- Arrival Process: Poisson with parameter $\lambda = 0.04/\text{min}$.
- Service Time $= 10\text{min}$ (deterministic) $\bar{x} = 10\text{min}$
- # of servers $=1$. $\sigma_x = 0$; $\overline{x^2} = 100\text{min}^2$
- System Capacity $= 4$.

Let Service Time $= T = 10\text{min}$.

$\Rightarrow$ System $= M/D/1/K$ $\quad \underline{\underline{K=4.}}$

1. $X(t)$: # of customers in the system at time t (Not Markov)

X_i : # of customers left in the system when i^{th} customer departs.

To Prove: X_i is a Markovian Chain.

A_{n+1} = # of customers that arrive during the $(n+1)^{th}$ customer service time.

$$X_{n+1} = \begin{cases} \cancel{\max}\ \min\{X_n - 1 + A_{n+1}, k-1\} & X_n \geq 1 \\ \cancel{\max}\ \min\{A_{n+1}, k-1\} & X_n = 0 \end{cases} \quad \text{— (1)}$$

$\longrightarrow$ By this definition its ensured that $X_n \in \{0, \dots k-1\}$ & $X(t) \in \{0, \dots k\}$

A_{n+1} is independent of n $\because$ service time is fixed & arrival process is Poisson (memoryless).

$\Rightarrow A_{n+1} = A.$

$$\Pr\{A = a\} = \frac{e^{-\lambda T}(\lambda T)^a}{a!} \quad \text{— (2)}$$

$T = 10\text{min}$
$\lambda = 0.04/\text{min}$
$a = 0, 1, 2 \dots$

$$\Rightarrow \quad X_{n+1} = \begin{cases} \cancel{\max}\ \min\{X_n - 1 + A, k-1\} & X_n \geq 1 \\ \min\{A, k-1\} & X_n = 0 \end{cases} \quad \text{— (3)}$$

For the given system K is constant & for the Random stochastic Process X_n, X_{n+1} depends only on its previous state X_n (apart from A and $\cancel{X_{n-1} \ldots X_0}$) $\xrightarrow{K}$ independent of states.

Hence X_n is a Markov Chain. Ans.

State Space of the Markov Chain is $\{0, 1, \ldots K-1\}$.

$\Rightarrow \{0, 1, 2, 3\}$ in our specific case. Ans.

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2. As derived in 1)

$$X_{n+1} = \begin{cases} \min\{X_n - 1 + A, K-1\} & X_n \geq 1 \\ \min\{A, K-1\} & X_n = 0 \end{cases}$$

$$Pr\{A = a\} = \frac{e^{-\lambda T}(\lambda T)^a}{a!} \qquad a = 0, 1, 2 \ldots$$

$$P = \{P_{ij}\} \Rightarrow P_{ij} = Pr\{X_{n+1} = j / X_n = i\}. \quad \boxed{i, j \in \{0, \ldots K-1\}}$$

Case 1: $i=0$.

$$Pr\{X_{n+1}=j \mid X_n=0\} = Pr\{\min\{A, k-1\}=j\}$$

$$= \begin{cases} Pr\{A=j\} & j=0,1,\dots k-2 \\ Pr\{A=k-1\} + Pr\{A=k\} + Pr\{A=k+1\} + \dots & j=k-1 \end{cases}$$

$$\Rightarrow P_{0j} = \begin{cases} Pr\{A=j\} & j=0,1\dots k-2 \\ \sum_{a=k-1}^{\infty} Pr\{A=a\} & j=k-1 \end{cases} \quad \text{--- (9)}$$

Case 2: $i \in \{1,2\dots k-1\}$

$$P_{ij} = Pr\{\min\{i-1+A, k-1\}=j\}$$

$j \neq 0$.

