FUNCTIONAL ANALYSIS: NOTES AND PROBLEMS

Abstract. These are the notes prepared for the course MTH 405 to be offered to graduate students at IIT Kanpur.

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1. Basic Inequalities

Exercise 1.1 : (AM-GM Inequality) Consider the set

$$A_n = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = n,\ x_i \geq 0 \text{ every } i\},$$

and the function $g : \mathbb{R}^n \to \mathbb{R}_+$ given by $g(x_1, \cdots, x_n) = x_1 \cdots x_n$. Verify:

(1) A_n is a compact subset of $\mathbb{R}^n$, and g is a continuous function.

(2) Let $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$ be such that $\max_{x \in A_n} g(x) = g(z)$. Then $z_i = 1$ for all i.

(**Hint**. Let $z_p = \min z_i$ and $z_q = \max z_i$ for some $1 \leq p, q \leq n$. Define $y = (y_1, \cdots, y_n) \in A_n$ by $y_p = (z_p + z_q)/2 = y_q$ and $y_i = z_i$ for $i \neq p, q$. If $z_p < z_q$ then $g(y) > g(z)$.)

(3) Let $x \in A_n$. Set $\alpha := \frac{1}{n}\sum_{i=1}^n x_i$ and $y := (x_1/\alpha, \cdots, x_n/\alpha) \in A_n$. Then $g(y) \leq 1$.

Conclude that $\left(\prod_{i=1}^n x_i\right)^{1/n} \leq \frac{1}{n}\sum_{i=1}^n x_i$.

Exercise 1.2 : (Characterization of differentiable convex functions) Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. For $a < x < y < b$, verify:

(1) If f is convex then f' is monotonically increasing.
Hint. Let $h > 0$ be such that $x + h < y$. Then
$$\frac{f(x+h) - f(x)}{h} \leq \frac{f(y) - f(x)}{y - x} \leq \frac{f(y) - f(y-h)}{h}.$$
(2) If f' is monotonically increasing then f is convex.
Hint. Let $z = (1-t)x + ty$. Then there exists $c_1 \in (x, z)$ and $c_2 \in (z, y)$ such that
$$f(z) = f(x) + f'(c_1)(z - x), \; f(y) = f(z) + f'(c_2)(y - z).$$

Exercise 1.3 : (Characterization of twice differentiable convex functions) Let $f : [a, b] \to \mathbb{R}$ be a twice differentiable function. Show that f is convex (resp. strictly convex) if and only if $f'' \geq 0$ (resp. $f'' > 0$).

Hint. Necessary part follows from the last exercise. For sufficiency part, use Taylor's mean value theorem.

Remark 1.4 : The exponential is strictly convex.

Exercise 1.5 : Let a_1, a_2 be positive numbers and let $p_1, p_2 > 0$ be such that $p_1 + p_2 = 1$. Prove that $a_1^{p_1} a_2^{p_2} \leq p_1 a_1 + p_2 a_2$. Equality holds iff $a_1 = a_2$.

Hint. The logarithm $\log \frac{1}{x}$ is strictly convex.

Exercise 1.6 : (Young's Inequality) Let $p, q > 1$ be conjugate exponents (that is, $1/p + 1/q = 1$). For positive numbers a, b prove that
$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$
Equality holds iff $a^p = b^q$.

Hint. Let $a_1 = a^p, a_2 = b^q$ and $p_1 = 1/p, p_2 = 1/q$ in the preceding exercise.

Exercise 1.7 : (Geometric Proof of Young's Inequality) Let $p, q > 1$ be conjugate exponents (that is, $1/p + 1/q = 1$). Given positive real numbers $a \leq b$, consider
$$\begin{aligned} D_1 &:= \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, \; 0 \leq y \leq x^{p-1}\} \\ D_2 &:= \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq b, \; 0 \leq x \leq y^{q-1}\}. \end{aligned}$$
Verify the following:
(1) The intersection of D_1 and D_2 is $\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a \;, y = x^{p-1}\}$ (**Hint.** Since $(p-1)(q-1) = 1$, $y = x^{p-1}$ iff $x = y^{q-1}$).
(2) The rectangle $\{(x, y) \in \mathbb{R} : 0 \leq x \leq a, \; 0 \leq y \leq b\}$ is contained in the union $D_1 \cup D_2$.

Conclude that $ab \leq a^p/p + b^q/q$. Equality holds iff $a^p = b^q$.

Exercise 1.8 : Assume the Hölder's (resp. Minkowski's) inequality for finite sequences, and derive it for sequences.

Exercise 1.9 : (Hölder's Inequality for measurable functions) Let $p, q > 1$ be conjugate exponents. Let f and g be Lebesgue measurable complex-valued functions. Then fg is measurable such that

$$\left| \int f(x)g(x)dx \right| \leq \left(\int |f(x)|^p dx \right)^{1/p} \left(\int |g(x)|^q dx \right)^{1/q}.$$

Hint. Let $\|f\|_p := \left(\int |f(x)|^p dx \right)^{1/p}$, $\tilde{f} = f/\|f\|_p$. By Young's Inequality,

$$|\tilde{f}(x)||\tilde{g}(x)| \leq |\tilde{f}(x)|^p/p + |\tilde{g}(x)|^q/q.$$

Now integrate both sides.

Exercise 1.10 : Prove Minkowski's Inequality for measurable functions.

2. Normed Linear Spaces: Examples

Throughout these notes, the field $\mathbb{K}$ will stand either for $\mathbb{R}$ or $\mathbb{C}$.

Exercise 2.1 : For $1 \leq p < \infty$, for $x = (x_1, \cdots, x_n) \in \mathbb{K}^n$, consider

$$\|x\|_p := \left(\sum_{j=1}^{n} |x_j|^p \right)^{1/p}.$$

Show that $(\mathbb{K}^n, \|\cdot\|_p)$ is a normed linear space.

Hint. Minkowski's Inequality.

Exercise 2.2 : What goes wrong in the last exercise if $0 < p < 1$?

Hint. If $p = 1/2$ then calculate the p-norms of $(1, 1), (1, 0)$ and $(2, 1)$.

Exercise 2.3 : How does a unit disc $\mathbb{D}_p := \{(x, y) \in \mathbb{R}^2 : |x|^p + |y|^p < 1\}$ in $(\mathbb{R}^2, \|\cdot\|_p)$ look like ? Whether $\mathbb{D}_\infty$ makes sense ?

Hint. $\mathbb{D}_p$ is invariant under reflections along X and Y axes: $(x, y) \in \mathbb{D}_p$ iff $(\alpha x, \beta y) \in \mathbb{D}_p$ for $\alpha, \beta \in \{\pm 1\}$. Now plot $x^p + y^p = 1$ for $x, y \geq 0$.

Exercise 2.4 : For $1 \leq p < \infty$, let l^p stand for

$$l^p := \{(a_n) : \sum_{n=1}^{\infty} |a_n|^p < \infty\}.$$

Show that l^p is a normed linear space with norm $\|(a_n)\|_p := \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p}$.

Exercise 2.5 : For a non-empty subset X of $\mathbb{R}$ (endowed with the subspace topology), let $b(X)$ denote the vector space of bounded functions $f : X \to \mathbb{K}$, and $C(X)$ denote the vector space of continuous functions $f : X \to \mathbb{K}$. Define $\|f\|_\infty := \sup_{x \in X} |f(x)|$ for $f \in C(X)$. Verify:

(1) $\|\cdot\|_\infty$ defines a norm on $b(X)$.
(2) $\|\cdot\|_\infty$ defines a norm on $C_b(X) := b(X) \cap C(X)$.

Exercise 2.6 : Discuss the last exercise in case X equals

(1) $\{1, \cdots, n\}$. What is $C(\{1, \cdots, n\})$?
(2) the set $\mathbb{N}$ of positive integers. In this case, $b(\mathbb{N}) = C_b(\mathbb{N})$. This is commonly denoted as l^∞.

Exercise 2.7 : Let $C^{(n)}(0,1)$ consist of functions $f(t)$ on $(0,1)$ having n continuous and bounded derivatives. Verify that

$$\|f\| := \sup\{\sum_{k=0}^{n} |f^{(k)}(t)| : 0 < t < 1\}$$

defines a norm on $C^{(n)}(0,1)$.

We say that (Lebesgue) measurable functions f *is equivalent to* g (for short, $f \sim g$) if f and g agree outside a set of (Lebesgue) measure 0. Let $[f]$ denote the equivalence class containing f, and let $\|[f]\|_p := \left(\int |f(x)|^p dx\right)^{1/p}$.

Exercise 2.8 : Verify the following:

(1) $\sim$ is an equivalence relation.
(2) If two measurable functions f, g are equivalent then $\|[f]\|_p = \|[g]\|_p$.

For simplicity, we denote the equivalence class $[f]$ containing f by f itself.

Exercise 2.9 : For a measurable set X, let $L^p(X)$ denote the set of all (equivalence classes of) measurable functions f for which $\|f\|_p < \infty$. Show that $L^p(X)$ is a normed linear space with norm $\|f\|_p$.

A complex-valued function f is said to be μ*-essentially bounded* if $\|\phi\|_{m,\infty}$ is finite, where $\|\phi\|_{m,\infty} \equiv \inf\{M \in \mathbb{R}_+ : |\phi(z)| \leq M$ outside set of measure $0\}$.

Exercise 2.10 : If $f \sim g$ then $\|f\|_\infty$ may be not be equal to $\|g\|_\infty$. Show that $\|f\|_{m,\infty} = \|g\|_{m,\infty}$.

Exercise 2.11 : For a measurable set X, let $L^\infty(X)$ denote the set of all (equivalence classes of) measurable functions f for which $\|f\|_{m,\infty} < \infty$. Show that $L^\infty(X)$ is a normed linear space with norm $\|f\|_{m,\infty}$.

Exercise 2.12 : Let t, u, v, w be generators of $\mathbb{R}^3$. For $x \in \mathbb{R}^3$, define

$$\|x\| := \inf\{|a| + |b| + |c| + |d| : a, b, c, d \in \mathbb{R} \text{ such that } x = at + bu + bcv + dw\}.$$

Show that $(\mathbb{R}^3, \|\cdot\|)$ is a normed linear space.

Hint. If $\|x\| = 0$ then there exists a sequence $|a_n| + |b_n| + |c_n| + |d_n| \to 0$ such that $x = a_n t + b_n u + c_n v + d_n w$. To see $\|x_1 + x_2\| \leq \|x_1\| + \|x_2\|$, note

$$\begin{aligned} &\{|a| + |b| + |c| + |d| : x_1 + x_2 = at + bu + cv + dw\} \supseteq \\ &\quad \{|a_1 + a_2| + |b_1 + b_2| + |c_1 + c_2| + |d_1 + d_2| : x_i = a_i t + b_i u + c_i v + d_i w\}. \end{aligned}$$

3. Normed Linear Spaces: Elementary Properties

Exercise 3.1 : (The norm determined by the unit ball)) Let $(X, \|\cdot\|)$ be a normed linear space. Let $\mathbb{B}(x_0, R) := \{x \in X : \|x - x_0\| < R\}$ be the ball centered at x_0 and of radius R. Show that the norm is determined completely by $D(0, R)$: $\|x\| = \inf\{R > 0 : x \in D(0, R)\}$.

Hint. If $x \in D(0, R)$ then $\|x\| < R$. Thus $\|x\|$ is at most the entity on RHS. To see the reverse inequality, argue by contradiction.

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a normed linear space X are *equivalent* if the identity mapping I from $(X, \|\cdot\|)$ onto $(X, \|\cdot\|')$ is a homoemorphism (that is, I is continuous with continuous inverse).

Remark 3.2 : A subset of X is open with respect to $\|\cdot\|$ topology if and only if it is open with respect to $\|\cdot\|'$ topology.

Exercise 3.3 : Prove that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a normed linear space X are equivalent if and only if there exist positive scalars C_1 and C_2 such that $C_1\|x\| \leq \|x\|' \leq C_2\|x\|$ for all $x \in X$.

