A

Answers, Solution Outlines and Comments to Exercises

Chapter 1

Preliminary Test (page 3)

1. $\sqrt{7}$. $[c^2 = a^2 + b^2 - 2ab\cos C.]$ (5 marks)

2. $\frac{x}{-4/3} + \frac{y}{16} = 1$. [Verify that the point is on the curve. Find slope $\frac{dy}{dx} = 12$ (at that point) and the tangent $y + 8 = 12(x + 2)$. (5 marks)
Rearrange the equation to get it in intercept form, or solve $y = 0$ for x-intercept and $x = 0$ for y-intercept.] (5 marks)

3. One. [Show that $g'(t) < 0\ \forall t \in R$, hence g is strictly decreasing. Show existence. Show uniqueness.] (5 marks)
Comment: Graphical solution is also acceptable, but arguments must be rigorous.

4. $4\pi(9 - 2\sqrt{6})$ or 16.4π or 51.54 cm^3. $[V = \int_0^{2\pi}\int_0^{R_h} 2\sqrt{R^2 - r^2}\ r dr d\theta$, $R = 3$, $R_h = \sqrt{3}$. Sketch domain. (5 marks)
$V = 4\pi\int_0^{R_h}\sqrt{R^2 - r^2}\ r dr$, substitute $R^2 - r^2 = t^2$. (5 marks)
Evaluate integral $V = -4\pi\int_R^{\sqrt{R^2 - R_h^2}} t^2 dt$.] (5 marks)

5. (a) (2,1,8). [Consider $\vec{p} - \vec{q} = \vec{s} - \vec{r}$.] (5 marks)
(b) $\sqrt{3/5}$. $[\cos Q = \frac{\vec{QP}\cdot\vec{QR}}{\|\vec{QP}\|\ \|\vec{QR}\|}$.] (5 marks)
(c) $\frac{9}{5}(\mathbf{j} + 2\mathbf{k})$. [Vector projection $= (\vec{QP}\cdot\hat{QR})\hat{QR}$.] (5 marks)
(d) $6\sqrt{6}$ square units. [Vector area $= \vec{QP} \times \vec{QR}$.] (5 marks)
(e) $7x + 2y - z = 8$.
[Take normal $\mathbf{n}$ in the direction of vector area and $\mathbf{n}\cdot(\mathbf{x} - \mathbf{q})$.] (5 marks)
Comment: Parametric equation $\mathbf{q} + \alpha(\mathbf{p} - \mathbf{q}) + \beta(\mathbf{r} - \mathbf{q})$ is also acceptable.
(f) 14, 4, 2 on yz, xz and xy planes. [Take components of vector area.] (5 marks)

6. 1.625%. $[A = \pi ab \Rightarrow \frac{dA}{A} = \frac{da}{a} + \frac{db}{b}$. (5 marks)
Put $a = 10$, $b = 16$ and $da = db = 0.1$.] (5 marks)

7. 9/2. [Sketch domain: region inside the ellipse $x^2 + 4y^2 = 9$ and above the x-axis. (5 marks)
By change of order, $I = \int_{-3}^{3}\int_0^{\frac{1}{2}\sqrt{9-x^2}} y dy dx$. (5 marks)

Then, evaluate $I = \int_{-3}^{3} \frac{9-x^2}{8} dx$.] (5 marks)

8. Connect $(\infty, 1)$, $(0, 1)$, $(-1/2, 0)$, $(0, -1)$, $(\infty, -1)$ by straight segments. [Split: for $y \geq 0$, $y = 1 + x - |x|$ and for $y < 0$, $y = -1 - x + |x|$. (5 marks)
Next, split in x to describe these two functions.] (5 marks)
Comment: y is undefined for $x < -\frac{1}{2}$.

Chapter 2 (page 13)

1. **Comment:** They show three different 'cases'. In one case, both products are of the same size, but the elements are different. In the second, the products are of different sizes. In the third, one of the products is not even defined.

2. $a = 2$, $b = 1$, $c = 2$, $d = 1$, $e = 0$, $f = i$. [Equate elements of both sides in an appropriate sequence to have one equation in one unknown at every step.]
Comment: In the case of a successful split with all *real* entries, we conclude that the matrix is positive definite. Cholesky decomposition of positive definite matrices has great utility in applied problems. In later chapters, these notions will be further elaborated. In the present instance, the given matrix is *not* positive definite.

3. (a) Domain: R^2; Co-domain: R^3. (b) Range: linear combination of the two mapped vectors. Null space: $\{\mathbf{0}\}$. [In this case, an inspection suffices; because two linearly independent vectors are all that we are looking for.] (c) $\begin{bmatrix} -7/2 & 3 \\ 9 & -6 \\ -5/2 & 2 \end{bmatrix}$. [Write as $\mathbf{A}[\mathbf{x}_1 \quad \mathbf{x}_2] = [\mathbf{y}_1 \quad \mathbf{y}_2]$ and determine $\mathbf{A}$.]
(d) $[-4 \;\; 12 \;\; -3]^T$. (e) The matrix representation changes to $\mathbf{AP}$, where the 2×2 matrix $\mathbf{P}$ represents the rotation (see Chaps. 3 and 10).

Chapter 3 (page 19)

1. (a) The null space is non-trivial, certainly containing $(\mathbf{x}_1 - \mathbf{x}_0)$ and its scalar multiples.
(b) The set of pre-images is an infinite set, certainly containing (possibly as a subset) all vectors of the form $\mathbf{x}_0 + \alpha(\mathbf{x}_1 - \mathbf{x}_0)$.

2. [Let $Rank(\mathbf{A}) = r$. Identify r linearly independent columns of $\mathbf{A}$ in basis $\mathcal{B}$. Express other $(n-r)$ dependent columns (together referred to as $\mathbf{D}$) in this basis, note the $r \times (n-r)$ coefficients c_{jk} and establish $[\mathbf{D} \quad \mathcal{B}] \begin{bmatrix} -\mathbf{I}_{n-r} \\ \mathbf{C} \end{bmatrix} = \mathbf{0}$. The left operand here contains columns of $\mathbf{A}$, hence the columns of the right operand gives $(n-r)$ linearly independent members of $Null(\mathbf{A})$. (Coordinates possibly need to be permuted.) So, nullity is at least $n-r$. In the first $n-r$ coordinates, all possibilities are exhausted through the span. So, allowing for additional possibility in the lower coordinates, propose $\mathbf{x} = \begin{bmatrix} -\mathbf{y} \\ \mathbf{Cy} + \mathbf{p} \end{bmatrix}$ and show that $[\mathbf{D} \quad \mathcal{B}]\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{p} = \mathbf{0}$. This establishes the nullity to be $n-r$.]
Comment: Alternative proofs are also possible.

3. (a) No. (b) Yes.

4. (a) $\mathbf{v}_1 = [0.82\ \ 0\ \ -0.41\ \ 0.41]^T$. (b) $\mathbf{v}_2 = [-0.21\ \ 0.64\ \ 0.27\ \ 0.69]^T$.
(c) $\mathbf{v}_3 = [0.20\ \ -0.59\ \ 0.72\ \ 0.33]^T$. (d) $\mathbf{v}_4 = [-0.50\ \ -0.50\ \ -0.50\ \ 0.50]^T$.
(e) Setting $\mathbf{v}_1 = \mathbf{u}_1/\|\mathbf{u}_1\|$ and $l = 1$; for $k = 2, 3, \cdots, m$,
$$\tilde{\mathbf{v}}_k = \mathbf{u}_k - \textstyle\sum_{j=1}^{l}(\mathbf{v}_j^T\mathbf{u}_k)\mathbf{v}_j; \quad \text{if } \tilde{\mathbf{v}}_k \neq \mathbf{0}, \text{ then } \mathbf{v}_{l+1} = \tilde{\mathbf{v}}_k/\|\tilde{\mathbf{v}}_k\| \text{ and } l \leftarrow l+1.$$

5. (a) $\mathbf{C} = \begin{bmatrix} \frac{\cos 10^\circ}{5} & -15\sin 5^\circ \\ -\frac{\sin 10^\circ}{5} & 15\cos 5^\circ \end{bmatrix}$. (b) $\mathbf{C}^{-1} = \frac{1}{3\cos 15^\circ}\begin{bmatrix} 15\cos 5^\circ & 15\sin 5^\circ \\ \frac{\sin 10^\circ}{5} & \frac{\cos 10^\circ}{5} \end{bmatrix}$.

6. (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$. (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -a & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Chapter 4 (page 27)

1. Rank = 3, Nullity = 2, Range = $< \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3 >$, Null space = $< \mathbf{n}_1, \mathbf{n}_2 >$, where $\mathbf{n}_1 = [-0.7\ \ 0.1\ \ 2.1\ \ 1\ \ 0]^T$, $\mathbf{n}_2 = [0.1\ \ 0.7\ \ 2.7\ \ 0\ \ 1]^T$;
Particular solution: $\bar{\mathbf{x}} = [2.3\ \ -2.9\ \ -13.9\ \ 0\ \ 0]^T$, General solution: $\mathbf{x} = \bar{\mathbf{x}} + \alpha_1\mathbf{n}_1 + \alpha_2\mathbf{n}_2$.