$$= Pr\{\min\{A, k-i\} = \cancel{j+i-1}\ j+1-i\}$$

$$\Rightarrow \underset{(j\neq 0)}{P_{ij}} = \begin{cases} \Pr\{A = j+1-i\} & j+i+1 = 0 \ldots k-i-1 \\ & \Leftrightarrow j = \{i-1, i, \ldots k-2\} \\ \sum_{a=k-i}^{\infty} \Pr\{A = a\} & j \cancel{+i} +1 = k - \cancel{i} \\ & \Leftrightarrow j = k-1. \end{cases}$$

Simplifying above expression:-

$$\underset{\substack{j\neq 0 \\ j \in \{1, \ldots k-1\}}}{P_{ij}} = \begin{cases} \Pr\{A = j+1-i\} & j \in \{i-1, i \ldots k-2\} \\ \sum_{a=k-i}^{\infty} \Pr\{A = a\} & j = k-1 \end{cases} \quad \text{—— (5)}$$

Hence The Generalized One-step probability Transition Matrix is given by $P = \{P_{ij}\}$ where P_{ij}'s are given by (4) & (5).

$$\begin{bmatrix} P_{10} & P_{11} & & \\ 0 & P_{21} & \cdots & P_{2\,k-1} \\ \vdots & \cdots & 0 & P_{k-2\,k-1} \; P_{k-1\,k-1} \end{bmatrix} \quad \text{--- (6)}$$

Ans to 3).

For the specific case of $\left.\begin{array}{l} \lambda = 0.04/\text{min} \\ \& \; T = 10 \text{min}. \\ \& \; k = 4. \end{array}\right\} \lambda T = 0.4$

$P_{00} = Pr\{A=0\} = e^{-0.4}\, 0.4^0/1 = e^{-0.4}$

$P_{01} = Pr\{A=1\} = e^{-0.4} \dfrac{0.4^1}{1!} = e^{-0.4} \times 0.4$

$P_{02} = Pr\{A=2\} = e^{-0.4} \dfrac{0.4^2}{2!} = e^{-0.4} \times 0.08$

$P_{03} = 1 - P_{00} - P_{01} - P_{02} = 1 - e^{-0.4}\left(1 + \dfrac{0.4}{1!} + \dfrac{0.4^2}{2!}\right)$

$P_{10} = Pr\{A=0\} = e^{-0.4}\, 0.4/0 = e^{-0.4}$

$P_{11} = Pr\{A=1\} = e^{-0.4} \dfrac{0.4^1}{1!} = e^{-0.4} \times 0.4$

$P_{12} = Pr\{A=2\} = e^{-0.4} \times .4^2/2! = e^{-0.4} \times 0.08$

$P_{13} = \cancel{Pr\{A=} \; 1 - P_{10} - P_{11} - P_{12} = 1 - e^{-0.4}(1 + 0.4 + 0.08)$

$P_{20} = 0$

$P_{21} = Pr\{A=0\} = e^{-0.4}$

$P_{22} = Pr\{A=1\} = e^{-0.4} \times 0.4$.

$P_{23} = \cancel{Pr\{} \; 1 - P_{20} - P_{21} - P_{22} = 1 - e^{-0.4}(1+0.4)$.

$P_{30} = 0$

$P_{31} = 0$

$P_{32} = Pr\{A=0\} = e^{-0.4}$

$P_{33} = 1 - e^{-0.4}$

$$\Rightarrow P = \begin{bmatrix} e^{-0.4} & 0.4e^{-0.4} & 0.08e^{-0.4} & 1 - 1.48e^{-0.4} \\ e^{-0.4} & 0.4e^{-0.4} & 0.08e^{-0.4} & 1 - 1.48e^{-0.4} \\ 0 & e^{-0.4} & 0.4e^{-0.4} & 1 - 1.4e^{-0.4} \\ 0 & 0 & e^{-0.4} & 1 - e^{-0.4} \end{bmatrix}$$

An

3) Ans to 3) has been derived as intermediate step in 2).

Clearly there is ~~an~~ pattern in the matrix s.t. (6) can be further simplified.