Exercise 3.4 : (Geometric interpretation of equivalence of norms) Show that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a normed linear space X (with balls $\mathbb{B}(x_0, r)$ and $\mathbb{B}'(x_0, R)$) are equivalent if and only $\mathbb{B}'(0, 1) \subseteq \mathbb{B}(0, r)$ and $\mathbb{B}(0, 1) \subseteq \mathbb{B}'(0, R)$ for some positive constants r and R.

Exercise 3.5 : Show that the equivalence of norms is an equivalence relation. Conclude that all norms $\|\cdot\|_p$ on $\mathbb{K}^n$ are equivalent by verifying $\|x\|_\infty \leq \|x\|_p \leq n^p\|x\|_\infty$..

Exercise 3.6 : (All norms on $\mathbb{K}^n$ are equivalent) Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{K}^n$. Verify:

(1) Let $e_1, \cdots, e_n$ denote the standard basis of $\mathbb{K}^n$. Then
$$\|x\| \leq (\sum_{i=1}^{n} \|e_i\|)\|x\|_\infty.$$
(2) The function $f : (\mathbb{K}^n, \|\cdot\|_\infty) \to \mathbb{R}$ given by $f(x) = \|x\|$ is continuous.
(3) There exists $a \in \mathbb{K}^n$ such that $\|a\|_\infty = 1$ such that $\|x\| \geq \|a\| > 0$ for every $x \in \mathbb{K}^n$ whenever $\|x\|_\infty = 1$.
(4) The norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.

Exercise 3.7 : (Inequivalent norms on $\mathbb{K}[x]$) Let $\mathbb{K}[x]$ denote the vector space of polynomials in x over $\mathbb{K}$. Let $p(x) \in \mathbb{K}[x]$ be given by $p(x) = \sum_{n=0}^{k} a_n x^n$. For $c := \{c_n\}_{n=0}^{\infty}$, define $\|p\|_c := \sum_{n=0}^{k} |c_n||a_n|$. Verify:

(1) $\|\cdot\|_c$ defines a norm on $\mathbb{K}[x]$.
(2) Let $c_n = 1/n$ and $d_n = n$. Then $\|\cdot\|_c$ and $\|\cdot\|_d$ are not equivalent.

Exercise 3.8 : For $1 \leq p < r \leq \infty$, verify the following:

(1) $\|x\|_p \geq \|x\|_r$ for every finite or infinite sequence x.
(2) There does not exist $K > 0$ such that $\|x\|_p \leq K\|x\|_r$ for every infinite sequence x.

Conclude that $l^r \subsetneq l^p$.

Hint. For the first, let $y =: x/\|x\|_p$ which has norm equal to 1. Note that $1 = \|y\|_p^p \geq \|y\|_r^r$, and hence $\|y\|_r \leq 1$. For the second part, find a sequence x such that $\|x\|_r < \infty$ but $\|x\|_p = \infty$ (Try: $x_k = 1/k^q$).

Exercise 3.9 : Let $1 \leq p < r \leq \infty$. For measurable function f on $[0,1]$, let $\|f\|_p := \int_{[0,1]} |f(x)|^p dm$, where dm denotes the normalized Lebesgue measure. Verify:

(1) $\|f\|_p \leq \|f\|_r$.
(2) There does not exist $K > 0$ such that $\|f\|_r \leq K\|f\|_p$ for every measurable function f on $[0,1]$.

Conclude that $L^r[0,1] \subsetneq L^p[0,1]$.

Hint. Choose $q > 1$ such that $p/r + 1/q = 1$. Now apply Hölder's inequality to $f(x) = |x|^p$ and $g(x) = 1$.

Exercise 3.10 : Let $1 \leq p \leq \infty$. Show that l^p is separable iff $p < \infty$.

Hint. Let $\{a_n\}$ denote a countable dense subset of $\mathbb{K}$. Then

$$\{a_{n_1}e_1 + \cdots + a_{n_k}e_k : n_k \in \mathbb{N},\ k \geq 1\}$$

is dense in l^p if $p < \infty$. If $p = \infty$ then any two distinct points in the uncountable set $\{x : x_n = 0 \text{ or } 1\}$ are at distance 1.

Exercise 3.11 : Show that a proper subspace of a normed linear space has empty interior.

Hint. If a proper subspace has non-empty interior then it contains a ball.

4. Complete Normed Linear Spaces

A sequence $\{x_n\}$ in a normed linear space X is *Cauchy* if $\|x_n - x_m\| \to 0$ as $n, m \to \infty$. We say that $\{x_n\}$ is convergent in X if there exists $x \in X$ such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Exercise 4.1 : If $\{x_n\}$ in a normed linear space X then show that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\|x_{n_k} - x_{n_l}\| \leq 2^{-k}$ for all $l \geq k$. Prove further that $\{x_n\}$ is convergent iff $\{x_{n_k}\}$ is convergent.

A normed linear space X is said to be *complete* if every Cauchy sequence is convergent in X. Complete normed linear spaces are also known as *Banach spaces*.

Remark 4.2 : Let X be a normed linear space X with norm $\|\cdot\|$. Then X is complete iff the metric space X with metric $d(x,y) := \|x-y\|$ is complete.

Exercise 4.3 : Verify the following:

(1) A Banach space is closed.
(2) A closed subspace of a Banach space is complete.

Remark 4.4 : If E is a subset of a Banach space X then the closure of the linear span linspan$E$ of E is also a Banach space.

Exercise 4.5 : Let X be a normed linear space with two equivalent norms $\|\cdot\|$ and $\|\cdot\|'$. Show that $(X, \|\cdot\|)$ is complete iff $(X, \|\cdot\|')$ is complete.

Exercise 4.6 : Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{K}^n$. Show that the normed linear space $(\mathbb{K}^n, \|\cdot\|)$ is complete.

Hint. In view Exercise 4.5, we need to check that $(\mathbb{K}^n, \|\cdot\|_\infty)$ is complete.

Let us see an example of incomplete normed linear space.

Exercise 4.7 : Consider the vector space $\mathbb{K}[x]$ of polynomials $p(x)$ in x with the norm $\|p\|_\infty := \sup_{x\in[0,1]} |p(x)|$. Let $f_k(x) = \sum_{n=0}^{k} (x/2)^n$ be a sequence of polynomials in $\mathbb{K}[x]$. Verify:

(1) $\{f_k\}$ is Cauchy.
(2) There is no polynomial $g(x)$ such that $\|f_k - g\|_\infty \to 0$ as $k \to \infty$.

Hint. To see (2): uniform convergence implies point-wise convergence.

A series $\sum_{n=0}^{n} x_n$ in a normed linear space X is said to be *convergent* if there exists $x \in X$ such that $\|\sum_{n=0}^{k} x_n - x\| \to 0$ as $k \to \infty$. We say that $\sum_{n=0}^{n} x_n$ is *absolutely convergent* if the series $\sum_{n=0}^{n} \|x_n\|$ is convergent.

Exercise 4.8 : Show that the series $\sum_{n=1}^{\infty} e_n/n$ is convergent in l^2 but not absolutely convergent, where $e_n \in l^2$ is the sequence with nth entry equal to 1 and all remaining entries 0.

Remark 4.9 : If $\{a_k := \sum_{n=0}^{k} \|x_n\|\}$ is Cauchy then so is $\{y_k := \sum_{n=0}^{k} x_n\}$:

$$\|y_k - y_l\| \leq |a_k - a_l|.$$

Exercise 4.10 : In a Banach space, show that every absolutely convergent series is convergent.

Exercise 4.11 : Let f_k be as in Exercise 4.7. Show that $\sum_{k=0}^{\infty} f_k$ is absolutely convergent, but not convergent.

Exercise 4.12 : Consider the vector space $\mathbb{R}[x]$ of polynomials in x is an with the norm $\|p\|_c$ with $c = \{1, 1, \cdots, \}$ (see Exercise 3.7). Consider $f(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Verify the following:

(1) $f(x)$ is absolutely convergent.
(2) For any $g(x) \in \mathbb{K}[x]$ of degree d and for any integer $k > d$,

$$\|\sum_{n=1}^{k} \frac{x^n}{n^2} - g(x)\| \geq \frac{1}{(d+1)^2}.$$

Exercise 4.13 : Let $(x^{(n)} := \{x_1^{(n)}, x_2^{(n)}, \cdots, \})$ be a Cauchy sequence in l^p. For $\epsilon > 0$, verify:

(1) $(x_i^{(n)}) \subseteq \mathbb{K}$ converges to some x_i for each i.
(2) For $k \geq 1$, there exists $n_0 \geq 1$ (independent of k) such that

$$\sum_{i=1}^{k} |x_i^{(n)} - x_i|^p \leq \epsilon \text{ for all } n \geq n_0.$$

(3) For $k \geq 1$, $\sum_{i=1}^{k} |x_i|^p \leq (\epsilon + \|x^{(n_0)}\|_p)^p$.
(4) The normed linear space l^p is complete.

Exercise 4.14 : Show that the normed linear space l^∞ is complete.

Exercise 4.15 : Let X be a normed linear space with the property that every absolutely convergent series is convergent. Let $\{x_n\}$ be a sequence in a normed linear space X such that $\|x_n - x_m\| \leq 2^{-n}$ for $m \geq n$. Show that $\{x_n\}$ is convergent.

Hint. Let $x_0 = 0$ and $y_k = x_k - x_{k-1}$ for $k \geq 1$. Examine $\sum_{k=1}^{\infty} y_k$.

Theorem 4.16. *A normed linear space is complete iff every absolutely convergent series is convergent.*

As an application let us see that $L^p[0,1]$ is complete.

Exercise 4.17 : (Riesz-Fischer) Let $\sum_{n=1}^{\infty} f_n$ be an absolutely convergent series in $L^p[0,1]$, where $1 \leq p < \infty$. Define the increasing sequence $g_n(x) := \sum_{k=1}^{n} |f_k(x)|$ and let $g(x) := \sum_{k=1}^{\infty} |f_k(x)| \in [0, \infty]$. Verify:

(1) $g \in L^p[0,1]$ (**Hint.** Show that there is $M > 0$ such that $\|g_n\|_p \leq M$ for every $n \geq 1$. Now apply Fatou Lemma).
(2) $g(x)$ is finite for all $x \in [0,1]$ outside a set E of measure zero. In particular, for $x \notin E$, $s_n(x) := \sum_{k=1}^{n} f_k(x)$ is convergent.
(3) Define the measurable function s by

$$s(x) = \begin{cases} \lim_{n\to\infty} s_n(x) \text{ for } x \notin E \\ 0 \text{ for } x \in E. \end{cases}$$

Then $|s(x)| \leq g(x)$ for all $x \in [0,1]$, and hence $s \in L^p[0,1]$.
(4) $\|s_n - s\|_p \to 0$ as $n \to \infty$ (**Hint.** Lebesgue Convergence Theorem).

Exercise 4.18 : Let X be a normed linear space with norm $\|\cdot\|$. Show that X is complete iff the unit sphere $\mathbb{S}_X$ in X is a complete metric space with the metric $d(x, y) := \|x - y\|$.

Hint. Since $\mathbb{S}_X$ is closed, the necessary part is immediate. For sufficiency part, let $\{x_n\}$ be a Cauchy sequence in X. By passing to a subsequence, we may assume that $x_n \neq 0$ for every n. Now consider two cases (1) $\inf \|x_n\| = 0$ and (2) $\inf \|x_n\| > 0$.

5. Various Notions of Basis

Recall that a subset of B of a vector space V is a *basis* if every $x \in V$ is a linear combination of finitely many elements in B. This notion of basis from Linear Algebra is also known as *Hamel Basis*.

Exercise 5.1 : Let $\{s_n\}$ be a Schauder basis for an n-dimensional space X. Show that the co-ordinate functionals α_i $(i = 1, \cdots, n)$ are continuous.

Hint. Prove that the linear transformation $T : X \to \mathbb{K}^n$ given by $T(x) = (\alpha_1(x), \cdots, \alpha_n(x))$ is a homeomorphism.

Exercise 5.2 : Let X be a Banach space. For a sequence $\{x_n\}$ in X, let $Y_m := \text{linspan}\{x_1, \cdots, x_m\}$. Verify:

(1) The complement $X \setminus Y_m$ of the closed set Y_m is dense in X.
(2) The intersection $\bigcap X \setminus Y_m$ is dense in X.
(3) $\text{linspan}\{x_n\}$ is a proper subspace of X.
(4) X can not a have countable Hamel basis.