2. $a = 5$, $b = 2$, $c = 1$, $d = 2$, $e = 2$, $f = -3$, $g = 1$, $h = -1$, $i = 3$. [Equate elements to determine the unknowns in this sequence, to get one equation in one unknown at every step.]

3. He can if n is odd, but cannot if n is even. [For the coefficient matrix, determinant = $\frac{1-(-1)^n}{2^n}$.]

4. Real component = $(\mathbf{A}_r + \mathbf{A}_i\mathbf{A}_r^{-1}\mathbf{A}_i)^{-1}$; Imaginary component = $-\mathbf{A}_r^{-1}\mathbf{A}_i(\mathbf{A}_r + \mathbf{A}_i\mathbf{A}_r^{-1}\mathbf{A}_i)^{-1}$.

5. $x_3 = 20/17$, $x_4 = 1$. [Partition $\mathbf{x} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$ and then solve for $\mathbf{y}_2 = [x_3\ \ x_4]^T$.]

6. $\epsilon = \frac{1}{\mathbf{a}^T\mathbf{a} - \mathbf{a}^T\mathbf{A}^T\mathbf{P}\mathbf{A}\mathbf{a}}$, $\mathbf{b} = -\mathbf{P}\mathbf{A}\mathbf{a}\epsilon$, $\mathbf{Q} = \mathbf{P}(\mathbf{I}_m - \mathbf{A}\mathbf{a}\mathbf{b}^T)$. [Equate blocks in $(\bar{\mathbf{A}}\bar{\mathbf{A}}^T)\bar{\mathbf{P}} = \mathbf{I}_{m+1}$.]
In the second case, with $\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{q} \\ \mathbf{q}^T & \alpha \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} \bar{\mathbf{A}} \\ \mathbf{a}^T \end{bmatrix}$, $\bar{\mathbf{P}} = \mathbf{Q} - \mathbf{q}\mathbf{q}^T/\alpha$.

7. $\mathbf{Q} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & -0.5 & -0.5 \end{bmatrix}$ and $\mathbf{R} = \begin{bmatrix} 12 & 6 & 0 & 4 \\ 0 & 4 & -2 & 2 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.
Comment: As a diagonal entry of $\mathbf{R}$ turns out as zero, the corresponding column of $\mathbf{q}$ becomes just undefined or indeterminate (not non-existent) and the only constraint that remains is the orthogonality of $\mathbf{Q}$. Any vector orthonormal to previous columns is acceptable. Based on the choice, however, results of the next steps may differ. In a way, the QR decomposition is an extension of Gram-Schmidt orthogonalization. See more detail on this decomposition in Chap. 12.

8. These elements are used to construct the basis vectors for the null space, by appending '-1' at the coordinate corresponding to the column number. These elements serve as the coefficients in the expression of non-basis coordinates in terms of the basis coordinates. [For the proof,

consider only those columns (to which the premise makes reference) in the RREF as well as the original matrix. Use invertibility of elementary matrices to establish equality of their ranks.]

9. [Identify m linearly independent vectors orthogonal to $\mathcal{M}$, from definition. Representing a general normal $\mathbf{w}$ in that basis, determine it to satisfy $\mathbf{Pv} = \mathbf{v} - \mathbf{w} \in \mathcal{M}$. This would yield $\mathbf{w} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}\mathbf{A}\mathbf{v}$ and lead to the result.]

Chapter 5 (page 34)

1. $\begin{bmatrix} -0.09 \\ 0.73 \end{bmatrix}$, $\begin{bmatrix} 2.27 \\ -0.18 \end{bmatrix}$, $\begin{bmatrix} -0.73 \\ -0.18 \end{bmatrix}$, $\begin{bmatrix} 0.36 & -0.27 \\ 0.09 & 0.18 \end{bmatrix}$, $\begin{bmatrix} 1.45 & -0.09 \\ 1.36 & 0.73 \end{bmatrix}$.

2. $\mathbf{L}^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{b}{ad} & \frac{1}{d} & 0 \\ \frac{be-cd}{adf} & -\frac{e}{df} & \frac{1}{f} \end{bmatrix}$.

3. $q_1 = b_1$. For $1 < j \le n$, $p_j = \frac{a_j}{q_{j-1}}$ and $q_j = b_j - p_j c_{j-1}$. FLOP: $3(n-1)$.
Forward substitution: $2(n-2)$. Back-substitution: $3n-2$.
$q_1 = b_1$, $w_1 = c_1/b_1$, $\rho_1 = r_1/b_1$. For $1 < j < n$, $q_j = b_j - a_j w_{j-1}$, $w_j = c_j/q_j$ and $\rho_j = \frac{r_j - a_j\rho_{j-1}}{q_j}$. Finally, $q_n = b_n - a_n w_{n-1}$, $\rho_n = \frac{r_n - a_n\rho_{n-1}}{q_n}$. FLOP: $6n-5$.

4. (a) Rank = 1. (b) The formula is used to make small updates on an already computed inverse. [For derivation, expand left-hand-side in binomial series and sandwich the scalar quantity $(1 - \mathbf{v}^T\mathbf{A}^{-1}\mathbf{u})$ in every term.]

5. $-\mathbf{A}^{-1}\frac{d}{dt}(\mathbf{A})\mathbf{A}^{-1}$. [Differentiate $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$.]

Chapter 6 (page 41)

1. $\sum_{j=1}^{n} a_{ij}y_j$, $\sum_{j=1}^{m} a_{ji}x_j$; $\mathbf{Ay}$, $\mathbf{A}^T\mathbf{x}$; $\sum_{j=1}^{n}(a_{ij} + a_{ji})x_j$; $(\mathbf{A} + \mathbf{A}^T)\mathbf{x}$.

2. Yes. [There are various approaches to this problem, viz. Sylvester's criterion, Cholesky decomposition, using Sturm sequence property of characteristic polynomials (see Chap. 11) or actual evaluation of the eigenvalues!]

3. Matrix $\mathbf{I}_n + c\mathbf{A}^T\mathbf{A}$ is of size $n \times n$, larger than the size of $\mathbf{I}_m + c\mathbf{A}\mathbf{A}^T$, hence inversion in terms of the latter gives computational advantage. Besides, the former matrix is ill-conditioned, with m eigenvalues of large magnitudes and the rest equal to unity.
Comment: Such matrices appear in the solution steps with a penalty-based formulation of constrained optimization problems.

4. **Comment:** Can you relate this problem with the previous one?

5. 3, 4, 4, 3, 3, 3, 2, 2. [This is a *cyclic tridiagonal system*, that differs from an ordinary tridiagonal case in that the elements at $(1, n)$ and $(n, 1)$ locations in the coefficient matrix are non-zero. Use Sherman-Morrison formula to reduce it to a tridiagonal system.]

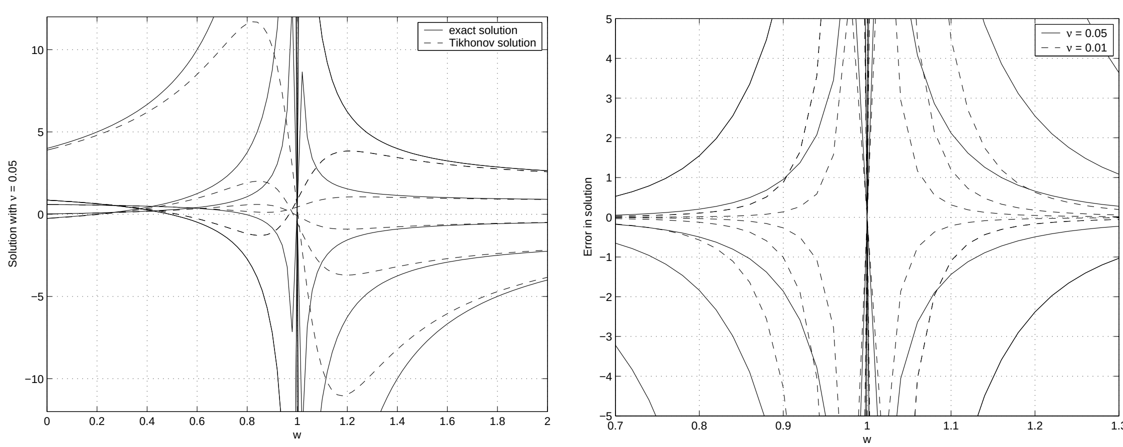
6. $\begin{bmatrix} -0.84 & 0.34 & 0.41 & 0.39 & -0.08 \\ 0.69 & -0.19 & -0.34 & -0.06 & -0.13 \\ -0.38 & -0.12 & 0.19 & 0.11 & 0.26 \\ 0.55 & -0.05 & -0.32 & -0.27 & 0.09 \\ -0.02 & 0.02 & 0.14 & -0.07 & -0.01 \end{bmatrix}$, $\begin{bmatrix} -0.89 & 0.37 & 0.44 & 0.40 & -0.08 \\ 0.72 & -0.20 & -0.36 & -0.06 & -0.13 \\ -0.39 & -0.11 & 0.19 & 0.12 & 0.26 \\ 0.57 & -0.06 & -0.33 & -0.28 & 0.09 \\ -0.02 & 0.02 & 0.15 & -0.07 & -0.01 \end{bmatrix}$.

