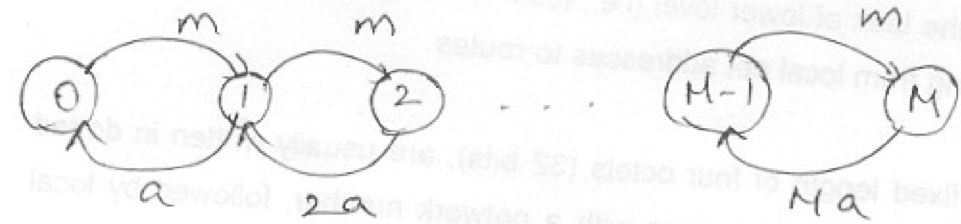
1) $X(t)$ — # of machines up at time t

Markov Chain is as under:



$$A = \begin{bmatrix} -m & m & 0 & \cdots & \cdots \\ a & -(a+m) & m & \cdots & \\ 0 & 2a & -(2a+m) & m & \cdots \\ \vdots & \vdots & & & \\ \vdots & \vdots & & & \end{bmatrix}$$

2) a) Avg. message transmission time $\frac{1}{\mu} = \frac{L}{C}$

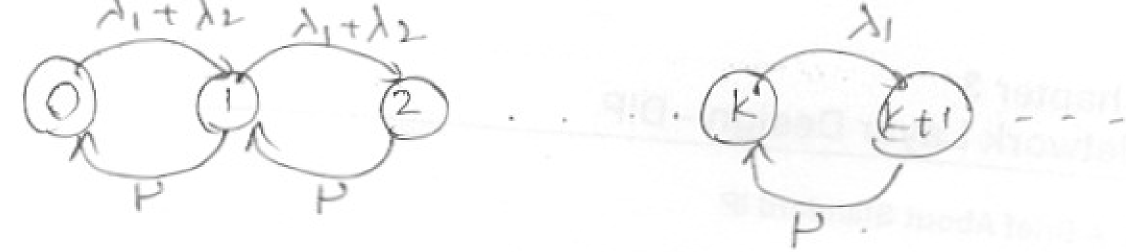
When # of pkts in the system is larger than K, arrival rate is λ_1

For the # of Packets in the system to stay bounded, we need

$$0 \le \lambda_1 < \mu$$
$$0 \le \lambda_2$$

Note that there is no restriction on λ_2, since they are naturally dropped when system size is larger than K.

b) The Markov chain for the process is as below



$$P_n = \rho^n \cdot P_0 \quad (n \le K) \qquad \rho = \frac{\lambda_1+\lambda_2}{\mu}$$

$$= \rho_1^{\,n-K} \cdot \rho^K \cdot P_0 \quad (n > K) \qquad \rho_1 = \frac{\lambda_1}{\mu}$$

$\sum_{n=0}^{\infty} P_n = 1$ gives $P_0 = \frac{(1-\rho)(1-\rho_1)}{1-\rho_1-\rho^K(\rho-\rho_1)}$ $(\rho<1$

$$P_0 = \frac{1-\rho_1}{1+K(1-\rho_1)} \qquad (\rho = 1)$$

c) For a packet from node 1, the time it spends in the system is given by

$$T_1 = \frac{L}{\mu} + \frac{1}{\mu}$$

where $L = \sum_{n=0}^{\infty} n P_n$ (average number in system upon arrival)