Although any vector space has a Hamel basis, it is too big to have any utility in case of Banach spaces. Thus the notion of Hamel basis is not appropriate in the study of Banach spaces. This motivates another notion of basis named after Schauder:

Definition 5.3 : A sequence $\{s_n\}$ in a Banach space X is said to be a *Schauder basis* if for every $x \in X$ there exists a unique sequence $\{c_n(x)\}$ of scalars $c_n(x)$ depending on x such that $\|x - \sum_{n=0}^{k} c_n(x)s_n\| \to 0$ as $k \to \infty$.

Remark 5.4 : For finite-dimensional spaces, the notions of Schauder basis and Hamel basis coincide.

Exercise 5.5 : Let $\{s_n\}$ be a Schauder basis for a Banach space X. Show that the mapping $c_n : x \longrightarrow c_n(x)$ is well-defined and linear.

The mapping c_n as given in the last exercise are known as the *co-ordinate functionals* corresponding to the Schauder basis $\{s_n\}$.

Exercise 5.6 : Let e_n denote the sequence with nth entry 1 and all remaining entries equal to 0. Show that $\{e_n\}$ forms a Schauder basis for l^p for $1 \leq p < \infty$.

Hint. To get uniqueness, use the definition of $\|\cdot\|_p$. Note that your argument fails in case $p = \infty$.

Exercise 5.7 : Prove that a Banach space with a Schauder basis must be separable. Conclude the following:

(1) l^p is separable for $1 \leq p < \infty$.
(2) l^∞ can not admit a Schauder basis.

We recall the notion of orthogonal vectors in $\mathbb{K}^n$. Two vectors x and y in $\mathbb{K}^n$ are *orthogonal* if $\langle x,\, y\rangle = 0$, where

$$\langle x,\, y\rangle := \begin{cases} \sum_{i=1}^n x_i y_{i2} \text{ if } \mathbb{K} = \mathbb{R}; \\ \sum_{i=1}^n x_i \bar{y}_{i2} \text{ if } \mathbb{K} = \mathbb{C}. \end{cases}$$

The *orthogonal basis* for $\mathbb{K}^n$ is nothing but a Hamel basis consisting mutually orthogonal basis vectors. The well-known Gram-Schmidt process allows one to construct an orthogonal basis from a given Hamel basis.

Exercise 5.8 : For $x, y \in \mathbb{K}^n$, define $\langle x,\, y\rangle_2 := \sum_{i=1}^n x_i y_i$. Show that $\mathbb{K}^n$ is an inner-product space with inner-product $\langle x,\, y\rangle_2$ iff $\mathbb{K} = \mathbb{R}$.

Exercise 5.9 : Let $A \in M_n(\mathbb{K})$ be such that A is one-one. Prove that $\langle x,\, y\rangle_A := \langle Ax,\, Ay\rangle_2$ defines an inner-product on $\mathbb{K}^n$.

Find $A \in M_2(\mathbb{R})$ for which $\langle (x_1, x_2),\, (y_1, y_2)\rangle_A := x_1x_2 + x_1y_2 + x_2y_1 + 2y_1y_2$ defines an inner-product on $\mathbb{R}^2$.

If $A \in M_n(\mathbb{K})$ then

$$A^* := \begin{cases} \text{transpose of } A \text{ if } \mathbb{K} = \mathbb{R}; \\ \text{conjugate transpose of } A \text{ if } \mathbb{K} = \mathbb{C}. \end{cases}$$

Exercise 5.10 : Let $A \in M_n(\mathbb{K})$ be such that $A^* = A$. Show that

$$\langle x,\, y\rangle := \langle Ax,\, y\rangle_2$$

defines an inner-product on $\mathbb{K}^n$ iff the eigen-values of A are positive.

Hint. Use Spectral Theorem for Symmetric/Self-adjoint matrices.

Exercise 5.11 : Let X denote an inner-product space with the inner-product $\langle \cdot,\, \cdot\rangle$ and let $\|\cdot\| = \sqrt{\langle \cdot,\, \cdot\rangle}$. Show that, for any $x, y \in X$, the following hold true:

(1) (Parallelogram Law)

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(2) (Polarization Identity)

$$\langle x,\, y\rangle = \begin{cases} \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2\right) \ (x, y \in X) \text{ if } \mathbb{K} = \mathbb{R}; \\ \frac{1}{4}\left(\|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2\right) \text{ if } \mathbb{K} = \mathbb{C}. \end{cases}$$

(3) (Pythagorean Identity) If $\langle x, y\rangle = 0$ then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

Hint. Write $\|x+y\|^2$ as $\langle x+y, x+y\rangle$ and then use properties of the inner-product.

Here are two applications of the Parallelogram Law:

Exercise 5.12 : Suppose that $n \geq 2$. Show that there does not exist invertible matrix A in $M_n(\mathbb{K})$ such that A maps $\{x \in \mathbb{K}^n : \|x\|_\infty = 1\}$ onto $\{x \in \mathbb{K}^n : \|x\|_2 = 1\}$.

Exercise 5.13 : Let Y be a subspace of an inner-product space X and let $x \in X$. Verify:

(1) There exists a Cauchy sequence $\{y_n\}$ in Y such that $\|x - y_n\| \to \inf\{\|x-y\| : y \in Y\}$ as $n \to \infty$ (**Hint.** Parallelogram Law).
(2) If $\{y_n\}$ converges to $y \in Y$ then $\langle x-y, z\rangle = 0$ for every $z \in Y$ (**Hint.** Assume that $\|z\| = 1$, and note that $\|x-y\| \leq \|x-w\|$, where $w := y + \langle x-y, z\rangle z$.

Theorem 5.14. *(Cauchy-Schwarz Inequality) Let X be an inner-product space with the inner-product $\langle\cdot, \cdot\rangle$ and the induced norm $\|\cdot\| = \sqrt{\langle\cdot, \cdot\rangle}$. Then $|\langle x, y\rangle| \leq \|x\|\|y\|$ for every $x, y \in X$. Moreover, equality holds iff x, y are linearly dependent.*

Proof. We may assume that $\langle x, y\rangle \neq 0$. Letting $t = \frac{\langle y, y\rangle}{\langle x, y\rangle}$ in $\langle y - tx, y - tx\rangle \geq 0$, we obtain Cauchy-Schwarz inequality. Equality holds iff $\langle y - tx, y - tx\rangle = 0$ iff $y = tx$. □

Corollary 5.15. *Let X denote an inner-product space with the inner-product $\langle\cdot, \cdot\rangle$. Show that $\|\cdot\| = \sqrt{\langle\cdot, \cdot\rangle}$ defines a norm.*

Proof. The only difficult part is the triangle inequality, which follows from the Cauchy-Schwarz Inequality:

$\|x+y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 \leq (\|x\| + \|y\|)^2$. □

Remark 5.16 : Note that equality holds in the Cauchy-Schwarz Inequality iff that holds in the triangle inequality.

Exercise 5.17 : Let X be an inner-product space and let $0 \leq t \leq 1$. If $x, y \in X$ are linearly independent then $\|tx + (1-t)y\| < 1$ iff $0 < t < 1$.

Hint. Use the equality part of Cauchy-Schwarz Inequality.

Corollary 5.18. *The inner-product is jointly continuous.*

Proof. Use the continuity of norm and either the Polarization Identity or Cauchy-Schwarz Inequality. □

Not all norms are induced by an inner-product.

Exercise 5.19 : (Jordan and von Neumann) Let X denote a normed linear space with the norm $\|\cdot\|$. If X satisfies the Parallelogram Law then the expression $\langle x, y\rangle$, as given in the Polarization Identity, satisfies $\sqrt{\langle\cdot, \cdot\rangle} = \|\cdot\|$ and defines an inner-product on X.

In other words, norm on any normed linear space is induced by an inner-product if and only if it satisfies the Parallelogram Law.

Hint. We divide the verification into four steps:

(1) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ $(x, y \in X)$.
(2) $\langle x/2, y\rangle = 1/2\langle x, y\rangle$ $(x, y \in X)$.
(3) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle =$ $(x, y \in X)$.
 Note that $x + y + z = x + y/2 + y/2 + z$.
(4) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for any $\alpha \in \mathbb{C}$.
 Use density of $\{m/2^n : m \in \mathbb{Z}, \ n \in \mathbb{N} \cup \{0\}\}$ in $\mathbb{R}$ to conclude that $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for any real α.

A *Hilbert space* is a complete inner-product space.

Exercise 5.20 : Verify the following:

(1) l^p is a Hilbert space iff $p = 2$.
(2) $L^p[0, 1]$ is a Hilbert space iff $p = 2$.

Hint. $\|\cdot\|_p$ satisfies the Paralleogram Law iff $p = 2$.

The *Hardy space* H^2 of the unit disc is a normed linear space of complex-valued functions f holomorphic on the unit disc $\mathbb{D}_1$ for which

$$\|f\|^2_{H^2} := \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Exercise 5.21 : For the Hardy space H^2, verify the following:

(1) H^2 is an inner-product space endowed with the inner-product

$$\langle f, g\rangle_{H^2} := \frac{1}{4}\left(\|f + g\|^2_{H^2} - \|f - g\|^2_{H^2} + i\|f + ig\|^2_{H^2} - i\|f - ig\|^2_{H^2}\right)$$

 (**Hint.** Verify that the norm on H^2 satisfies the Paralleogram Law).
(2) If $\{f_n\}$ is a Cauchy sequence in H^2 then

$$|f_n(z) - f_m(z)| \leq \frac{\|f_n - f_m\|_{H^2}}{R - r}$$

 for any $|z| \leq r < R < 1$ (**Hint.** By the Cauchy Integral Formula,

$$f_n(z) - f_m(z) = \int_{|z|=R} \frac{f(w)}{w - z} dw \ (|z| < R < 1).$$

(3) If $\{f_n\}$ is a Cauchy sequence in H^2 then there exists a holomorphic function $f : \mathbb{D}_1 \to \mathbb{C}$ such that $\|f_n - f\|_{\infty,K}$ for every compact subset K of the unit disc $\mathbb{D}_1$ (**Hint.** Weierstrass Convergence Theorem).
(4) H^2 is a Hilbert space.

Remark 5.22 : In the discussion to follow, we confine ourselves to separable Hilbert spaces. In particular, according to our definition, orthonormal basis is necessarily (finitely or infinitely) *countable*. Even after this assumption, practically, we do not loose much.

Exercise 5.23 : Find an example of a non-separable Hilbert space.

Let X be an inner-product space with inner product $\langle \cdot, \cdot \rangle$. A subset E of X is *orthogonal* if $\langle x, y \rangle = 0$ for every $x, y \in E$. A subset E of X is *orthonormal* if every element in E has unit norm and $\langle x, y \rangle = 0$ for every $x, y \in E$. We say that y is *orthogonal* to E if $\langle x, y \rangle = 0$ for every $x \in E$.

Exercise 5.24 : Let X be a separable inner-product space. Any orthonormal subset of X is countable.

Hint. Suppose $\{x_\alpha\}$ is orthonormal. Then $\{\mathbb{B}(x_\alpha, 1/\sqrt{2})\}$ is a collection of disjoint open balls in H. Now if $\{y_n\}$ is countable dense subset of H then each $B(x_\alpha, 1/\sqrt{2})$ must contain at least one y_n.