7. (a) $-\frac{\mathbf{d}_k^T \mathbf{e}_k}{\mathbf{d}_k^T \mathbf{A} \mathbf{d}_k}$. (b) $\alpha_1 \mathbf{d}_0^T \mathbf{A} \mathbf{d}_1$, $\alpha_1 \mathbf{d}_0^T \mathbf{A} \mathbf{d}_1 + \alpha_2 \mathbf{d}_0^T \mathbf{A} \mathbf{d}_2$ and $\alpha_2 \mathbf{d}_1^T \mathbf{A} \mathbf{d}_2$.
   (c) For $i < j-1$ and $2 \le j \le k$, $\mathbf{d}_i^T \mathbf{e}_j = \sum_{l=i+1}^{j-1} \alpha_l \mathbf{d}_i^T \mathbf{A} \mathbf{d}_l$. (d) $\mathbf{d}_i^T \mathbf{A} \mathbf{d}_j = 0$ for $i < j \le k$.

Chapter 7 (page 50)

1. (a) $[1 + \frac{u_2}{2u_1} \quad 1 - \frac{u_2}{2u_1}]^T$. (b) $\sqrt{2 + u_1^2 + \sqrt{4 + u_1^4}}$ and $\frac{2+u_1^2+\sqrt{4+u_1^4}}{2u_1}$. (c) $[1 \quad 1]^T$, 2.000025 and 200.005. (d) The solution is insensitive to small changes in u_1 and with u_2 it changes as $\delta \mathbf{x} = 50[1 \quad -1]^T \delta u_2$.
   Comment: The given system exhibits the rudimentary features of ill-conditioning, as discussed in the chapter and depicted in Fig. 7.1.

2. **Comment:** At $w = 1$, note the singularity and how the Tikhonov regularization handles it. See Fig. A.1(a) for a comparison with the exact solution. Across a singular zone, it gives smooth solutions that are usable as working solutions for many practical applications. But, even away from singularity, the resulting solution has an error, where the exact solution is quite well-behaved. There is a trade-off involved. Reduction of the parameter ν results in better accuracy of solutions away from a singularity, at the cost of smoothness around it! See Fig. A.1(b).



(a) Comparison with exact solution

(b) Accuracy and smoothness

Figure A.1: Solution of an ill-conditioned system: Tikhonov regularization

3. First iteration: $[17.83 \;\; 2.02 \;\; 10.92]^T$, second iteration: $[15.68 \;\; 4.68 \;\; 11.79]^T$ etc.
Comment: Note the fast progress. In the third iteration, the error moves to second place of decimal! However, with the changed order, the algorithm diverges, because the coefficient matrix is no more diagonally dominant.

4. (a) $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 10000 & -7500 \\ -7500 & 5625 \end{bmatrix}$; Eigenvalues: $\lambda_1 = 15625$, $\lambda_2 = 0$; Eigenvectors: $\mathbf{v}_1 = \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}$. (b) $\Sigma = \begin{bmatrix} 125 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. (c) $\mathbf{U} = \begin{bmatrix} 0.80 & 0.00 & 0.60 \\ 0.36 & 0.80 & -0.48 \\ -0.48 & 0.60 & 0.64 \end{bmatrix}$.
[Keeping the first column $\mathbf{u}_1$ fixed, any orthogonal matrix will be valid.]
(d) $< \mathbf{v}_2 >$. (e) $< \mathbf{u}_1 >$. (f) $\Sigma\mathbf{y} = \mathbf{c}$, where $\mathbf{y} = \mathbf{V}^T\mathbf{x}$ and $\mathbf{c} = \mathbf{U}^T\mathbf{b}$.

Chapter 8 (page 57)

1. (a) Eigenvalues: -2, 3, 3; Eigenvectors: $[1 \;\; -1 \;\; 1]^T$, $[3 \;\; 4 \;\; 0]^T$, $[1 \;\; 0 \;\; 2]^T$.
Comment: Linear combinations of the last two are also eigenvectors.
(b) Eigenvalues: 1, 2, 2; Eigenvectors: $[1 \;\; 1 \;\; -1]^T$, $[1 \;\; 2 \;\; 1]^T$.
Comment: The third eigenvector does not exist. The given matrix is defective.

2. $\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n$. Significance: For every n-th degree polynomial, we can construct an $n \times n$ matrix, called its *companion matrix*, whose eigenvalues are the roots of the given polynomial.
Comment: This result is the basis of one of the commonly used methods for solving polynomial equations. (See Chap. 19.)

3. The solution is trivial. For discussion, see the text of the next chapter (Chap. 9).

4. (a) $\overline{\mathbf{A}}_1 = \begin{bmatrix} \lambda_1 & \mathbf{b}_1^T \\ \mathbf{0} & \mathbf{A}_1 \end{bmatrix}$, $\mathbf{b}_1^T = \mathbf{q}_1^T\mathbf{A}\overline{\mathbf{Q}}_1$, $\mathbf{A}_1 = \overline{\mathbf{Q}}_1^T\mathbf{A}\overline{\mathbf{Q}}_1$. (b) $\tilde{\mathbf{Q}}_2 = [\mathbf{q}_2 \;\; \overline{\mathbf{Q}}_2]$, $\mathbf{Q}_2 = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{Q}}_2 \end{bmatrix}$.
(c) For a real matrix $\mathbf{A}$, having real eigenvalues, there exists an orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}^T\mathbf{A}\mathbf{Q}$ is upper triangular, with the eigenvalues at the diagonal.

5. [For the first part, use
$\mathbf{A}^T(\mathbf{I}_m + c\mathbf{A}\mathbf{L}^{-1}\mathbf{A}^T) = \mathbf{A}^T + c\mathbf{A}^T\mathbf{A}\mathbf{L}^{-1}\mathbf{A}^T = (\mathbf{I}_n + c\mathbf{A}^T\mathbf{A}\mathbf{L}^{-1})\mathbf{A}^T = (\mathbf{L} + c\mathbf{A}^T\mathbf{A})\mathbf{L}^{-1}\mathbf{A}^T$.
For the second part, operate $\mathbf{A}\mathbf{L}^{-1}\mathbf{A}^T$ on its eigenvector and note that
$(\mathbf{I}_m + c\mathbf{A}\mathbf{L}^{-1}\mathbf{A}^T)\mathbf{v} = \mathbf{v} + c\sigma\mathbf{v} = (1 + c\sigma)\mathbf{v}$.]

6. Eigenvalue $\lambda = 28$ and corresponding eigenvector $\mathbf{v} = [-0.41 \;\; 0.41 \;\; -0.82]^T$ converges within one place of decimal, in seven iterations (in this case).

Chapter 9 (page 64)

1. (a) Jordan form: $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. [In the second column of the similarity transformation matrix, use *generalized eigenvector* $\mathbf{w}_2$, given by $(\mathbf{A} - 2\mathbf{I})\mathbf{w}_2 = \mathbf{v}_1$.] (b) Diagonal form:

diag $(3,1,1)$. (c) Jordan form: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. [$\mathbf{S} = [\mathbf{v}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3]$.] (d) Diagonal form in C: diag $(-1, i, -i)$. [In this case, eigenvectors are also complex and, hence, so is the similarity transformation.]

2. $\mathbf{A} = \begin{bmatrix} 1.25 & -0.20 & -0.15 \\ 0.10 & 0.90 & -0.10 \\ 0.25 & -0.20 & 0.85 \end{bmatrix}$ and $\mathbf{A}^6 = \begin{bmatrix} 2.63 & -1.24 & -0.85 \\ 0.47 & 0.53 & -0.47 \\ 1.63 & -1.24 & 0.15 \end{bmatrix}$.
[Use $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$ and $\mathbf{A}^6 = \mathbf{S}\Lambda^6\mathbf{S}^{-1}$.]

3. [If a diagonalizable matrix has a full set of *orthogonal* eigenvectors, then the associated orthogonal similarity transformation will preserve the symmetry of a diagonal matrix and the given matrix turns out to be symmetric. In the non-diagonalizable case, the resulting Jordan form is not symmetric, so the matrix can have all its eigenvectors (less than the full set) orthogonal.]

4. Deflated matrix: $\begin{bmatrix} 0.64 & 0.48 & 0 \\ 0.48 & 0.36 & 0 \\ 0 & 0 & 0 \end{bmatrix}$; other eigenvalues: 1, 0; and corresponding eigenvectors: $[0.8 \;\; 0.6 \;\; 0]^T$, $[-0.36 \;\; 0.48 \;\; 0.8]^T$ respectively.

