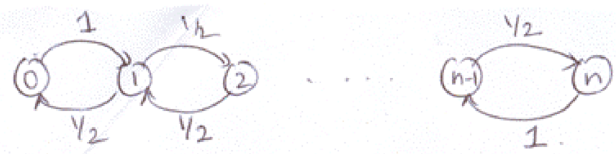
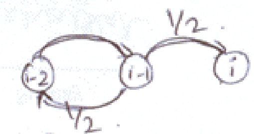
1)



Let T_i denote the # of steps to go from vertex $i-1$ to i.
Since there is only one step to go from 0 to 1 w.p. 1
we have

$$E[T_1] = 1.$$



for $i > 1$,

$$E[T_i] = \frac{1}{2} \cdot 1 + \frac{1}{2} E[1 + T_{i-1} + T_i] \quad \ldots (1)$$

The reasoning is as follows. We have 1/2 a prob of going to i when in state "$i-1$", we have 1/2 a prob of going to i (the number of steps is 1). This accounts for the first part. We have 1/2 a prob of going to "$i-2$". The number of steps then will be 1 (from $i-1$ to $i-2$) + T_{i-1} (steps from $i-2$ to $i-1$] + T_i (steps from $i-1$ to i). to return ~~back~~ to i from $i-1$.

Expanding Eq. 1, we have.

$$E[T_i] = 2 + E[T_{i-1}] \Rightarrow E[T_i] - E[T_{i-1}] = 2.$$

Using recursion.

$$\left.\begin{aligned} E[T_2] - E[T_1] &= 2 \\ E[T_3] - E[T_2] &= 2 \\ &\vdots \\ E[T_i] - E[T_{i-1}] &= 2. \end{aligned}\right\} i-1 \qquad \begin{aligned} E[T_i] - E[T_1] &= 2(i-1) \\ \Rightarrow E[T_i] &= E[T_1] + 2(i-1) = \underline{\underline{2i-1}}. \end{aligned}$$

Summing them yields.

We were asked to determine expected # of steps it takes to reach n starting from 0. This is nothing

$E\left[\sum_{i=1}^{n} T_i\right]$ since the only way to reach n involves going from 0 to 1 to 2 & so on to n

$\therefore E$ [Number of steps to go from 0 to n]

$$= \sum_{i=1}^{n} E[T_i] = \sum_{i=1}^{n} (2i-1) = \frac{\cancel{2}n(n+1)}{\cancel{2}} - n$$

$$= \underline{\underline{n^2}}.$$

2) a) Since the inter-arrival time of the trains is exponential, regardless of when you arrive at the station, the arrival of the next train will be exponentially distributed with parameter "a".

Let T be the time of arrival at home.

$$E[T] = (r+W)\,P_r\{\cancel{\text{train}} \text{ didn't arrive within } r\}$$
$$+ (T+u)\,P_r\{\text{train arrived within } r \text{ say at } u,\ u \leq r\}.$$

$$= e^{-ar}(r+W) + \int_0^r a e^{-au}\, du\, (T+u) \qquad \left[\, a e^{-au} \rightarrow \text{pdf of exponential r.v} \right.$$

Solving this yields. (second part is solved by integrating by parts)

$$= \cancel{re^{-ar}} + We^{-ar} + T - Te^{-ar} - \cancel{re^{-ar}} - \frac{e^{-ar}}{a} + \frac{1}{a}.$$

$$= e^{-ar}\left(W - T - \frac{1}{a}\right) + T + \frac{1}{a}$$

b). If $W > T + \frac{1}{a}$, $E[T]$ is decreasing wrt r

Therefore it is minimized at $r = \infty$.

If $W < T + 1/a$, $E[T]$ is increasing wrt r

Therefore it is minimized at $r = -\infty$

However since $r \geq 0$, we set $r = 0$.

If $W = T + 1/a$, $E[T] = T + 1/a$, result is independent of r.

c) This is because of the memoryless property of exponential distribution. Regardless of the previous arrivals, whenever you arrive, the expected arrival of the next train is $1/a$. Suppose you waited for t & the train did not arrive. Still, the expected time of next arrival is $1/a$. Waiting does not decrease your expected travel time.

Therefore, whenever you arrive at the station, you evaluate the two expected times for immediately start walking and waiting for bus & decide accordingly.

3) a) Let the states be 0,1,2 where 0 denotes 'Tom & Jerry in the same location. 1 denotes 'Tom at location a & Jerry at location b'. 2 denotes 'Tom at location b & Jerry at location a'.

The way the states are defined, knowing the current states is sufficient for predicting the next. Therefore it is a Markov Chain.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.54 & 0.28 & 0.18 \\ 0.54 & 0.18 & 0.28 \end{bmatrix}$$

Since $P_{10} = 0.7 \times 0.6 + 0.3 \times 0.4$ $\quad P_{20} = 0.7 \times 0.6 + 0.3 \times 0.4$

$P_{11} = 0.7 \times 0.4$ $\quad P_{21} = 0.3 \times 0.6$

$P_{12} = 0.3 \times 0.6$ $\quad P_{22} = 0.7 \times 0.4$

b) $P_{11}^{n} = P_r\{$Tom & Jerry are at the initial location after n steps$\}$.

$$= P_r\{X_n = 1 \mid X_0 = 1,\ X_i \neq 0\ \forall i \in (1, \dots n)\} \cdot P_r\{X_i \neq 0,\ \forall i \in (1, \dots n) \mid X_0 = 1\}.$$

$$= P_r\{X_n = 1 \mid X_0 = 1,\ X_i \neq 0\ \forall i \in 1, \dots n\}\ (0.46)^{n}.$$

[0.46 is the Prob of not making transition to state 0].

let $Q_{ij}^{n} := P_r\{X_n = j \mid X_0 = i,\ X_i \neq 0 \quad \forall i = 1, \dots n\}$

$(i, j \in \{1, 2\})$

The Q matrix is given by (obtained by normalizing probabilities)

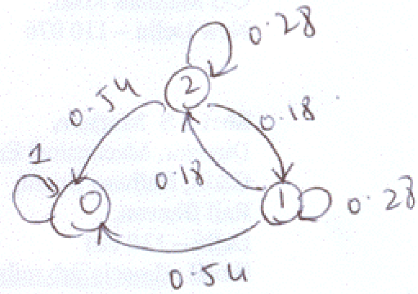
$$Q = \begin{bmatrix} \frac{0.28}{0.46} & \frac{0.18}{0.46} \\ \frac{0.18}{0.46} & \frac{0.28}{0.46} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

The Q matrix is of the form $\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

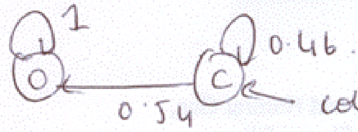
$$\therefore Q_{11}^{n} = \frac{1}{2} + \frac{1}{2}(2p-1)^{n}$$

$$\therefore P_{11}^{n} = (0.46)^{n}\left[\frac{1}{2} + \frac{1}{2}\left(\frac{28}{23} - 1\right)^{n}\right]$$

$$=$$



c) Since $P_{10} = P_{20}$, states 1 & 2 can be collapsed to a single state giving a Markov chain as shown



collapsed chain.

When starting in state C, the probability of ~~chase~~ ending is 0.54

The trials can be viewed as a geometric random variable where prob of success is 0.54.

Mean value of the geometric r.v is $1/0.54$

$\therefore$ duration of chase = $\underline{\underline{1.85}}$