Theorem 5.25. *(Orthogonal Decomposition of a Hilbert Space) Let H be a Hilbert space and let Y be a closed subspace of H. Then every $x \in H$ can be written uniquely as $y + z$, where $y \in Y$ and z is orthogonal to Y.*

Proof. By Exercise 5.13(1), there exists a Cauchy sequence $\{y_n\}$ such that $\|x - y_n\|$ converges to the distance d between x and Y. Note that d is positive since Y is closed subspace of H. Also, since H is complete and Y is closed, $\{y_n\}$ converges to some $y \in Y$. By Exercise 5.13(2), $x - y$ is orthogonal to Y. To see the uniqueness part, note that $y_1 + z_1 = y_2 + z_2$ implies $y_1 - y_2 = z_2 - z_1$ belongs to Y and orthogonal to Y. It follows that $y_1 = y_2$ and $z_1 = z_2$. □

Example 5.26 : The orthogonal decomposition may not be unique in case Y is not a closed subspace of H. For instance, consider $H := l^2$ and $c_{00} := \{(x_n) \in l^2 : x_n \text{ is non-zero for finitely many } n\}$. Then, for each $k \geq 1$, $\sum_{k=1}^{\infty} \frac{e_k}{k} \in l^2$ can be decomposed as $\sum_{k=1}^{n} \frac{e_k}{k} + z$, where $z := \sum_{k=n+1}^{\infty} \frac{e_k}{k}$ is orthogonal to $\sum_{k=1}^{n} \frac{e_k}{k}$. ■

An *orthonormal basis* for X is a Schauder basis which is also an orthonormal subset of X.

Example 5.27 : Let e_n denote the sequence with nth entry equal to 1 and all other entries 0. Then $\{e_n\}$ forms an orthonormal basis for l^2. ■

Exercise 5.28 : Show that $\{z^n\}$ forms an orthonormal basis for the Hardy space H^2.

Exercise 5.29 : Let $\{e_n\}$ be an orthonormal subset of an inner-product space X. Verify the following:

(1) (Bessel's Inequality) $\sum_{n=0}^{\infty} |\langle x, e_n \rangle|^2 \leq \|x\|^2$.

(2) (Riemann-Lebesgue Lemma) $\lim_{n\to\infty}\langle x, e_n\rangle = 0$.

Hint. Apply Pythagorean Identity to $y := x - \sum_{n=0}^{k}\langle x, e_n\rangle e_n$ and $z := \sum_{n=0}^{k}\langle x, e_n\rangle e_n$.

Exercise 5.30 : For $n \in \mathbb{Z}$, let $E_n(t) := e^{int}$. Verify:

(1) $\{E_n\}$ is an orthonormal subset of $L^2[0, 2\pi]$.
(2) $\lim_{n\to\infty}\int_{[0,2\pi]} f(t)\cos(nt)dt = 0 = \lim_{n\to\infty}\int_{[0,2\pi]} f(t)\sin(nt)dt$ for any $f \in L^2[0, 2\pi]$.

Exercise 5.31 : Let $\{e_n\}$ be an orthonormal basis of a Hilbert space H. Verify the following:

(1) Every $h \in H$ takes the form $\sum_{n=0}^{\infty}\langle h, e_n\rangle e_n$ (**Hint.** If $h = \sum_{n=0}^{\infty}\alpha_n e_n$ then $\alpha_n = \langle h, e_n\rangle$).
(2) The co-ordinate functionals corresponding to the orthonormal basis $\{e_n\}$ are continuous.
(3) (Parseval's Identity) $\|h\|^2 = \sum_{n=0}^{\infty}|\langle h, e_n\rangle|^2$.

Exercise 5.32 : (Maximal Orthonormal Set) Let $\{e_n\}$ be an orthonormal set in a Hilbert space H with the property: If $x \in H$ such that $\langle x, e_n\rangle = 0$ for all n then $x = 0$. Show that $\{e_n\}$ is an orthonormal basis.

Hint. Examine $\langle x - \sum_{n=0}^{\infty}\langle x, e_n\rangle e_n, e_m\rangle$.

Exercise 5.33 : For $n \in \mathbb{Z}$, let $E_n(t) := e^{int}$. Let $f \in L^1[0, 2\pi]$ be such that $\langle f, E_n\rangle_2 = 0$ for every $n \in \mathbb{Z}$. Define $F(t) := \int_{[0,t]} f(s)ds$. Verify:

(1) There is a scalar $a \in \mathbb{K}$ such that $\langle F - a, E_n\rangle_2 = 0$ for every $n \in \mathbb{Z}$ (**Hint.** Integration by Parts).
(2) There exists a sequence of trigonometric polynomials p_n such that $\|F - a - p_n\|_\infty \to 0$ as $n \to \infty$ (**Hint.** Stone-Weierstrass Theorem).
(3) $F = a$ a. e. (**Hint.** $\|F - a\|_2 \leq \|F - a - p_n\|_\infty\|F - a\|_1$).
(4) f is zero almost everywhere.

Conclude that $\{E_n\}$ is an orthonormal basis for $L^2[0, 2\pi]$.

Exercise 5.34 : Show that, for every $f \in L^2[0, 2\pi]$, we have

$$\int_{[0,2\pi]} |f(s) - \sum_{n=-k}^{k}\langle f, e^{int}\rangle e^{-ins}|^2 ds \to 0 \text{ as } k \to \infty.$$

Scholium 5.35 : Every separable Hilbert space has an orthonormal basis.

Proof. This is an application of the Zorn's Lemma: *Suppose a partially ordered set $\mathcal{P}$ has the property that every totally ordered subset has an upper bound in $\mathcal{P}$. Then the set $\mathcal{P}$ contains at least one maximal element.*

Let $\mathcal{P}$ be the collection of all orthonormal subsets of H ordered by inclusion. If $h \neq 0$ then $\{h/\|h\|\} \in \mathcal{P}$, so that $\mathcal{P}$ is non-empty. If $\mathcal{Q}$ be a totally ordered subset of $\mathcal{P}$, then $\cup_{A\in\mathcal{Q}}A$ is an upper bound of $\mathcal{Q}$ in $\mathcal{P}$. By Zorn's Lemma, $\mathcal{P}$ contains a maximal element $B \in \mathcal{P}$. Since H is separable, by Exercise 5.24, B is countable. We contend that B is an orthonormal basis for H. To see that, let $x \in H$ be such that $\langle x, y\rangle = 0$ for every $y \in B$. If $x \neq 0$ then $B \cup \{x/\|x\|\} \in \mathcal{P}$, which is impossible. It follows that $x = 0$, and hence by Exercise 5.32, B is an orthonormal basis of H. □

Exercise 5.36 : Show that there exists an isometric isomorphism from a separable Hilbert space H onto l^2. In case H is $L^2[0, 2\pi]$, can you see what will be the isometric isomorphism ?

Hint. For second part, recall the definition of Fourier transform.

6. Bounded Linear Transformations

A linear transformation $T : X \to Y$ is said to be *bounded* if there exists $M > 0$ such that $\|Tx\| \leq M\|x\|$ for every $x \in X$.

Exercise 6.1 : Show that every linear transformation from a finite dimensional normed linear space is bounded.

Exercise 6.2 : Let X be a normed linear space with Hamel basis $\{e_n\}$ consisting unit vectors e_n. Define $Te_n = ne_n$ and extend T linearly to X. Verify that T is bounded iff X is finite-dimensional.

Exercise 6.3 : Use Riesz Lemma to show that if X contains an infinite linearly independent subset $\{z_n\}$ then the unit sphere in X is not compact.

Hint. Let $Z_n := \text{span}\{z_1, \cdots, z_n\}$. Choose $x_n \in Z_{n+1}$ such that $\|x_n\| = 1$ and $d(x_n, Z_n) \geq 1/2$.

Exercise 6.4 : For any subspace Y of H, show that $Y^\perp := \{x \in H : \langle x, y\rangle = 0 \text{ for all } y \in Y\}$ is a closed subspace of H.

Exercise 6.5 : (Orthogonal Projection) Let H be a Hilbert space with a closed subspace Y. In view of Theorem 5.25, every $x \in H$ may be uniquely written as $y + z$, where $y \in Y$ and $z \in Y^\perp$. Verify:

(1) $P : H \to Y$ given by $Px = y$ defines a bounded linear map.
(2) $P^\perp : H \to Y^\perp$ given by $P^\perp x = z$ defines a bounded linear transformation such that $P + P^\perp = I$.

Theorem 6.6. *Let $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

(1) *T is bounded.*
(2) *T is continuous.*
(3) *T is continuous at* 0.
(4) *T maps null-sequences to bounded sequences.*

By a *linear functional*, we mean a linear transformation from a normed linear space into $\mathbb{K}$.

Exercise 6.7 : Let f be a linear functional be such that the kernel of f is closed. Show that f is bounded.

Theorem 6.8. *(Riesz Representation Theorem) Let H be a separable Hilbert space and let $f : H \to \mathbb{K}$ be a bounded linear functional. Then there exists a unique $y \in H$ (representative of f) such that $f(x) = \langle x,\, y\rangle$ for all $x \in H$.*

Proof. Let $\{e_n\}$ be an orthonormal basis for H as ensured by Scholium 5.35. Consider $y_k := \sum_{n=1}^{k} \overline{f(e_n)} e_n \in H$. Note that $f(y_k) = \sum_{n=1}^{k} |f(e_n)|^2 = \|y_k\|^2$. Also, there is $M > 0$ such that $|f(y_k)| \leq M\|y_k\|$. Combining last two estimates, we obtain $\|y_k\| \leq M$ for every $k \geq 1$. In particular, $y := \sum_{n=1}^{\infty} \overline{f(e_n)} e_n \in H$. Check that $f(x) = \langle x,\, y\rangle$.

To see the uniqueness part, note that $\langle x,\, y_1\rangle = \langle x,\, y_2\rangle$ implies $\langle x,\, y_1 - y_2\rangle$ for every $x \in H$, and hence for $x := y_1 - y_2$. This gives $y_1 = y_2$. $\square$

Exercise 6.9 : Consider the Hilbert space H_k of trigonometric polynomials spanned by the orthogonal set $\{e^{int} : -k \leq n \leq k\}$ with $\|\cdot\|_2$ norm. Find representative of the linear functional $f : H_k \to \mathbb{C}$ given by $f(p) = p'(0)$.

Exercise 6.10 : Consider the Hilbert space $L^2[0, 2\pi]$. Find representative of the linear functional $\phi : L^2[0, 2\pi] \to \mathbb{C}$ given by $\phi(f) = f(0)$.

Exercise 6.11 : Consider the linear functional $f : c_{00} \to \mathbb{K}$ given by $f(x) = \sum_{k=1}^{\infty} x_k/k$. Verify:

1. $|f(x)| \leq \sqrt{\pi/6}\|x\|_2$.
2. f does not have any representative in c_{00}.

Let $T : X \to Y$ be a bounded linear transformation. The *operator norm* of T is given by $\|T\| := \sup\{\|Tx\| : \|x\| = 1\}$. Since T is bounded, $\|T\|$ is finite. Further:

Exercise 6.12 : Show that $\|T\| = \inf\{M \in \mathbb{R}_+ : \|Tx\| \leq M\|x\|\}$.

Exercise 6.13 : What is the operator norm of (1) a diagonal operator (2) an isometry (3) an orthogonal projection (4) f appearing in RRT ?

Exercise 6.14 : Let A be an $n \times n$ self-adjoint matrix with complex entries. If $\lambda_1, \cdots, \lambda_n$ are eigenvalues of A then show that $\|A\| = \max\{|\lambda_1|, \cdots, |\lambda_n|\}$.

Hint. If A, B are two $n \times n$ matrices then $\|AB\| \leq \|A\|\|B\|$. Now apply the Spectral Theorem.

Theorem 6.15. *The set $BL(X, Y)$ of bounded linear transformations from X into Y is a normed linear space with norm as the operator norm. If Y is complete then so is $BL(X, Y)$.*

Remark 6.16 : Let X' denote the normed linear space of bounded linear functionals on a normed linear space X. Then X' is always complete.

Exercise 6.17 : Let H be a separable Hilbert space. Define $\phi : H \to H'$ by $\phi(y) = f_y$, where $f_y(x) = \langle x, y\rangle$ $(x \in H)$. Show that ϕ is an isometric isomorphism. Conclude that the norm on H' is induced by an inner-product.

In the following two exercises, we need the fact that every bijective bounded linear operator on l^2 has bounded inverse (This will be proved later!)

Exercise 6.18 : Define $D : l^2 \to l^2$ by $De_n = 1/ne_n$. Show that D is one-one with dense range, but not surjective.

Exercise 6.19 : Let $B : l^2 \to l^2$ be defined by $Be_1 = 0$ and $Be_n = e_{n-1}$ for $n \geq 2$. Show that $B-I$ is injective with dense range, which is not surjective.