Chapter 10 (page 71)

1. Eigenvalues and eigenvectors: $0.620 + 0.785i$, $[-0.115 - 0.500i \quad -0.115 + 0.500i \quad 0.688]^T$; $0.620 - 0.785i$, $[-0.115 + 0.500i \quad -0.115 - 0.500i \quad 0.688]^T$; 1, $[0.688 \quad 0.688 \quad 0.229]^T$.
$\det(\mathbf{Q}) = 1$.
Yes, $\mathbf{Q}$ represents a rotation. [Check orthogonality.]
Plane orthogonal to the third eigenvector, or the plane of the first two eigenvectors, of which the real plane can be obtained with the basis $\{[-1 \;\; -1 \;\; 6]^T, \;\; [1 \;\; -1 \;\; 0]^T\}$.

2. 3.4142, 2, 0.5858 (after two sweeps, diagonal entries: 3.4134, 2.0008, 0.5858).

3. $\begin{bmatrix} 5.00 & 3.00 & 0 & 0 \\ 3.00 & -2.11 & -3.28 & 0 \\ 0 & -3.28 & 4.53 & 1.22 \\ 0 & 0 & 1.22 & 1.58 \end{bmatrix}$.

4. $\begin{bmatrix} 9.00 & 3.61 & 0 & 0 & 0 \\ 3.61 & 9.54 & 1.69 & 0 & 0 \\ 0 & 1.69 & 6.87 & -2.44 & 0 \\ 0 & 0 & -2.44 & 5.01 & -1.14 \\ 0 & 0 & 0 & -1.14 & 7.58 \end{bmatrix}$. [Use $\mathbf{P}_{23}$, $\mathbf{P}_{34}$ and $\mathbf{P}_{45}$.]

5. In $\mathbf{QB} = \mathbf{AQ}$, the first column gives $b_{11}\mathbf{q}_1 + b_{21}\mathbf{q}_2 = \mathbf{A}\mathbf{q}_1$. Determine $b_{11} = \mathbf{q}_1^T\mathbf{A}\mathbf{q}_1$ and separate out $b_{21}\mathbf{q}_2$, the norm of which gives $b_{21} \neq 0$ (from hypothesis) and leads to $\mathbf{q}_2$. This determines first column (and row) of $\mathbf{B}$ and second column of $\mathbf{Q}$.
For $1 < k < n$, k-th column gives $b_{k-1,k}\mathbf{q}_{k-1} + b_{k,k}\mathbf{q}_k + b_{k+1,k}\mathbf{q}_{k+1}$, in which previous columns and rows of $\mathbf{B}$ and up to k-th column of $\mathbf{Q}$ are already known. Determine $b_{k,k}$, $b_{k,k+1}$ and

$\mathbf{q}_{k+1}$, in that order, as above.
Finally, $b_{n,n} = \mathbf{q}_n^T \mathbf{A} \mathbf{q}_n$.

6. (a) $\begin{bmatrix} 0.897 & -0.259 & 0.359 \\ 0.341 & 0.921 & -0.189 \\ -0.281 & 0.291 & 0.914 \end{bmatrix}$. [Find unit vector $\mathbf{c}_1$ along the line. Assume $\mathbf{c}_2$ and $\mathbf{c}_3$ to complete orthonormal basis. Rotation matrix $= \mathbf{C}\Re_{yz}^T\mathbf{C}^T$. Analytical multiplication and subsequent reduction can be used to eliminate $\mathbf{c}_2$ and $\mathbf{c}_3$. Alternatively, a consistent pair can be chosen, e.g. through the Gram-Schmidt process (see Chap. 3), and the solution follows from straightforward computation.]
(b) First verify that the given matrix is orthogonal and its determinant is +1. The analytical expression of the matrix in part (a) shows that its trace gives the angle. Then the difference of the given matrix and its transpose gives the axis. Alternatively, one can simply find its eigenvalues and eigenvectors. Eigenvalues are 1 and $e^{\pm i\theta}$. Eigenvector corresponding to the real unit eigenvalue is the axis of rotation and θ is the angle. The *plane* of rotation can be developed either as the subspace orthogonal to the axis or through real combinations of eigenvectors corresponding to the complex eigenvalues $e^{\pm i\theta}$.
(c) Direct problem: form orthogonal matrix $\mathbf{C}$ with pair(s) of columns spanning the plane(s) of rotation, then $\Re = \mathbf{C}\mathbf{P}_{12}^T\mathbf{C}^T$. Inverse problem: Same as part (b) except that there is no relevance of an *axis*, and there may be several plane rotations involved in mutually orthogonal planes.
Comment: In higher dimensions, the second approach looks cleaner in each of the direct and inverse problems.

Chapter 11 (page 79)

1. $\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$.
Comment: For this problem, alternative solutions are possible, because the problem does not *ask* for a Householder matrix.

2. (a) One eigenvalue is -1 with eigenvector $\mathbf{w}$ with the notation of page 73. All other eigenvalues are 1 with corresponding eigenvectors forming an eigenspace orthogonal to $\mathbf{w}$.
Determinant $= -1$.
(b) All eigenvalues are of absolute value of unity. First, there can be one or more unit (+1) eigenvalue with real eigenvectors. Besides, on 2-d planes of rotation, there may be pairs of eigenvalues in the form $e^{\pm i\theta}$ with associated eigenvectors in the form $\mathbf{u} \pm i\mathbf{w}$. In a special case of $\theta = \pi$, $\mathbf{u}$ and $\mathbf{w}$ separate out as individual (real) eigenvectors. Finally, there may remain a single negative eigenvalue -1 signifying reflection.

3. We cannot. With such an attempt, zeros set in left multiplication will not be preserved through the right multiplication by the Householder matrix.

4. Eigenvalues of $\mathbf{B}$: $0.59, 2, 3.41$. Eigenvalues of $\mathbf{C}$: $-4.33, 1.29, 4.98, 7.05$.

5. Five. [Examine the Sturmian sequence at zero.]

Chapter 12 (page 88)

1. 5.81, 3.53, 1.66. Eigenvectors are the columns of cumulative product $\mathbf{Q}$ of the orthogonal matrices arising out of the QR iterations.

2. Eigenvalues: 6.31, 2.74, 16.46, 18.48.
   Comment: The problem asks for a lot more. Follow the prescribed steps. Here, eigenvalues are supplied merely for verification.

3. Eigenvalues: 8, 14.8, 5.8, -4.6, 10, 8 (appearing in this order).
   Comment: QR algorithm does not iterate across subspaces which are already decoupled!

4. Eigenvalues: 12.1025, -8.2899, 4.7895, 2.9724, 1.4255.
   Comment: The reader should develop computer programs and experiment on the issue to get the feel for the different methods.

Chapter 13 (page 95)

1. $\begin{bmatrix} 6 & 2 & 4 & 1 \\ 4 & 6 & 4 & 0 \\ 0 & -4 & -1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$.
   It is possible, in principle, to convert it into a tridiagonal form through the Gaussian elimination process. In the procedure described in the chapter, interchange the roles of rows and columns, and carry through only the elimination step (do not pivot) to get zeros above the super-diagonal. However, (a) this process is numerically not stable, and (b) in this part, the ban on pivoting makes it unusable as a general *algorithm*. Even the benefit is not much. Without symmetry, the tridiagonal structure will not be preserved when it is passed through QR iterations!

2. Eigenvalues: 5.24, 5.24, 0.76 and 0.76.
   5 iterations and 21 iterations. After a sub-diagonal member becomes extremely small, it is better to split the matrix into two smaller matrices and proceed with them separately. In this particular case, each of them is a 2×2 matrix, which can be handled directly from definition.

3. Eigenvalue = 30.289; Eigenvector = $[0.38 \;\; 0.53 \;\; 0.55 \;\; 0.52]^T$. [Use inverse iteration.]
   For ill-conditioned matrix, take zero as an approximate eigenvalue and use inverse iteration again. The approximate zero eigenvalue gets refined to the correct eigenvalue 0.010 and the corresponding eigenvector is obtained as $[0.83 \;\; -0.50 \;\; -0.21 \;\; 0.12]^T$.
   Finally, extend these two eigenvectors to an orthogonal basis and transform the matrix to that basis. The remaining problem is just 2×2.
   Comment: One can explicitly use deflation at this stage.

Chapter 14 (page 105)

1. [From SVD, $\mathbf{A}^T\mathbf{A} = \mathbf{V}\Sigma^2\mathbf{V}^T$; while from symmetry, $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$ and $\mathbf{A}^T\mathbf{A} = \mathbf{A}^2 = \mathbf{V}\Lambda^2\mathbf{V}^T$.]

