q(t),\dot{q}(t))$	$L_d(q_k,q_{k+1})$
$\mathfrak{S}(q) := \int_0^{t_N} L(q(t),\dot{q}(t))dt$ using $q(0)=q_0, q(t_N)=q_N$	$\mathfrak{S}_d^N(\{q_k\}_{k=0}^N) := \sum_{k=0}^{N-1} L_d(q_k,q_{k+1})$
$\frac{d}{dt}(D_2L(q(t),\dot{q}(t))) - D_1L(q(t),\dot{q}(t)) = 0$	$D_2L_d(q_{k-1},q_k) + D_1L_d(q_k,q_{k+1}) = 0$
Evolution of $q(t)$ from q_0 to q_N in the configuration space Q	Evolution of q_k from q_0 to q_N in the configuration space Q
Legendre Transform $\mathcal{F}L(q,\dot{q}) = D_2L(q,\dot{q})$	$\mathcal{F}L_d^+(q_k,q_{k+1}) = D_2L_d(q_k,q_{k+1})$ $\mathcal{F}L_d^-(q_k,q_{k+1}) = -D_1L_d(q_k,q_{k+1})$
$H(q,p) = (p\dot{q} - L(q,\dot{q}))$ using $\dot{q} \leftrightarrow p = \mathcal{F}L(q,\dot{q})$	$L_d(q_k,q_{k+1})$ or $(H_d^+(q_k,p_{k+1}) = p_{k+1}q_{k+1} - L_d(q_k,q_{k+1}))$ using $q_{k+1} \leftrightarrow p_{k+1} = \mathcal{F}L_d^+(q_k,q_{k+1}))$ or $(H_d^-(p_k,q_{k+1}) = -p_kq_k - L_d(q_k,q_{k+1}))$ using $q_{k+1} \leftrightarrow p_k = \mathcal{F}L_d^-(q_k,q_{k+1}))$
Hamilton's equations $\begin{cases} \dot{q}(t) &= D_2H(q(t),p(t)), \\ \dot{p}(t) &= -D_1H(q(t),p(t)) \end{cases}$	$\begin{cases} p_k &= -D_1L_d(q_k,q_{k+1}), \\ p_{k+1} &= D_2L_d(q_k,q_{k+1}) \end{cases}$ or $\begin{cases} q_{k+1} = D_2H_d^+(q_k,p_{k+1}), \\ p_k = D_1H_d^+(q_k,p_{k+1}) \end{cases}$ or $\begin{cases} q_k &= -D_1H_d^-(p_k,q_{k+1}), \\ p_{k+1} &= -D_2H_d^-(p_k,q_{k+1}) \end{cases}$
Continuous evolution of trajectory $(q(t),p(t))$ from (q_0,p_0) in the phase space T^*Q	Discrete evolution of trajectory (q_k,p_k) from (q_0,p_0) in the phase space T^*Q
$S(q,t) = \int_0^t (p(s)\dot{q}(s) - H(q(s),p(s)))ds$ using (q,p) as solution of Hamilton's equations with $q(t)=q, q(0)=q_0$	$S_d^k(q_k) = \sum_{l=0}^{k-1} L_d(q_l,q_{l+1})$ or $S_d^k(q_k) = \sum_{l=0}^{k-1}(p_{l+1}q_{l+1} - H_d^+(q_l,p_{l+1}))$ or $S_d^k(q_k) = \sum_{l=0}^{k-1}(-p_lq_l - H_d^-(p_l,q_{l+1}))$ using (q_k,p_k) as solution of discrete Hamilton's equations
$\frac{\partial S}{\partial q}(q,t)dq + \frac{\partial S}{\partial t}(q,t)dt$	$S_d^{k+1}(q_{k+1}) - S_d^k(q_k)$
$p(t)dq - H(q(t),p(t))dt$	$L_d(q_k,q_{k+1})$ or $p_{k+1}q_{k+1} - H_d^+(q_k,p_{k+1})$ or $-p_kq_k - H_d^-(p_k,q_{k+1})$
Hamilton-Jacobi equation $\frac{\partial S}{\partial t}(q,t) + H(q,\frac{\partial S}{\partial q}(q,t)) = 0$	$\begin{cases} S_d^{k+1}(q_{k+1}) - S_d^k(q_k) - DS_d^{k+1}(q_{k+1})q_{k+1} \\ +H_d^+(q_k,DS_d^{k+1}(q_{k+1})) = 0 \end{cases}$ or $\begin{cases} S_d^{k+1}(q_{k+1}) - S_d^k(q_k) + DS_d^k(q_k)q_k \\ +H_d^-(DS_d^k(q_k),q_{k+1}) = 0 \end{cases}$

Table 1: Correspondence [26] between some ingredients in continuous and discrete theories; $\mathbb{R}_0^+$ is the set of non-negative real numbers and $\mathbb{Z}_0^+$ is the set of nonnegative integers.

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