Hint. Verify: (1) $Bx = x$ implies $x = 0$. (2) $\{e_n\} \subseteq \text{ran}(B)$. (3) If there is $\alpha > 0$ such that $\|(B-I)x\| \geq \alpha\|x\|$ then try $x = \sum_{n=1}^{\infty}(1-1/k)^n e_n \in l^2$.

The following exercise shows how the information about operators can be used to know more about spaces.

Exercise 6.20 : Suppose that there exists an injective operator $T \in B(\mathcal{K})$ such that $\text{ran}(T) \subsetneq \mathcal{K}$ is dense in $\mathcal{H}$. Prove that there exist a Hilbert space $\mathcal{H}$ and closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ such that $\mathcal{M} + \mathcal{N} \subsetneq \mathcal{H}$ is dense in $\mathcal{H}$ and $\mathcal{M} \cap \mathcal{N} = \{0\}$.

Hint. Let $\mathcal{H} := \mathcal{K} \oplus \mathcal{K}$ (with inner-product $\langle (x \oplus y, x' \oplus y'\rangle = \langle x, x'\rangle + \langle y, y'\rangle)$, $\mathcal{M} := \{x \oplus Tx : x \in \mathcal{K}\}$ and $\mathcal{N} := \mathcal{K} \oplus \{0\}$.

Exercise 6.21 : Let X be a normed linear space. Let M be a closed linear subspace of X, let N be a finite-dimensional subspace, and let $x \in X$. Prove:

(1) $M + \mathbb{K}x$ is a closed subspace of X.
(2) $M + N$ is a closed subspace of X.

Conclude that any finite-dimensional subspace of X is closed in X.

Hint. If $y_n + \alpha_n x$ is a Cauchy sequence in $M + \mathbb{K}x$ then so is α_n : $\|y_n + \alpha_n x - (y_m + \alpha_m x)\| \geq |\alpha_n - \alpha_m| d(M, x)$.

7. Three Basic Facts in Functional Analysis

A mapping $T : X \to Y$ is said to be open if $T(U)$ is open in Y for every open subset U of X.

Exercise 7.1 : Show that translation and dilation are open mappings.

Exercise 7.2 : Show that every linear open map is surjective.

If T has continuous inverse then clearly T is open. What is surprising is that this is true even if T is not invertible. This is the content of the open mapping theorem. Before we state and prove it, let us see a handy characterization of linear open mappings.

Lemma 7.3. *Let $T : X \to Y$ be a linear transformation. Then the following are equivalent:*

1. *T sends the open unit ball to an open subset of Y.*
2. *T is an open mapping.*
3. *There exists $c > 0$ such that for each $y \in Y$ there corresponds $x \in X$ with the properties $\|x\| \leq c\|y\|$ and $Tx = y$.*

Proof. (1) implies (2): Let U be an open subset of X. Let $y = Tx \in T(U)$. Then $\mathbb{B}(x, R) \subseteq U$ for some $R > 0$. Then $\mathbb{B}(0, 1) \subseteq \frac{U-x}{R} := \{\frac{y-x}{R} : y \in U\}$. Since $T(\mathbb{B}(0, 1))$ is open in Y, so is $R \cdot T(\mathbb{B}(0, 1)) + Tx = T(\mathbb{B}(x, R))$ in $T(U)$.

(2) implies (3): Note that $0 \in T(\mathbb{B}(0, 1))$ is open, and hence $\mathbb{B}(0, R) \subseteq T(\mathbb{B}(0, 1))$ for some $R > 0$. Thus for every $y \in Y$, $r\frac{y}{\|y\|} \in B(0, R)$ for any $0 < r < R$. Then there exists $x_0 \in \mathbb{B}(0, 1)$ such that $Tx_0 = r\frac{y}{\|y\|}$. Check that $Tx = y$, where $x = x_0\|y\|/r$. Check that $\|x\| \leq R^{-1}\|y\|$, so that (3) holds with $c = R^{-1}$.

(3) implies (1): For any $r > 0$, check that $\mathbb{B}(y_0, rc^{-1}) \subseteq T(\mathbb{B}(x_0, r))$, where $y_0 = Tx_0$. □

Exercise 7.4 : Let X, Y be Banach spaces and let $T : X \to Y$ be a surjective bounded linear transformation and let $y \in Y$ be a unit vector. Verify:

1. $Y = \cup_{n=1}^{\infty} \overline{T(\mathbb{B}(0, n))}$.
2. There is an integer $k \geq 1$ such that $\overline{T(\mathbb{B}(0, k))}$ contains a non-empty open set W.
3. Let $y_0 \in W$ be such that $\mathbb{B}(y_0, R) \subseteq W$ for some $R > 0$. If $z \in \overline{\mathbb{B}(y_0, R)}$ then there exists $\{u_n\} \subseteq \overline{\mathbb{B}(0, 2k)}$ such that $Tu_n \to z - y_0$.
4. There exists $x_1 \in X$ such that $\|x_1\| \leq \frac{2k}{R}$ and $\|y - Tx_1\| < 1/2$.
5. There exists a sequence $\{x_n\}$ such that $\|x_n\| \leq \frac{2k}{R}\frac{1}{2^{n-1}}$ and $\|y - (Tx_1 + \cdots + Tx_n)\| \leq 1/2^n$.
6. The sequence $\{x_1 + \cdots + x_n\}$ converges to some $x \in X$. Moreover, $\|x\| \leq 4k/R$ and $Tx = y$.

The preceding exercise and Lemma 7.3 immediately give the following:

Theorem 7.5. *(Open Mapping Theorem) Every bounded linear transformation from a Banach space onto a Banach space is open.*

The following is often known as the Bounded Inverse Theorem (for short, BIT).

Corollary 7.6. *(Algebraic invertibility implies topological invertibility) A bijective bounded linear transformation is a homeomorphism.*

Example 7.7 : Let (a_n) be a sequence of positive real numbers. Define $\|\cdot\|_a$ on l^∞ by $\|(b_n)\|_a := \sum_n a_n|b_n|$. Note that $\|\cdot\|_a$ satisfies all conditions of a norm except that $\|\cdot\|_a$ is $[0,\infty]$-valued function. We contend that for no (a_n), $\|(b_n)\|_a < \infty$ iff $(b_n) \in l^\infty$. We prove this by contradiction.

Note that $\|\cdot\|_a$ defines a norm, which makes l^∞ complete. In particular, $\sum_n a_n < \infty$. It follows that $\|(b_n)\|_a \leq \|(b_n)\|_\infty \sum_n a_n$. Thus the identity transformation from l^∞ onto the Banach space $(l^\infty, \|\cdot\|_a)$ is continuous. By Bounded Inverse Theorem, there exists $M > 0$ such that $\|(b_n)\|_\infty \leq M\|(b_n)\|_a$ for all $(b_n) \in l^\infty$. Letting $(b_n) = e_n$, we obtain $1 \leq Ma_n$, which implies that $\sum_n a_n = \infty$. ■

Exercise 7.8 : Use BIT to show that $C[0,1]$ is incomplete in the $\|\cdot\|_p$ norm for any $1 \leq p < \infty$.

Corollary 7.9. *(Closed Graph Theorem) Let X and Y be Banach spaces. Let $T : X \to Y$ be a linear operator, which is closed in the sense that if $x_n \to x$ and $Tx_n \to y$ then $Tx = y$. Then T is continuous.*

Proof. Define the norm $|x|$ on X by $|x| := \|x\| + \|Tx\|$ $(x \in X)$. Then $(X, |\cdot|)$ is complete. Apply BIT to identity mapping from X onto $(X, |\cdot|)$. □

Exercise 7.10 : Define a linear operator D by $De_n = ne_n$. Extend D linearly to $\{x = (x_n) \in l^2 : \sum_{n=1}^\infty n^2|x_n|^2 < \infty\}$ by setting $Dx = \sum_{n=1}^\infty nx_n$. Show that D is closed but not continuous. What goes wrong with CGT ?

Exercise 7.11 : (Multiplication Operators) Let $\phi \in L^p[0,1]$ be such that $\phi f \in L^p[0,1]$ whenever $f \in L^p[0,1]$. Define a linear operator $M_\phi : L^p[0,1] \to L^p[0,1]$ by $M_\phi(f) = \phi f$. Show that M_ϕ is a bounded linear operator.

Hint. Closed Graph Theorem.

Exercise 7.12 : Show that, up to equivalence of norms, the sup norm is the only norm on $C[0,1]$, which makes $C[0,1]$ complete and which also implies the point-wise convergence.

Exercise 7.13 : Show that, up to equivalence of norms, the $\|\cdot\|_p$ norm is the only norm on $L^p[0,1]$, which makes $L^p[0,1]$ complete and which also implies the point-wise convergence almost everywhere of a subsequence.

Exercise 7.14 : Suppose X is a Banach space, Y is a normed linear space, and $\mathcal{F} \subseteq B(X,Y)$. For $n \geq 1$, let

$$V_n := \{x \in X : \text{there exists } T \in \mathcal{F} \text{ for some } \|Tx\| > n\}.$$

Verify the following:

(1) V_n is an open subset of X.
(2) If V_n is dense for every $n \geq 1$, then there exists a dense subset E of X such that $\sup_{T\in\mathcal{F}} \|Tx\| = \infty$ for all $x \in E$ (**Hint.** Take $E := \cap_n V_n$).
(3) If there is an $N \geq 1$ such that V_N is not dense then $\sup_{T\in\mathcal{F}} \|T\| < \infty$.

The following is often referred to as the Uniform Boundedness Principle (for short, UBP).

Theorem 7.15. *Suppose X is a Banach space, Y is a normed linear space, and $\mathcal{F} \subseteq B(X, Y)$. Then only one of the following holds true:*

1. $\sup_{T \in \mathcal{F}} \|T\| < \infty$.
2. *There exists a dense set $E \subseteq X$ such that* $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$ *for all* $x \in E$.

Proof. Use the last exercise. $\square$

Corollary 7.16. *If $\{T_n\}$ is a sequence of bounded linear operators from a Banach space X into a normed linear space Y such that* $\lim_{n \to \infty} T_n x$ *exists for every $x \in X$. Then the linear operator $T : X \to Y$ defined by* $Tx := \lim_{n \to \infty} T_n x$ $(x \in X)$ *is a bounded linear operator.*

Proof. Apply UBP to $\mathcal{F} := \{T_n\}$ to conclude that $\sup_n \|T_n\| < \infty$. For given $\epsilon > 0$, choose $N \geq 1$ (depending on x) such that $\|T_n x - Tx\| < \epsilon$. Then, for any unit vector $x \in X$, $\|Tx\| \leq \epsilon + \sup_n \|T_n\|$. $\square$

Exercise 7.17 : Let $f \in L^1[-\pi, \pi]$ with $\|f\|_1 := \int_{[-\pi,\pi]} |f(t)| \frac{dt}{2\pi}$. Let $\hat{f}(k) = \int_{[-\pi,\pi]} f(t) e^{-ikt} \frac{dt}{2\pi}$ for $k \in \mathbb{Z}$. Verify the following:

1. $\sum_{k=-n}^{n} \hat{f}(k) e^{ikx} = \int_{[-\pi,\pi]} f(t) D_n(x - t) \frac{dt}{2\pi}$, where
$$D_n(t) := \sum_{k=-n}^{n} e^{ikt} = \frac{\sin((n + 1/2)t)}{\sin(t/2)}.$$
2. $\|D_n\|_1 \geq \frac{4}{\pi} \sum_{k=1}^{n} \frac{1}{k}$ (**Hint.** $|D_n(t)| \geq 2|\sin((n + 1/2)t)|/|t|$).
3. If $\Lambda_n : C[-\pi, \pi] \to \mathbb{C}$ is given by $\Lambda_n(f) := \sum_{k=-n}^{n} \hat{f}(k)$ then Λ_n is a bounded linear functional with $\|\Lambda_n\| = \|D_n\|_1$.
4. There exists $f \in C[-\pi, \pi]$ such that the Fourier series $\sum_{k=-\infty}^{\infty} \hat{f}(k) e^{ikt}$ diverges at any given point in $[-\pi, \pi]$.