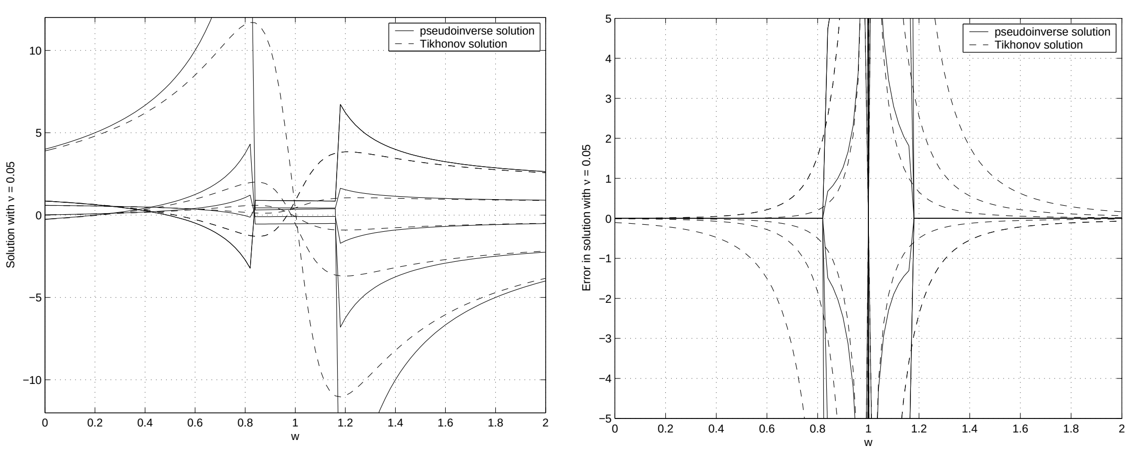
2. [(a) With SVD in hand, examine the ranks of matrices Σ, Σ^T, $\Sigma^T\Sigma$ and $\Sigma\Sigma^T$. (b) First show that ranges of $\mathbf{A}^T\mathbf{A}$ and $\mathbf{A}^T$ are the same. (c) Examine diagonal entries of $\Sigma^T\Sigma$ and $\Sigma\Sigma^T$.]

3. False. See a counter-example in the next problem.

4. Ellipse: major radius 5.728 in direction $[0.9029 \quad 0.4298]^T$ and minor radius 3.492 in direction $[-0.4298 \quad 0.9029]^T$. Principal scaling effects: 5.728 on $[0.7882 \quad 0.6154]^T$ and 3.492 on $[-0.6154 \quad 0.7882]^T$. Eigenvectors: $[1 \quad 0]^T$ with eigenvalue 5 and $[-0.8944 \quad 0.4472]^T$ with eigenvalue 4.

5. 99.25 % . [Use Eqn. 14.4 and cut-off lower singular values after a gap of order of magnitude, in this case only nine singular values are retained.]

6. **Comment:** Near $w = 1$, the exact solution approaches infinity, as expected. The pseudoinverse gives a viable solution by cutting off small singular values, which basically results into an automatic trimming of the solution. See comparison with Tikhonov solution in Fig. A.2. To emphasize the role and effect of the parameter ν, the plots are shown for a high value of ν, namely 0.05. In practical applications, much lower values are normally used. While the Tikhonov regularization gives a continuous and more acceptable solution close to the singularity, pseudoinverse solution has the merit of being exact above the cut-off value (ν) of singular values.



(a) Solution about a singularity

(b) Deviation from exact solution

Figure A.2: Ill-conditioned systems: pseudoinverse and Tikhonov regularization

7. (a) $[0.16 \quad -0.12]^T$. (b) 0.04 and 25. (c) [Show $\Delta E = (100\alpha - 75\beta)^2$.] (d) [Show $\Delta\|\mathbf{x}\|^2 = 25k^2$ with $\alpha = 3k$ and $\beta = 4k$.]

Chapter 15 (page 115)

1. No. Yes, the set of $n \times n$ *non-singular* matrices.

2. (a) [Assume two identity elements e_1, e_2 and compose them, once using the identity property of e_1 and the other time using the same property of e_2.]
(b) [For $a \in G$, assume two inverses a_1, a_2 and use $a_1 = a_1 \cdot e = a_1 \cdot (a \cdot a_2) = (a_1 \cdot a) \cdot a_2 = e \cdot a_2 = a_2$.]

3. [Verify unique definition. Work out the effect of two such rigid body motions on a general point, and verify that the result is the same as a rigid body motion, to establish closure. Consider three rigid body motions and show associativity. Point out the identity element. Establish the inverse of a general rigid body motion. Finally, show non-commutativity.]

4. $(Z_e, +)$ is a subgroup, but $(Z_o, +)$ is not.

5. (b) The result in part (a) establishes the closure of the addition operation of linear transformations and also that of the scalar multiplication. [Further, verify commutativity and associativity of the addition of two linear transformations through application on a general vector. Identify the zero transformation, which maps all vectors in $\mathbf{V}$ to $\mathbf{0} \in \mathbf{W}$ and define the 'negative' of a transformation. Unity is obviously the identity element of the scalar multiplication. For this operation, establish associative and distributive laws to complete the proof.]
(c) A set of mn transformations $\mathbf{E}_{ij},\ 1 \le i \le m,\ 1 \le j \le n$; mapping j-th basis member of $\mathbf{V}$ to i-th basis member of $\mathbf{W}$ and all other basis members of $\mathbf{V}$ to the zero element of $\mathbf{W}$, where $n = \dim \mathbf{V}$ and $m = \dim \mathbf{W}$. Matrix of $\mathbf{E}_{ij}$ has '1' at the i, j location and '0' everywhere else.

6. [Hint in the question leads to Cauchy-Schwarz inequality. Use this result in the direct computation of $\|\mathbf{a} + \mathbf{b}\|^2$, to establish the triangle inequality.]

7. [Substitution of definition of norm in $\|f + g\| > \|f\| + \|g\|$ leads to $|(f, g)|^2 > \|f\|^2\ \|g\|^2$. In this inequality (Cauchy-Schwarz), use dummy variable y in one integral and z in the other, on both sides to develop the inequality in terms of double integrals as

$$\int_a^b \int_a^b w(y)w(z)f(y)g(y)f(z)g(z)\,dy\,dz > \int_a^b \int_a^b w(y)w(z)f^2(y)g^2(z)\,dy\,dz.$$

Interchanging y and z, obtain another similar equation. Add these two equations and simplify to show that

$$\int_a^b \int_a^b w(y)w(z)[f(y)g(z) - f(z)g(y)]^2\,dy\,dz < 0,$$

which is absurd.]

Chapter 16 (page 126)

1. (a) $\alpha\,[\nabla f(\mathbf{x})]^T\,\mathbf{q}_j$.
(c) [Find out the k-th component. Then, either show it to be $\frac{\partial f}{\partial x_k}$ or assemble by rows.]

2. $\frac{\partial s}{\partial x} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}}$, $\quad \frac{\partial t}{\partial x} = -\frac{\begin{vmatrix} F_s & F_x \\ G_s & G_x \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}}$, $\quad \frac{\partial s}{\partial y} = -\frac{\begin{vmatrix} F_y & F_t \\ G_y & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}}$, $\quad \frac{\partial t}{\partial y} = -\frac{\begin{vmatrix} F_s & F_y \\ G_s & G_y \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}}$.

3. Independent x, y, z: $\frac{\partial w}{\partial x} = 2x - 1$. Independent x, y, t: $\frac{\partial w}{\partial x} = 2x - 2$. Independent x, z, t: impossible.

4. $\sqrt{\pi}$. $[I = \int_{-\infty}^{\infty} e^{-x^2} dx = \int_{-\infty}^{\infty} e^{-y^2} dy$. Multiply the two to develop I^2 as a double integral and change to polar coordinates.]

5. (a) [Expand $\phi(x + \delta x)$ by splitting the interval $[u(x + \delta x), v(x + \delta x)]$ into three sub-intervals: $[u(x + \delta x), u(x)]$, $[u(x), v(x)]$ and $[v(x), v(x + \delta x)]$.]
(b) [For the first component, apply mean theorem on the integrand with respect to x. For others, apply it on the integrals with respect to t.]

6. Gradient = $\begin{bmatrix} 7.389 \\ 29.556 \end{bmatrix}$ (four function values used). Hessian = $\begin{bmatrix} 8 & 44.25 \\ 44.25 & 148 \end{bmatrix}$ (nine function values used).
Comment: The tabulated values were obtained from the function $e^{x_1 x_2^2}$. Check with the actual values obtained from analytical derivatives.

7. [Show that $g_j'(\mathbf{x}') = \frac{\partial x_i}{\partial x_j'} g_i(\mathbf{x})$.]

8. $5x_1^2 + 3x_1 x_2 - 20x_1 - 4x_2 + 28$; hyperbolic contours. See Fig. A.3.