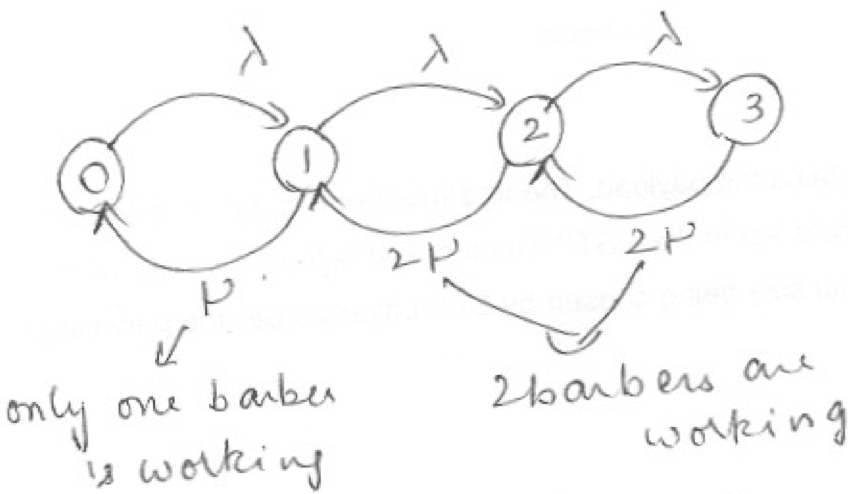
d) For a packet from node 2, the time spent in system is given by

$$T_2 = \frac{L'}{\mu} + \frac{1}{\mu} \qquad \text{where } L' = \frac{\sum_{n=0}^{K-1} n P_n}{\sum_{n=0}^{K-1} P_n}$$

Above prob. have been normalized by the prob. that a packet from node 2 finds acceptance to the system

3.

a)



$\lambda = 8, \ \mu = 4 \quad \therefore \ \lambda/\mu = 2$

$$\pi_n = \frac{\lambda^n}{\mu_n \cdots \mu_1} \cdot \pi_0$$

$$\pi_1 = \frac{\lambda}{\mu} \cdot \pi_0, \quad \pi_2 = \frac{\lambda^2}{2\mu^2} \pi_0, \quad \pi_3 = \frac{\lambda^3}{4\mu^3} \cdot \pi_0$$

$$\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$$

$$\Rightarrow \pi = [\,1/7, 2/7, 2/7, 2/7\,]$$

b) $\pi_0 = 1/7$

c) $1 \cdot 2/7 + 2 \cdot 2/7 + 3 \cdot 2/7 = 12/7$

d) $1 - \pi_3 = 5/7.$

4.

a) $\lambda_n = b_n \cdot \lambda$ where $b_n = e^{-\alpha n/\mu}$

$\mu_n = \mu.$

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}$$

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n \prod_{i=1}^{n} b_{i-1}$$

$$= P_0 \cdot \left(\frac{\lambda}{\mu}\right)^n \prod_{i=1}^{n} e^{-\alpha \frac{(i-1)}{\mu}}$$

$$= P_0 \left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\alpha (n-1) n}{2\mu}}$$

$$P_0 = \frac{1}{\sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n e^{-\frac{\alpha (n-1) n}{2\mu}}}$$

b) If $\alpha > 0$, steady state always exists even when $\lambda > \mu$. This is because as n grows, $e^{-\frac{\alpha (n-1) n}{2\mu}}$ will decrease faster than $\frac{\lambda}{\mu}$ & goes to zero.

If $\alpha = 0$, then $\lambda < \mu$ (regular M/M/1)

c) If $\alpha \to \infty$.

$$P_1 = P_0 \cdot \frac{\lambda}{\mu}$$

$P_n \to 0$ as $\alpha \to \infty$ for $n \geq 2$

$\therefore\ P_0 + P_1 = 1 \Rightarrow P_0 + \frac{\lambda}{\mu} \cdot P_0 = 1$

$\Rightarrow P_0 = \frac{\mu}{\lambda + \mu}, \quad P_1 = \frac{\lambda}{\lambda + \mu}.$

5)

Let $P_m = Pr\{$ the 1^{st} m servers are busy $\}$

This can be obtained from the M/M/m/m model

$$= \frac{r^m/m!}{\sum_{i=0}^{m} r^i/i!}, \quad r = \frac{\lambda}{\mu}.$$

Let r_m – Arrival rate to server (m+1) & above

λ_m – Arrival rate to server m

$$r_m = \lambda . P_m$$

$$\lambda_m = r_{m-1} - r_m = (P_{m-1} - P_m)\lambda$$

fraction of time server m is busy

$$\rho_m = \frac{\lambda_m}{\mu}$$