8. The Hahn-Banach Extension Theorem

In this section, we discuss the norm-preserving extension problem, that is, the problem of extending a given linear functional defined on a subspace linearly and continuously to the given space such that norm is preserved: Let Y be a subspace of X, and let $f : Y \to \mathbb{K}$ be a bounded linear functional. We wish to find a linear functional $g : X \to \mathbb{K}$ such that

1. $g(y) = f(y)$ for every $y \in Y$, and
2. $\|g\| = \|f\|$.

If this happens then we say that g is a *norm-preserving extension* of f.

Remark 8.1 : (1) above implies that $\|f\| \leq \|g\|$.

Let us see a couple of special cases in which the norm-preserving extension problem can be solved.

Exercise 8.2 : If Y is a dense subspace of X then any bounded linear functional $f : Y \to \mathbb{K}$ extends uniquely to a bounded linear functional $g : X \to \mathbb{K}$ such that $\|g\| = \|f\|$.

Exercise 8.3 : If Y is a subspace of a Hilbert space H and $f : Y \to \mathbb{K}$ then there exists a linear functional $g : X \to \mathbb{K}$ such that $g(y) = f(y)$ for every $y \in Y$, and $\|g\| = \|f\|$.

Hint. $g := f \circ P_{\overline{Y}}$, where $P_{\overline{Y}}$ is the orthogonal projection of H onto $\overline{Y}$.

To treat the general case, let us first solve the problem for a subspace of co-dimension 1 over real field.

Exercise 8.4 : Let Y be a subspace of the normed linear space X over $\mathbb{R}$ such that $X \setminus Y$ is a 1 dimensional space spanned by x_1. Let $f : Y \to \mathbb{R}$ be a bounded linear functional. Verify the following:

(1) For $y_1, y_2 \in Y$,
$$-f(y_2) - \|f\|\|y_2 + x\| \leq -f(y_1) + \|f\|\|y_1 + x\|.$$
(2) For $y \in Y$, there exists $\alpha \in \mathbb{R}$ (independent of y) such that
$$-f(y) - \|f\|\|y + x_1\| \leq \alpha \leq -f(y) + \|f\|\|y + x_1\|$$
(**Hint.** Take sup over left, and then inf over right in (1). Any α between the sup and inf works).
(3) $|\alpha + f(y)| \leq \|f\|\|y + x\|$ for every $y \in Y$.
(4) Define the functional $g : Y + \mathbb{R}x_1 \to \mathbb{R}$ by $g(y + \lambda x_1) = f(y) + \lambda\alpha$. Then g_1 is a well-defined linear functional such that $\|g\| = \|f\|$.

Exercise 8.5 : Let X be a normed linear space and let Y be a subspace of X (over $\mathbb{R}$). Given a bounded linear functional $f : Y \to \mathbb{R}$, consider
$$\mathcal{P} := \{(Z, g) : g : Z \to \mathbb{R} \text{ is a norm-preserving bounded linear extension of } f\}.$$
Verify the following:

(1) $\mathcal{P}$ is a non-empty partially ordered set with order defined by
$$(Z_1, g_1) \leq (Z_2, g_2) \text{ if } Z_1 \subseteq Z_2 \text{ and } g_1(z) = g_2(z) \text{ for all } z \in Z_1.$$
(2) Suppose $\mathcal{Q} := \{(Z_i, g_i) \in \mathcal{P} : i \in I\}$ is a totally ordered subset of $\mathcal{P}$. Then (Z, g) is a upper bound of $\mathcal{Q}$ in $\mathcal{P}$, where
$$Z := \bigcup_{i \in I} Z_i, \text{ and } g(z) = g_i(z) \text{ if } z \in Z_i.$$
(3) If (Z, g) is the maximal element of $\mathcal{P}$ then $Z = X$.
(4) There exists a norm-preserving bounded linear extension of f.

Exercise 8.6 : Let X be a normed linear space and let Y be a subspace of X (over $\mathbb{C}$). Let $f : Y \to \mathbb{C}$ be a bounded linear functional. Verify:

(1) There exists a bounded $\mathbb{R}$-linear functional $u : Y \to \mathbb{R}$ such that $f(y) = u(y) - iu(iy)$ for all $y \in Y$ and $\|f\| = \|u\|$ ((**Hint.** For $y \in Y$, find $\theta \in \mathbb{R}$ such that $|f(y)| = f(e^{i\theta}y) = u(e^{i\theta}y) \leq \|u\|\|y\|$).
(2) Let $v : X \to \mathbb{R}$ be a norm-preserving extension of u (as guaranteed by the previous exercise). Then $g(x) = v(x) - iv(ix)$ is a norm-preserving extension of f.

We combine last two exercises to obtain Hahn-Banach extension Theorem.

Theorem 8.7. *Let X be a normed linear space and let Y be a subspace of X (over $\mathbb{K}$). Then, for any bounded linear functional $\phi : Y \to \mathbb{K}$, there exists a bounded linear functional $\psi : X \to \mathbb{K}$ such that*

$$\psi(y) = \phi(y) \ (y \in Y) \ \textit{and} \ \|\psi\| = \|\phi\|.$$

Corollary 8.8. *If $x \neq y \in X$ then there exists a bounded linear functional $g : X \to \mathbb{K}$ such that $g(x) \neq g(y)$.*

Proof. Consider the one-dimensional subspace Y spanned by $x-y$ and define $f(t(x-y)) = t\|x-y\|$. Now apply HBT. $\square$

Exercise 8.9 : Show that

$$\|x\| = \sup\{|f(x)| : f \text{ is a bounded linear functional such that } \|f\| \leq 1\}.$$

Corollary 8.10. *If M is a closed subspace of X and $x \in X \setminus M$ then there exists a bounded linear functional $g : X \to \mathbb{K}$ of unit norm such that $g = 0$ on M and $g(x) = d(x, M)$.*

Proof. Consider the subspace $Y := M + \mathbb{K}x$, and define $f(y + \alpha x) = \alpha d(x, M)$ for $y \in M$ and $\alpha \in \mathbb{K}$. Now apply HBT. $\square$

Exercise 8.11 : Let $\{M_k\}_{k \geq 1}$ be a countable collection of closed subspaces of H. Show that

$$\{\cap_{k \geq 1} M_k\}^{\perp} = \bigvee_{k \geq 1} M_k^{\perp}.$$

Hint. Consider the two closed subspaces S_1 and S_2 of H:

$$S_1 := \bigvee_{k \geq 1} M_k^{\perp}, \ S_2 := \{\cap_{k \geq 1} M_k\}^{\perp}.$$

Verify that $S_1 \subseteq S_2$. Suppose, we have the strict inclusion, $S_1 \subsetneq S_2$, and apply the previous corollary.

Exercise 8.12 : Show that polynomials are dense in the Hardy space of the unit disc.

Hint. Consider $M_k = z^k H^2$. By the Identity Theorem, $\cap_{k \geq 1} M_k = \{0\}$. Apply the last exercise to conclude that $\bigvee_{k \geq 1} M_k^{\perp} = H^2$. Since $M_k^{\perp}$ is the vector space of polynomials of degree less than k, we are done.

Remark 8.13 : Apart from an application of HBT, this is not the best solution.

Exercise 8.14 : Let M be a closed subspace of X. Show that

$$M = \bigcap_{f \in \mathcal{F}} \ker f,$$

where $\mathcal{F}$ is the space of all bounded linear functionals f on X such that $M \subseteq \ker f$.

The last exercise may be used to give a proof of Runge's Theorem from $\mathbb{C}$-analysis [2, Chapter III, Section 8].

9. Dual Spaces

Let X be a normed linear space. The *dual space* X' of X is defined as the normed linear space of all bounded linear functionals $f : X \to \mathbb{K}$. We have seen that X' is a Banach space with norm $\|f\| := \sup\{|f(x)| : \|x\| \leq 1\}$ (compare this with the conclusion of Exercise 8.9).

Exercise 9.1 : $X \neq \{0\}$ if and only if $X' \neq \{0\}$.

Hint. Let $x \in X$ be non-zero. Then $f(\alpha x) = \alpha\|x\|$ is a bounded linear functional on $\mathbb{K}x$. Now apply HBT.

Exercise 9.2 : For $p < \infty$, find dual space of (1) $(\mathbb{K}^n, \|\cdot\|_p)$ (2) l^p.

Hint. Let q be such that $1/p + 1/q = 1$. Define $F((y_n)) = \phi_y$, where $\phi_y((x_n)) = \sum_n x_n \bar{y}_n$. Check that $\|\phi_y\| \leq \|y\|_q$. We will see in the class that $\|\phi_y\| = \|y\|_q$ if $y \in l^q$. If $\phi \in (l^p)'$ then $\phi = \phi_y$ with $y = (\phi(e_1), \phi(e_2), \cdots,)$. If $p = 1$ then $\|y\|_\infty \leq \|\phi\|\|e_i\|_1 = \|\phi\|$. Otherwise, let $z_m = (y_1, \cdots, y_m, 0, \cdots,)$ and note that $\|\phi_{z_m}\| = \|z_m\|_q$. Since $\|\phi_{z_m}\| \leq \|\phi\|$, we obtain $\|y\|_q = \lim_{m\to\infty} \|z_m\|_q \leq \|\phi\|$.

Exercise 9.3 : Let $\{f_1, f_2, \cdots,\}$ be a countable dense subset of the unit sphere in X'. Let Q be a dense subset of $\mathbb{K}$. Verify the following:

1. For all $n \geq 1$, there is $x_n \in X$ such that $\|x_n\| = 1$ and $|f_n(x_n)| > 1/2$.
2. $\{\sum_{i=1}^m k_i x_i : k_i \in Q\}$ is a countable dense subset of $Y := \text{linspan}\{x_n\}$.
3. Y is dense in X if and only if the following is true: For every $f \in X'$ such that $f(y) = 0$ for all $y \in Y$ implies $f = 0$.
4. Let $f \in X'$ be such that $f(y) = 0$ for all $y \in Y$. Then $f = 0$ (**Hint.** Suppose $\|f\| = 1$ and find $n \geq 1$ such that $\|f_n - f\| < 1/2$. However, $|f_n(x_n)| = |f_n(x_n) - f(x_n)| \leq \|f_n - f\|$).

Conclude that X is separable whenever so is X'.

Remark 9.4 : The dual of l^∞ can not be l^1.

Exercise 9.5 : Let Y be a dense subspace of X. Define $F : Y' \to X'$ by $F(f) = g$, where g is the unique continuous extension of f to X (discussed in the class). Show that F is an isometric isomorphism.

Remark 9.6 : For $p < \infty$, the dual of $(c_{00}, \|\cdot\|_p)$ is l^q.

Let us calculate the dual of the Banach space c_0 of sequences converging to 0 (with sup norm).

Exercise 9.7 : Verify the following:

(1) For $y \in l^1$, define $f_y : c_0 \to \mathbb{K}$ by $f_y(x) = \sum_{j=1}^{\infty} x_j \bar{y_j}$. Then $f_y \in (c_0)'$ such that $\|f_y\| \leq \|y\|_1$.
(2) If $x_j = \bar{y_j}/y_j$ for $1 \leq j \leq n$ and 0 otherwise. Then $f_y(x) \to \|y\|_1$.
(3) $\|f_y\| = \|y\|_1$.
(4) Every $f \in (c_0)'$ is of the form f_y for some $y \in l^1$.

Remark 9.8 : The dual of $(c_{00}, \|\cdot\|_\infty)$ is l^1.

Exercise 9.9 : Show that the dual of $L^2[0,1]$ is $L^2[0,1]$ itself.

Hint. Riesz Representation Theorem for Hilbert spaces.

Let us try to understand the dual of L^∞.