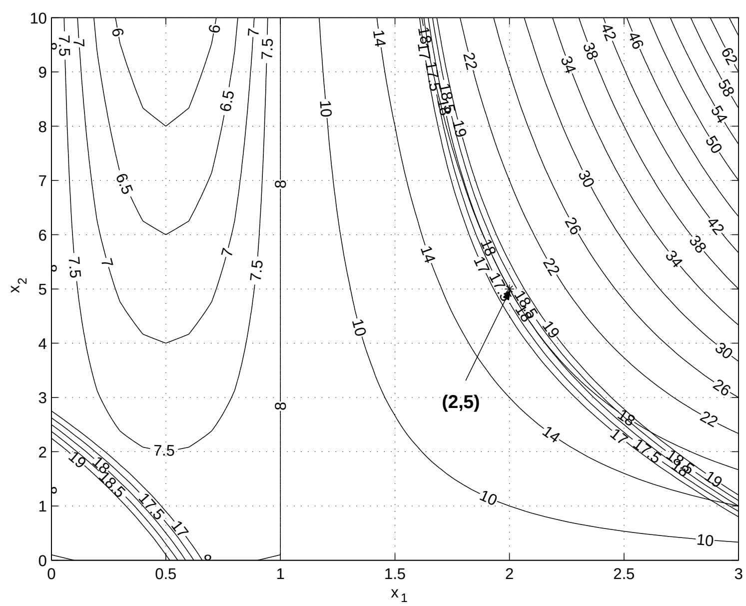


Figure A.3: Function contours and quadratic approximation

Chapter 17 (page 137)

1. $\frac{5}{6}\sqrt{6}$ feet or 2.04 feet. [Express end-points of the rope (say A and B), and end-points of the plank (say O and R) in some consistent frame of reference. Then, it is needed to find point C

on AB and point P on OR, such that distance CP is minimum, or vector CP is perpendicular to both vectors AB and OR.]

2. Parametric equations: (a) $\mathbf{r}(v,w) = (2-w)\mathbf{i} + \left(3+\frac{v}{\sqrt{2}}\right)\mathbf{j} + \left(4+\frac{v}{\sqrt{2}}\right)\mathbf{k}$, (b) $\mathbf{r}(v,w) = (2-v)\mathbf{i}+\left(3-\frac{w}{\sqrt{2}}\right)\mathbf{j}+\left(4+\frac{w}{\sqrt{2}}\right)\mathbf{k}$ and (c) $\mathbf{r}(v,w) = 2\mathbf{i}+\left(3-\frac{v}{\sqrt{2}}+\frac{w}{\sqrt{2}}\right)\mathbf{j}+\left(4+\frac{v}{\sqrt{2}}+\frac{w}{\sqrt{2}}\right)\mathbf{k}$.
Cartesian equations: (a) $z-y=1$, (b) $y+z=7$ and (c) $x=2$.

3. Value of $a=2$. Point of intersection (unique): (2,2,4), and included angle = $\cos^{-1}\frac{11\sqrt{2}}{18}$ or $30.2°$. [From $\mathbf{r}_1(t_1)=\mathbf{r}_2(t_2)$, solve for a, t_1 and t_2.]

4. [Using $g(t)=\frac{ds}{dt}=\|\mathbf{r}'(t)\|$, express $\mathbf{r}'(t)=g(t)\mathbf{u}$, hence $\mathbf{r}''(t)=g'(t)\mathbf{u}+[g(t)]^2\kappa\mathbf{p}$ and evaluate $\kappa(t)$. Similarly, develop $\mathbf{r}'''(t)$ and separate out $\tau(t)$.]

5. (a) [Differentiate $\psi(s)$ and use the Serret-Frenet formulae to show that the derivative vanishes.]
(b) $\psi(s)=\psi(0)=3 \Rightarrow \mathbf{u}_1\cdot\mathbf{u}_2=\mathbf{p}_1\cdot\mathbf{p}_2=\mathbf{b}_1\cdot\mathbf{b}_2=1$, i.e. the Serret-Frenet frames stay coincident for all values of s.
Comment: This exercise constructs a proof of the proposition (see page 133) stating the equivalence of curves with the same curvature and torsion functions.

6. Confirmation is the right conclusion.
[**Approach I** (based on intrinsic properties): Obviously, the first curve is a circle with radius of 3 units. Therefore, $\kappa_1(s)=1/3$ and $\tau_1(s)=0$. Now, reduce the second curve to arc-length parametrization to show that $\kappa_2(s)=1/3$ and $\tau_2(s)=0$.
Approach II (based on external transformations): From the starting positions and initial Serret-Frenet frames of both the curves, work out the candidate transformation (rotation and translation) that would map the initial segment of the second curve to that of the first one. Apply this transformation to the equation of the second curve to obtain its equation in the form $\bar{\mathbf{r}}_2(t)=3\cos(2t-\pi/2)\mathbf{i}+3\sin(2t-\pi/2)\mathbf{j}$, $t\geq\frac{\pi}{4}$. Now, a reparametrization of $\bar{\mathbf{r}}_2(t)$ with parameter $t'=2t-\pi/2$ yields its equation in exactly the same form as $\mathbf{r}_1(t)$.
Approach III (based on special case data): Again, we start from the knowledge that the first curve is a circle with a radius of 3 units. On the second curve, we pick up three arbitrary points P, Q, R at some convenient parameter values. First, we work out the normal to the plane PQR and verify that it is perpendicular to PX, where X is a general point on $\mathbf{r}_2(t)$. This establishes $\mathbf{r}_2(t)$ to be a planar curve. Next, we find out the candidate centre C as the point equidistant from P, Q, R and observe that $PC=QC=RC=3$. Finally, we verify that $CX=3$.]
Comment: The first two approaches would work on any set of candidate curves.

7. Quadratic approximation: $z(x,y)=0$.
Comment: If we replace x and y coordinates by $x_1=x-2y$, $y_1=2x+y$ (orthogonal frame), then we have the surface as $z=x_1^3$, which is invariant with respect to y_1 and has an inflection at the origin (along the x_1 direction).

8. $\frac{\sqrt{\kappa_1^2+\kappa_2^2-2\kappa_1\kappa_2\cos\theta}}{\sin\theta}$. [The normal $\mathbf{r}''(s)=\kappa\mathbf{p}$ to the curve C is in the plane of unit normals $\mathbf{n}_1$ and $\mathbf{n}_2$ to the surfaces, as all three share the same tangent. Hence, we can take $\mathbf{r}''=c_1\mathbf{n}_1+c_2\mathbf{n}_2$, work out the normal curvatures, solve for c_1, c_2 and substitute back.]

9. (a) [Show that $\mathbf{r}_u \cdot \mathbf{r}_v \neq 0$, in general.]
   (b) $\boldsymbol{\rho}(t) = \left(t^2 + \frac{1}{4}\right)\mathbf{i} - \frac{1}{4}\mathbf{j} + t\mathbf{k}$.
   (c) $x + y = z^2$, $\mathbf{r}(p, q) = \frac{q^2+p}{2}\mathbf{i} + \frac{q^2-p}{2}\mathbf{j} + q\mathbf{k}$ with $p = x - y$ and $q = z$.

Chapter 18 (page 149)

1. (a) [For Eqn. 18.8, one may evaluate the four terms in the right-hand side and add.]

2. Yes, $\mathbf{V}(x, y, z) = \nabla\phi(x, y, z)$, where the potential function is $\phi(x, y, z) = e^{xyz} - x^2 + yz + \sin y + 4z + c$. [$\phi(x, y, z) = \int V_x dx + \psi(y, z)$. Compare $\frac{\partial\phi}{\partial y}$ with V_y. Continue similarly. Alternatively, show that $\nabla \times \mathbf{V} = \mathbf{0}$, but that does not give the potential function.]

3. Potential function: $\phi(\mathbf{r}) = \frac{c}{\|\mathbf{r}-\mathbf{r}_0\|}$. [Show that $\nabla\phi = \mathbf{g}$. Alternatively, show that $\mathbf{g} \cdot d\mathbf{r}$ is an exact differential and integrate it.]

4. [Find unit normal $\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}$, evaluate $w\frac{\partial w}{\partial n}ds = w\nabla w \cdot \mathbf{n}ds$ and apply Green's theorem in the plane.]

5. (a) [First, show that $\int\int_T\int \nabla^2\psi dv = \int\int_S \frac{\partial\psi}{\partial n}dS$.]
   (b) [First, show that for a harmonic function $\psi(x, y, z)$ satisfying $\psi(x, y, z) = 0$ everywhere on the boundary S, $\psi(x, y, z) = 0$ everywhere in the domain T. Then, consider two solutions ψ_1, ψ_2 of a Dirichlet problem and their difference $\psi_1 - \psi_2 = \psi$.]
   (c) [Proceed as above with Neumann condition. In this case, you will find that $\psi = \psi_1 - \psi_2 = k$.]

6. (a) $\int\int_T\int \sigma\rho u dv$. (b) $-\int_S\int K\nabla u \cdot \mathbf{n}dS$. (c) $\sigma\rho\frac{\partial u}{\partial t} = \nabla K \cdot \nabla u + K\nabla^2 u$. (d) $\frac{\partial u}{\partial t} = \frac{K}{\sigma\rho}\nabla^2 u$.
   Comment: This is the well-known heat equation or the diffusion equation, one of the classic partial differential equations of mathematical physics.

7. (a) div $\mathbf{E} = \frac{1}{\epsilon}Q$, div $\mathbf{B} = 0$. [Apply Gauss's theorem on the last two equations.]
   curl $\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, curl $\mathbf{B} = \mu\epsilon\frac{\partial\mathbf{E}}{\partial t}$. [Apply Stokes's theorem on first two equations.]
   (b) $\frac{\partial^2\mathbf{E}}{\partial t^2} = c^2\nabla^2\mathbf{E}$ and $\frac{\partial^2\mathbf{B}}{\partial t^2} = c^2\nabla^2\mathbf{B}$. [Take curl of curl $\mathbf{E}$ and curl $\mathbf{B}$.]

8. (a) $\nabla \cdot (\rho\mathbf{U}) + \frac{\partial\rho}{\partial t} = 0$. [Use $\eta = 1$ and apply divergence theorem on the first term.]
   (b) $\int_S\int \mathbf{T}dA + \int\int_V\int \mathbf{B}\rho dv = \int_S\int \mathbf{U}(\rho\mathbf{U} \cdot \mathbf{n})dA + \frac{\partial}{\partial t}\int\int_V\int \rho\mathbf{U}dv$. [$\eta = \mathbf{U}$.]
   (c) $\nabla \cdot \boldsymbol{\tau} + \rho\mathbf{B} = \rho(\mathbf{U} \cdot \nabla)\mathbf{U} + \rho\frac{\partial\mathbf{U}}{\partial t}$,
   where $\nabla \cdot \boldsymbol{\tau} = [\nabla \cdot \boldsymbol{\tau}_1 \quad \nabla \cdot \boldsymbol{\tau}_2 \quad \nabla \cdot \boldsymbol{\tau}_3]^T$, with $\nabla \cdot \boldsymbol{\tau}_1 = \frac{\partial\tau_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z}$ etc; and the operator $\mathbf{U} \cdot \nabla \equiv u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}$. [Alternative derivation in tensor notation is possible, in which no separation of components is needed.]
   (d) $-\frac{\nabla p}{\rho} + \mathbf{B} = (\mathbf{U} \cdot \nabla)\mathbf{U} + \frac{\partial\mathbf{U}}{\partial t} = \frac{D\mathbf{U}}{Dt}$.
   Comment: Application to general state of stress would lead to Navier-Stokes equations.

Chapter 19 (page 160)

1. **Comment:** This exercise demonstrates the mechanism of a proof of the proposition on multiple roots, expressed in Eqn. 19.3.

2. [Odd degree of the equation ensures at least one real root. Descartes' rule of sign limits the positive real roots to either two or none. The same rule, applied after a substitution $x = -y$ gives one positive root for y, i.e. one negative root for x. Thus, from this much, we know that there are exactly two possibilities of the distribution of roots: (a) one negative real root and two conjugate pairs of complex roots and (b) three real roots (two positive and one negative) and a pair of complex roots.]
Comment: You may solve the equation to verify that one of these is indeed the case here.

3. Eigenvalues: 2.6185, -2.3032, 1.3028, 0.3820. Values of the polynomial at these values of x: 0.0067, 0.0202, -0.0001, 0.0002 (respectively).

4. Eliminant equation in x: $7x^4+48x^3+89x^2+30x+25=0$. At most, four solutions. And, the (x, y) solution points are: $(-0.0965 \pm 0.5588i, -0.4678 \mp 0.1959i)$, $(-3.3321 \pm 0.0639i, 0.6107 \mp 2.9176i)$. [Steps: (a) Resolvent of the quartic: $z^3 - 12.714z^2 + 15.102z - 4.665 = 0$. (b) Solutions of the resolvent cubic: 11.43, 0.57, 0.71. (c) Using $z = 11.43$ and completing squares in the quartic, $x^2 + \frac{24x}{7} + \frac{11.43}{2} = \pm(3.2356x + 5.3927)$. (d) Developing two quadratic equations and solving, $x = -0.0965 \pm 0.5588i$, $-3.3321 \pm 0.0639i$ and $y = -\frac{x^2+5x+5}{3x+10}$ available from the elimination step.]
Comment: All four solutions being complex shows that these two ellipses do not have any intersection in the entire x-y plane.

5. [Upon expansion (for example, by the first column), the result follows immediately.]

6. (a) $\frac{\partial r}{\partial p} = up - v$, $\frac{\partial r}{\partial q} = -u$, $\frac{\partial s}{\partial p} = uq$, and $\frac{\partial s}{\partial q} = -v$.
[$P(x)$ is a *given* polynomial and x is a dummy variable, hence they are independent of the *chosen* variables p and q. Keeping this in view while carrying out the required differentiation, and combining with the given expressions, we obtain $\frac{\partial r}{\partial q} = -u$, $\frac{\partial s}{\partial q} = -v$ and
$$x(x^2+px+q)P_2(x) + ux^2 + vx + (x^2+px+q)\frac{\partial P_1}{\partial p} + \frac{\partial r}{\partial p}x + \frac{\partial s}{\partial p} = 0.$$
Substituting α and β, roots of x^2+px+q, in this, we derive two linear equations in $\frac{\partial r}{\partial p}$, $\frac{\partial s}{\partial p}$ and solve them.]
(b) $\mathbf{J} = \begin{bmatrix} up-v & -u \\ uq & -v \end{bmatrix}$, $\begin{bmatrix} p_{k+1} \\ q_{k+1} \end{bmatrix} = \begin{bmatrix} p_k \\ q_k \end{bmatrix} - \frac{1}{\Delta_k}\begin{bmatrix} -v_k & u_k \\ -u_k q_k & u_k p_k - v_k \end{bmatrix}\begin{bmatrix} r_k \\ s_k \end{bmatrix}$, for $\Delta_k = v_k^2 - u_k p_k v_k + u_k^2 q_k \neq 0$.
[Since $\mathbf{J}\begin{bmatrix} \delta p \\ \delta q \end{bmatrix} = \begin{bmatrix} \delta r \\ \delta s \end{bmatrix} = \begin{bmatrix} -r \\ -s \end{bmatrix} \Rightarrow \begin{bmatrix} \delta p \\ \delta q \end{bmatrix} = -\mathbf{J}^{-1}\begin{bmatrix} r \\ s \end{bmatrix}$.]
(c) Roots: 4.37, -1.37, 8.77, 0.23, $3.5 \pm 2.4i$.
[After convergence of the first set of iterations, with $p = -3$ and $q = -6$, we find out the first two roots from x^2-3x-6 and also the quotient, which is found to be $x^4-16x^3+83x^2-176x+36$. The same process, repeated over this fourth degree polynomial, upon convergence, separates out two factors $x^2 - 9x + 2$ and $x^2 - 7x + 18$, that yield the other roots.]
Comment: Bairstow's method can be recognized as a special implementation of the Newton-Raphson method for solution of systems of nonlinear equations. (See the next chapter.)

7. Degree: 12. [$l_3 = c_1 l_2 - c_2 l_1$, $q_3 = c_1 q_2 - c_2 q_1$, $C_3 = c_1 C_2 - c_2 C_1$, $Q_3 = -c_2 Q_1$; $q_4 = c_1 q_2 - c_2 q_1$, $C_4 = c_1 C_2 + l_1 q_2 - c_2 C_1 - l_2 q_1$, $Q_4 = l_1 C_2 - c_2 Q_1 - l_2 C_1$, $\mathcal{Q}_4 = -l_2 Q_1$; $C_5 = C_3$, $Q_5 = Q_4$, $\mathcal{Q}_5 = q_1 C_2 - l_2 Q_1 - q_2 C_1$, $\mathcal{S}_5 = -q_2 Q_1$.]

Chapter 20 (page 168)

1. 0.57. [Bracket: $[0, 1]$; Fixed point iteration: 10 iterations; Method of false position: 4 iterations; Newton-Raphson: 3 iterations.]
   The solution is unique. $[f(x) = e^{-x} - x,\ f'(x) = -e^{-x} - 1 < 0$ monotonically decreasing.]

2. 0.17. (a) 3 iterations. [First, check the bracket.] (b) 2 iterations.
   (c) From $x_0 = 1.43$, slow progress, mild oscillations: 1.43, -1.24, 1.22, -0.47 etc; from $x_0 = 1.44$, almost periodic oscillations: 1.44, -1.2964, 1.4413, -1.3041 etc; from $x_0 = 1.45$, divergence with oscillations: 1.45, -1.35, 1.73, -5.23 etc.
   Comment: Fig. A.4 depicts this comparison of trajectories of search starting from the three almost coincident points of $x = 1.43, 1.44, 1.45$.

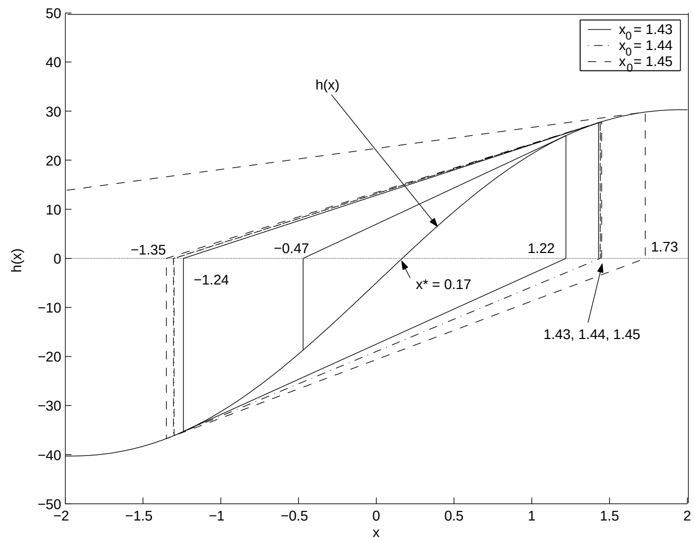


Figure A.4: Pathology of Newton-Raphson method

3. $[0.88\ \ 0.68\ \ 1.33]^T$.