Proposition 9.10. *(Dual of L^∞ is bigger than L^1) Let $g \in L^1[0,1]$. Then the linear functional $\phi_g : L^\infty[0,1] \to \mathbb{C}$ given by $\phi_g(f) := \int_{[0,1]} f(t)\overline{g(t)}dt$ is bounded. Furthermore, the mapping $F : L^1[0,1] \to (L^\infty)'$ given by $F(g) = \phi_g$ is injective but nor surjective.*

Proof. We prove only that F is not surjective. Consider the subspace $Y := C[0,1]$ of $X := L^\infty[0,1]$ and the bounded linear functional $\phi : Y \to \mathbb{C}$ given by $\phi(f) = f(0)$. By HBT, there exists $\psi : X \to \mathbb{C}$ such that $\psi(f) = \phi(f)$ $(f \in Y)$ and $\|\psi\| = \|\phi\|$. Suppose that there exists $g \in L^1[0,1]$ such that $\psi = \phi_g$. But then $\int_{[0,1]} f(t)g(t)dt = 0$ for every $f \in Y$ such that $f(0) = 0$. Let $f \in C[0,1]$. Given $\epsilon > 0$, let $f_\epsilon \in C[0,1]$ be a function such that $f_\epsilon(0) = 0$, $f_\epsilon = f$ on $[\epsilon, 1]$, and $\sup_{t \in [0,\epsilon]} |f(t) - f_\epsilon(t)| \leq 2\|f\|_\infty$. But then $|\int_{[0,1]} f(t)\overline{g(t)}dt| \leq 2\|f\|_\infty \int_{[0,\epsilon]} |g(t)|dt$. Since ϵ is arbitrary, by Dominated Convergence Theorem, we get $\int_{[0,1]} f(t)\overline{g(t)}dt = 0$ for every $f \in C[0,1]$. By similar argument, one can see that $\int_{[0,t]} \overline{g(t)}dt = 0$, and hence $\int_{[0,t]} g(t)dt = 0$. Differentiating both sides, we obtain $g(t) = 0$ almost eveywhere. But then ψ, and hence $\phi = 0$, which is absurd. $\square$

Let us see the dual of $L^1[0,1]$.

Exercise 9.11 : Let $g \in L^\infty[0,1]$. Define the linear functional $\phi_g : L^1[0,1] \to \mathbb{C}$ by $\phi_g(f) := \int_{[0,1]} f(t)g(t)dt$. For $\phi \in (L^p[0,1])'$, verify the following:

(1) $\phi_g \in (L^1[0,1])'$.

(2) For a measurable subset Δ of $[0,1]$, define $\mu(\Delta) = \phi(\chi_\Delta)$. Then μ is a countably additive measure with the property: If Δ is of Lebesgue measure 0 then $\mu(\Delta) = 0$ (**Hint.** $|\mu(\Delta)| = |\phi(\chi_\Delta)| \leq \|\phi\|\|\chi_\Delta\|_1 = \|\phi\|$(Lebesgue measure of Δ)).
(3) There exists a Lebesgue measurable function g such that $\mu(\Delta) = \int_\Delta g(t)dt$ for every Lebesgue measurable subset of $[0,1]$ (This follows from the Radon-Nikodym Theorem).
(4) $\|\phi_g\| = \|g\|_\infty$ an hence $g \in L^\infty[0,1]$ (**Hint.** For $\epsilon > 0$, let $A = \{x \in [0,1] : |g(x)| > \|\phi\| + \epsilon\}$ and let $f = \chi_A(\bar{g}/g)$. Calculate $\|f\|_1$ and examine $\phi_g(f)$).
(5) $\phi = \phi_g$ (**Hint** Check it for simple measurable functions and apply Dominated Convergence Theorem).

The *total variation* $V(g)$ of a function $g : [0,1] \to \mathbb{K}$ is defined as

$$\sup_P \sum_{i=1}^n |g(t_i) - g(t_{i-1})|$$

, where sup is taken over all partitions $P : \{t_0 < t_1 < \cdots < t_n\}$ of $[0,1]$. We say that g *is of bounded variation* if its total variation $V(g)$ is finite.

Recall that the vector space $B[0,1]$ of bounded linear functionals on $[0,1]$ is a normed linear space endowed with the sup norm.

Exercise 9.12 : Let $\phi \in (C[0,1])'$. Verify the following:

(1) If g is of bounded variation then $\phi_g(f) := \int_{[0,1]} f(t)dg(t)$ defines a bounded linear functional on $C[0,1]$. Moreover, $\|\phi_g\| \leq V(g)$.
(2) There exists a bounded linear functional $\psi : B[0,1] \to \mathbb{K}$ such that $\psi(f) = \phi(f)$ for all $f \in C[0,1]$ and $\|\psi\| = \|\phi\|$.
(3) Define $g : [0,1] \to \mathbb{K}$ by $g(0) = 0$ and $g(t) = \psi(\chi_{(0,t]})$. Then $V(g) \leq \|\psi\|$ (**Hint.** There exists a real θ such that $|g(t_i) - g(t_{i-1})| = e^{i\theta}(g(t_i) - g(t_{i-1})) = \psi(e^{i\theta}\chi_{(t_{i-1},t_i]})$).
(4) For $f \in C[0,1]$, consider $s_n := \sum_{r=1}^n f(r/n)\chi_{((r-1)/n,r/n]}$. Then s_n converges uniformly to f (**Hint.** Given $\epsilon > 0$, choose $n \geq 1$ such that $|f(s) - f(t)| < \epsilon$ for all x, y such that $|x - y| < 1/n$. Note that $s_n(t) - f(t) = \sum_{r=1}^n (f(r/n) - f(t))\chi_{((r-1)/n,r/n]}(t)$).
(5) $\phi = \phi_g$ for g of bounded variation (**Hint.** $\psi(s_n)$ converges to $\psi(f)$).

10. Weak Convergence and Eberlein's Theorem

As we have seen that the unit ball in an infinite-dimensional normed linear space can never be compact (in the norm topology). The question is whether the unit ball is compact in some topology "weaker" than norm topology. To answer this, we introduce a new convergence, which relies on the structure of the dual space.

We say that a sequence $\{x_n\}$ in X *converges weakly* to $x \in X$ if $x'(x_n) \to x'(x)$ for every $x' \in X'$.

Remark 10.1 : Every convergent sequence is weakly convergent.

Exercise 10.2 : Limit of a weakly convergent sequence is unique.

Hint. HBT.

Exercise 10.3 : Let $\{x_k\}$ be a sequence in $\mathbb{K}^n$. Show that every weakly convergent sequence is convergent.

Exercise 10.4 : Show that $\{e_n\}$ converges weakly to 0 in l^p for $1 < p < \infty$.

Exercise 10.5 : For $n \in \mathbb{Z}$, consider the function $E_n(t) = e^{int}$. Show that $\{E_n\}$ converges weakly to 0 in $L^2[0, 2\pi]$.

Exercise 10.6 : Show that any orthonormal sequence in a Hilbert space converges weakly to 0.

None of the sequences discussed in last three exercises converges in norm. It is natural to know whether there exists an infinite-dimensional normed linear space in which weak convergence is equivalent to normed convergence.

Theorem 10.7. *(Schur's Lemma) A sequence $\{x_n\}$ in l^1 is weakly convergent iff $\{x_n\}$ is convergent.*

Proof. Suppose there exists a weakly convergent sequence $\{x_n\}$, which is not convergent. Replacing x_n by $x_n - x$, we may assume that $\{x_n\}$ converges weakly to 0, and $\|x_n\| \nrightarrow 0$. Given $\epsilon > 0$, there exists a subsequence of $\{x_n\}$, denoted by $\{x_n\}$ itself, such that $\|x_n\|_1 \geq 5\epsilon$ for all $n \geq 1$. By the weak convergence of $\{x_n\}$, $x_n(j) \to 0$ as $n \to \infty$ $(j \geq 1)$.

Set $n_0 = 1 = m_0$. Let n_1 be the smallest integer greater than n_0 such that $\sum_{j=1}^{m_0} |x_{n_1}(j)| = |x_{n_1}(1)| < \epsilon$. Let m_1 be the smallest integer bigger than m_0 such that $\sum_{j=m_1+1}^{\infty} |x_{n_1}(j)| < \epsilon$. Inductively define $\{n_k\}$ and $\{m_k\}$ of natural numbers such that n_k is the smallest integer greater than n_{k-1} satisfying $\sum_{j=1}^{m_{k-1}} |x_{n_k}(j)| < \epsilon$, and m_k be the smallest integer greater than m_{k-1} such that $\sum_{j=m_k+1}^{\infty} |x_{n_k}(j)| < \epsilon$. Define $y \in l^\infty$ by $y(1) = 1$ and $y(j) = \overline{x_{n_k}(j)}/|x_{n_k}(j)|$ if $m_{k-1} < j \leq m_k$. Clearly, $f_y \in (l^1)'$. Since $\|y\|_\infty = 1$,

$$|f_y(x_{n_k}) - \|x_{n_k}\|_1| < 4\epsilon.$$

It follows that $|f_y(x_{n_k})| \geq \|x_{n_k}\|_1 - 4\epsilon \geq \epsilon$. However, since $\{x_n\}$ converges weakly to 0, so does $\{x_{n_k}\}$. This contradiction completes the proof. □

Exercise 10.8 : Show that $\{e_n\}$ is not weakly convergent in l^1.

For a normed linear space X, consider the dual X'' of the dual X' of X. Consider the mapping $J_X : X \to X''$ given by $J_X(x)(x') = x'(x)$. By Exercise 8.9, $\|J_X(x)\| = \sup_{\|x'\| \leq 1} |J_X(x)(x')| = \sup_{\|x'\| \leq 1} |x'(x)|| = \|x\|$. Thus J_X is a bounded linear isometry which embeds X into X''.

We will refer to J_X as the *canonical embedding* of X into X''.

Remark 10.9 : In general, J_X is not surjective. For example, if $X = l^1$ then $X' = l^\infty$, and $X'' = (l^\infty)' \neq l^1$. This shows that J_{l^1} is not surjective.

Example 10.10 : Consider a Hilbert space H. Recall that any $g \in H'$ is given by the inner-product $\langle \cdot, g\rangle$ for some $g \in H$. Thus the canonical embedding J_H is given by $J_H(f)(g) = \langle f, g\rangle$.

Exercise 10.11 : A weakly convergent sequence is bounded.

Hint. It suffices to check that $\{J_X x_n\}$ is bounded. Note that for every $x' \in X'$, $\{(J_X x_n)(x') = x'(x_n)\}$ is convergent, and hence bounded. Now apply UBP to $\{J_X x_n\}$.

Exercise 10.12 : Let $1 \leq p < \infty$. Show that the following are equivalent:

1. A sequence $\{x_n\}$ in weakly convergent to x in l^p.
2. $\{x_n\}$ is bounded and $x_n(j) \to x(j)$ for every integer $j \geq 1$.

Hint. To see that (2) implies (1), let $f \in (l^p)'$ be of the form f_y for $y \in l^q$. Given $\epsilon > 0$, there exists $N \geq 1$ such that $\sum_{j \geq N} |y(j)|^q \leq \epsilon^q$. Note that $|f_y(x_n) - f_y(x)| \leq (\sum_{j=1}^N |x_n(j) - x(j)|^p)^{1/p} \|y\|_q + \epsilon \sup \|x_n - x\|_p$. Now choose n sufficiently large, so that $(\sum_{j=1}^N |x_n(j) - x(j)|^p)^{1/p} < \epsilon$.

We say that a Banach space X is *reflexive* if J_X is surjective.

Example 10.13 : The spaces l^p and $L^p[0,1]$ are reflexive for $1 < p < \infty$. The spaces l^1 and $L^1[0,1]$ are not reflexive.

Exercise 10.14 : Suppose X is reflexive. Then X is separable if and only if X' is separable.

Hint. In view of Exercise 9.3, it suffices to check that if X is separable then so is X'. Note that X'' is separable if so is X. Now apply Exercise 9.3.

Exercise 10.15 : Show that $C[0,1]$ is not reflexive.

Hint. For $t \in [0,1]$, define $E_t : C[0,1] \to \mathbb{K}$ by $E_t(f) = f(t)$. Then $E_t \in (C[0,1])'$. Choose a continuous function f_0 with $\|f_0\|_\infty = 1$ such that for $s \neq t$, $f_0(s) = 1$ and $f_0(t) = 0$. It follows that that $\|E_t - E_s\| = \sup_{\|f\|=1} |f(t) - f(s)| \geq 1$. This shows that $(C[0,1])'$ is not separable.

Exercise 10.16 : Let X be a normed linear space and let Y be a subspace. Define $\phi : X' \to Y'$ by $\phi(x') = x'|_Y$. Show that ϕ is a surjective linear isometry.

Exercise 10.17 : Let X be a Banach space and let Y be a closed subspace of X. Let $y'' \in Y$. Verify the following:

1. For every $y' \in Y'$, there is $x' \in X'$ such that $x'|_Y = y'$.

(2) Define $x'' \in X''$ by $x''(x') := y''(x'|_Y)$. Let $x \in X$ be such that $J_X(x) = x''$. Then $x \in Y$ (**Hint.** HBT).
(3) $J_Y(x) = y''$. (**Hint.** $J_Y(x)(y') = y'(x) = x'|_Y(x) = x'(x) = J_X(x)(x') = x''(x') = y''(x'|_Y) = y''(y')$.).
(4) If X is reflexive then so is Y

The the main result of this section is the following result due to Eberlein. The proof will be presented in the next section.

Theorem 10.18. *If X is reflexive then every bounded sequence has weakly convergent subsequence.*

Corollary 10.19. *Let X be a reflexive space. If the norm convergence is equivalent to the weak convergence then X is finite-dimensional.*

Proof. Suppose weak convergence implies norm convergence. Let $\{x_n\}$ be a bounded sequence. By Eberlein's Theorem, $\{x_n\}$ has a weakly convergent subsequence, and hence by assumption, $\{x_n\}$ has a convergent subsequence. But then the unit sphere in X is (sequentially) compact. Hence, by Exercise 6.3, X must be finite-dimensional. □

11. Weak* Convergence and Banach's Theorem

A sequence $\{x'_n\}$ in X' is said to be *weak* convergent* to some $x' \in X'$ if for every $x \in X$, $x'_n(x) \to x'(x)$ as $n \to \infty$.

Remark 11.1 : If X is a Banach space then a weak* convergent sequence is bounded. This follows from UBP.

Example 11.2 : Let $\{x'_n\}$ be a sequence in a Hilbert space H. Thus there exists $y_n \in H$ such that $x'_n(x) = \langle x, y_n\rangle$. It follows that $\{x'_n\}$ is weak* convergent if and only there exists $y \in H$ such that $\langle x, y_n\rangle \to \langle x, y\rangle$.

Recall that the dual of $(c_0, \|\cdot\|_\infty)$ is isometrically isomorphic to l^1 : For every $f \in (c_0)'$, there exists unique $y \in l^1$ such that

$$f(x) = f_y(x) = \sum_{j=1}^{\infty} x(j)\bar{y(j)}.$$

Moreover, $\|f_y\| = \|y\|$.

Exercise 11.3 : Let $f_{y_n} \in (c_0)'$ for $y_n \in l^1$. Show that $\{f_{y_n}\}$ is weak* convergent to f_y iff $y_n(k) \to y(k)$ for all $k \geq 1$ and $\sup \|y_n\|_1 < \infty$.

Hint. For fixed $m \geq 1$, check that $|\sum_{j=1}^m x(j)\overline{y_n(j)}| \leq \|x\|_\infty \sup_n \|y_n\|_1$. Letting $n \to \infty$, we obtain

$$|\sum_{j=1}^{m} x(j)\overline{y(j)}| \leq \|x\|_\infty \sup_n \|y_n\|_1.$$

Now let $m \to \infty$ to get $\|f_y\| \leq \sup_n \|y_n\|_1 < \infty$. It follows that $y \in l^1$. Let $\epsilon > 0$ and $x \in c_0$. Choose $N \geq 1$ large enough so that $|x(j)| < \epsilon$ for $j \geq N$. Note that for large $n \geq 1$, $|y_n(j) - y(j)| < \epsilon/N$.

$$\begin{aligned} |f_{y_n}(x) - f_y(x)| &\leq \sum_{j=1}^{N} |x(j)||\overline{(y_n(j) - y(j))}| + \sum_{j=N}^{\infty} |x(j)||\overline{(y_n(j) - y(j))}| \\ &\leq \|x\|_\infty \epsilon + \epsilon(\|y_n\|_1 + \|y\|_1). \end{aligned}$$

The following is a special case of the Banach-Alagou Theorem.

Theorem 11.4. *If X is a separable then every bounded sequence in X' has a weak* convergent subsequence.*

Proof. Suppose that X has a countable dense subset $\{x_k\}$. Let $M := \sup_n \|x'_n\|$. Since $|x'_n(x_1)\| \leq M\|x_1\|$, $\{x'_n(x_1)\}$ is a bounded sequence in $\mathbb{K}$. By the Bolzano-Weierstrass Theorem, $\{x'_n(x_1)\}$ has a convergent subsequence $\{x'_{n1}(x_1)\}$. Since $|x'_{n1}(x_2)\| \leq M\|x_2\|$, $\{x'_{n1}(x_2)\}$ is a bounded sequence in $\mathbb{K}$. Again by the Bolzano-Weierstrass Theorem, $\{x'_{n1}(x_2)\}$ has a convergent subsequence $\{x'_{n2}(x_2)\}$. Inductively, for $k \geq 1$, $\{x'_{nk-1}(x_k)\}$ has a convergent subsequence $\{x'_{nk}(x_k)\}$.

For fixed $k \geq 1$, consider the sequence $\{x'_{nn}(x_k)\}$. Note that $x'_{nn}(x_k)$ belongs to the convergent sequence $\{x'_{nk}(x_k)\}$ for $n \geq k$. Hence $\{x'_{nn}(x_k)\}$ is also convergent. To complete the proof, let $x \in X$. Given $\epsilon > 0$, choose k large enough so that $\|x - x_k\| < \epsilon$. For $m, n \geq 1$, note that

$$\begin{aligned} |x'_{mm}(x) - x'_{nn}(x)| &\leq |x'_{mm}(x) - x'_{mm}(x_k)| + |x'_{mm}(x_k) - x'_{nn}(x_k)| \\ &+ |x'_{nn}(x_k) - x'_{nn}(x)| \\ &\leq 2M\epsilon + |x'_{mm}(x_k) - x'_{nn}(x_k)|. \end{aligned}$$

Thus $\{x'_{nn}(x)\}$ is a Cauchy sequence. It follows that x'_{nn} converges to a linear functional x'. Finally, note that $\|x'\| \leq M$. □

Remark 11.5 : If X is separable then the closed unit ball in X' is weak* sequentially compact.

Now we are in a position to prove Theorem 10.18.

Proof of Eberlein's Theorem. Suppose that X is reflexive, and let $\{x_n\}$ be a bounded sequence in X. Note that $Y = \overline{\{x_n\}}$ is closed and separable subspace of X. By Exercise 10.17, Y is reflexive, and hence Y' is separable. By 11.4, the bounded sequence $\{J_Y(x_n)\}$ in Y'' has a weak* convergent subsequence $\{J_Y(x_{n_k})\}$. Now if $x' \in X'$ then $x'(x_{n_k}) = x'|_Y(x_{n_k}) = J_Y(x_{n_k})(x'|_Y)$ converges as desired. □

12. Spectral Theorem for Compact Operators

Let X, Y be Banach spaces. A linear operator T from X into Y is said to be *compact* if for every bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has a convergent subsequence.

Remark 12.1 : If S is compact and T is bounded then ST, TS are compact.

Exercise 12.2 : Show that every finite-rank linear mapping is compact.

Proposition 12.3. *Compact operators form a closed subspace of $B(X, Y)$.*

Proof. Given $\epsilon > 0$, find $N \geq 1$ such that $\|T - T_N\| < \epsilon$. Now find a convergent sequence $\{T_N x_{n_k}\}$ using compactness of T_N. Check that $\{Tx_{n_k}\}$ is a Cauchy sequence. □

Exercise 12.4 : Let $\{a_n\}$ be a bounded sequence. Show that the diagonal operator on l^2 with diagonal entries $a_1, a_2, \cdots$ is compact iff $\{a_n\} \in c_0$.

Hint. If T is a diagonal operator with diagonal entries $b_1, b_2, \cdots$, then $\|T\| = \sup_n |b_n|$. The sufficient part follows from this. WLOG, assume that $\{a_n\}$ is bounded from below. To see the necessary part, note that $\|De_n - De_m\|^2 = a_n^2 + a_m^2 > c > 0$ if $\{a_n\} \notin c_0$.

Exercise 12.5 : Consider $(Tf)(x) = \int_0^x f(y)dy$ $(x \in [0, 1])$ and

$$(T_n f)(x) = \sum_{k=0}^{n-1} \chi_{[k/n,(k+1)/n)}(x) \int_0^{k/n} f(y)dy \ (x \in [0, 1]).$$

Verify the following:

(1) $T \in B(L^2(0, 1), L^2(0, 1))$.
(2) $T_n \in B(L^2(0, 1), L^2(0, 1))$ has n-dimensional range.
(3) $\|T - T_n\| \leq n^{-1/2}$.
(4) T is compact.

Exercise 12.6 : If T is compact then show that the closure of $\{Tx : \|x\| \leq 1\}$ is sequentially compact.

Exercise 12.7 : Consider the linear $T : L^2(0, \infty) \to L^2(0, \infty)$ given by

$$(Tf)(x) = \frac{1}{x} \int_0^x f(t)dt \ (x \in (0, \infty)).$$

Show that T is not compact.

Hint. Consider $f_n(t) = n$ if $0 < t \leq 1/n^2$, and 0 otherwise. Note that $\|Tf_n\| \geq 1$ for all n. However, $\langle Tf_n, g\rangle \to 0$ for all $g \in L^2(0, \infty)$.

Theorem 12.8. *Show that if T is compact and the range $R(T)$ is closed then the closed unit ball in $R(T)$ is sequentially compact. In particular, $R(T)$ is finite-dimensional in this case.*

Proof. Use Open Mapping Theorem. □

Exercise 12.9 : Let μ be a non-zero complex number and $T : X \to Y$ be compact. Then $T|_{\ker(T-\mu)}$ is compact with closed range $\ker(T-\mu)$.

We say that T has an *eigenvalue* μ if $\ker(T-\mu)$ is non-zero.

Exercise 12.10 : Show that the eigenspace $\ker(T-\mu)$ of a compact operator corresponding to non-zero eigenvalue is finite-dimensional.

A bounded linear operator on a Hilbert space H is *normal* if $T^*T = TT^*$. We say that T is *self-adjoint* if $T^* = T$.

The operator of multiplication by $\phi \in L^\infty$ on L^2 is normal. In fact, $M^*_\phi = M^*_{\bar{\phi}}$, where $\bar{\phi}(z) = \overline{\phi(z)}$. Note that M_ϕ is self-adjoint iff ϕ is real-valued.

Exercise 12.11 : If N is normal then so is $N - \lambda I$ for any scalar λ. Use this to deduce that the eigenspaces corresponding to distinct eigenvalues are orthogonal.

Exercise 12.12 : Consider the linear operator T on $L^2[0,1]$:

$$(Tf)(x) = (1-x)\int_0^x yf(y)dy + x\int_x^1 (1-y)f(y)dy \ (x \in [0,1]).$$

Verify the following:

(1) $T \in B(L^2(0,1), L^2(0,1))$ is compact and self-adjoint (**Hint.** Use Exercise 12.5 and Remark 12.1).
(2) If $Tf = \lambda f$ then for some integer $n \geq 1$, $f(x) = c\sin(n\pi x)$ for some scalar c and $\lambda = 1/n^2\pi^2$ (**Hint.** Calculate second derivative of Tf).

Theorem 12.13. *Let T be a normal operator on an infinite-dimensional Hilbert space. If T is compact then there exists an orthonormal basis $\{e_n\}$ of $\ker(T)^\perp$ and a sequence $\{\lambda_n\}$ of complex numbers (possibly repeated) such that $Te_n = \lambda_n e_n$. Moreover, the following hold:*

(1) *For each $n \geq 1$, $\ker(T-\lambda_n)$ is finite-dimensional.*
(2) *$\lambda_n \to 0$ as $n \to \infty$.*
(3) *$\{\lambda_n\}$ has no accumulation point except 0.*
(4) *For any $x \in H$, $Tx = \sum_{n=1}^\infty \lambda_n \langle x, e_n\rangle e_n$.*

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