4. (a) $y - f(x_0) = \frac{f(x_1)-f(x_0)}{x_1-x_0}(x - x_0)$. (b) $\left(x_0 - \frac{x_1-x_0}{f(x_1)-f(x_0)} f(x_0), 0\right)$, a secant method iteration. (c) $\left(x_0 - \frac{f(x_0)}{f'(x_0)}, 0\right)$, a Newton-Raphson iteration.

5. (a) $\mathbf{x}_4 = [-0.32\ \ -2.38]^T$ with errors: -4.62, 9.80. [Not yet even in the trend; converges to the solution in 9 iterations up to two places of decimal.]
   (b) $\mathbf{x}_4 = [-0.9497\ \ -2.0430]^T$ [Converged to four places of decimal.]
   Comment: It pays off to start with the correct Jacobian, through one-time computation; still avoiding costly Jacobian computation *within the loop*.

Chapter 21 (page 177)

1. $\phi(\frac{1}{2}) = 0$ maximum, $\phi(\frac{1}{2} \pm \frac{1}{5}) = -\frac{1}{25}$ minima. Other features: symmetry about $x = \frac{1}{2}$, zero-crossing at $x = \frac{1}{2} \pm \frac{\sqrt{2}}{5}$, etc. See Fig. A.5.

2. $[1.90, 1.92]$. $[f(1.90) > f(1.91) < f(1.92)$ by exhaustive search method.]
Comment: Other intervals with similar properties, enclosing the same minimum or a different minimum are also acceptable.

3. (a) $a = y_0 - bx_0 - cx_0^2$, $b = \frac{y_1-y_2}{x_1-x_2} - \frac{x_1+x_2}{x_0-x_2}\left[\frac{y_0-y_1}{x_0-x_1} - \frac{y_1-y_2}{x_1-x_2}\right]$, $c = \frac{1}{x_0-x_2}\left[\frac{y_0-y_1}{x_0-x_1} - \frac{y_1-y_2}{x_1-x_2}\right]$.
(b) $x_3 = -\frac{b}{2c}$.
(c) With three function values, obtain points (x_0, y_0), (x_1, y_1) and (x_2, y_2). Compute a, b and c by the above formulae. Evaluate $x_3 = -\frac{b}{2c}$, replace worst of x_0, x_1 and x_2 by x_3 and repeat till convergence.
Comment: Condition for the scheme to work: $c > 0$. (d) $x^* = 1.9053$, $f(x^*) = -7.8713$.

4. $x^* = 2.5$, $p(x^*) = -78.125$.
Comment: At the minimum point, the derivative of the function has a *triple* root. $[p'(x) = (2x-5)^3$.] Note how slowly the regula falsi iterations approach it. On the other side, though the golden section search made a function evaluation at $x = 2.4721$, really close to the minimum point, in the very first iteration, the operation of the method does not get any advantage of that proximity, and follows through the usual number of iterations!

5. Minima: $(0, 0)$, $(\pm\sqrt{3}, \mp\sqrt{3})$; Saddle points: $(\pm 1, \mp 1)$; no maximum point.
Comment: Rearranging the function as $f(\mathbf{x}) = \frac{1}{2}(x_1 + x_2)^2 + \frac{x_1^2}{6}(x_1^2 - 3)^2$ or $\phi(\mathbf{u}) = \frac{1}{2}u_2^2 + \frac{u_1^2}{6}(u_1^2 - 3)^2$, the critical points become clear. See Fig. A.6.

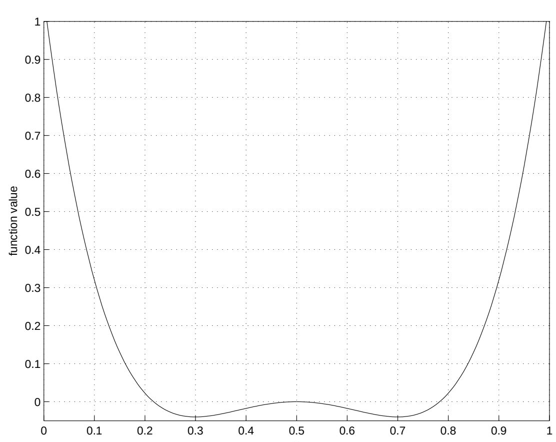


Figure A.5: Single-variable function example

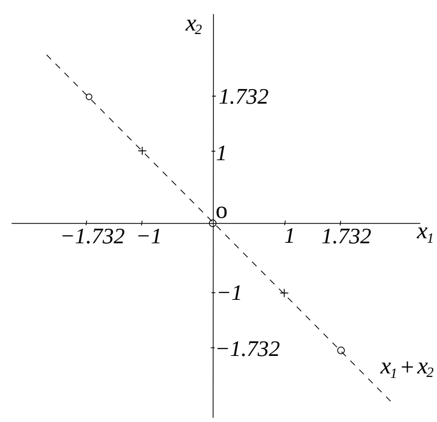
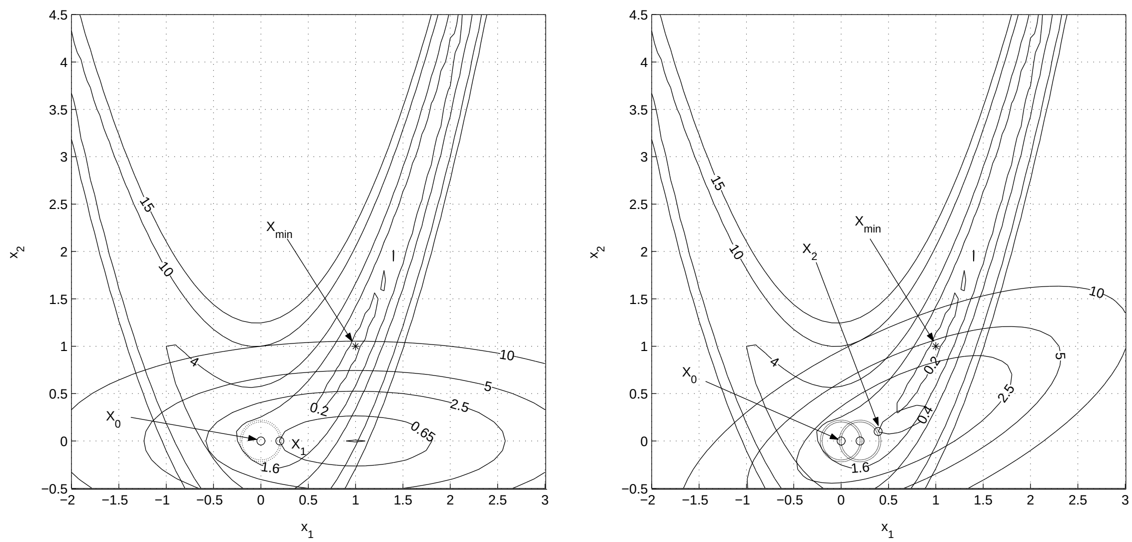


Figure A.6: Stationary points of a two-variable function (example)

6. (a) [Taylor's theorem in the remainder form needs to be used. Truncated series is not enough for the argument.]
(b) [In $\mathbf{x}$-space, take a line through $\mathbf{x}_2$ and use two points $\mathbf{y}$ and $\mathbf{z}$ on it on opposite sides from $\mathbf{x}_2$ as $\mathbf{x}_1$ in the above result.]
(c) $18x_1^2 + 1 \geq 18x_2$. [Impose positive semi-definiteness of the Hessian.]

7. (a) $p_1(\mathbf{x}) = x_1^2 + 9x_2^2 - 2x_1 + 1$. (b) See Fig. A.7(a). (c) $\mathbf{x}_1 : (0.2, 0)$.
(d) $p_2(x) = 3.16x_1^2 - 7.2x_1x_2 + 9x_2^2 - 2.576x_1 + 0.72x_2 + 1.0432$; see contours in Fig. A.7(b); $\mathbf{x}_2 : (0.39, 0.10)$.



(a) First step (b) Second step

Figure A.7: Trust region method

8. [Frame the given function in the form $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} - \mathbf{b}^T\mathbf{x} + c$, the gradient as $\mathbf{g}(\mathbf{x}) = \mathbf{Q}\mathbf{x} - \mathbf{b}$ and hence $\alpha_k = \frac{\mathbf{g}(\mathbf{x}_k)^T\mathbf{g}(\mathbf{x}_k)}{\mathbf{g}(\mathbf{x}_k)^T\mathbf{Q}\mathbf{g}(\mathbf{x}_k)}$. The minimum point is $(1, 1, 1)$ and hence $d_0 = \sqrt{3} = 1.732$.]
(a) 1, $[-2 \quad 0 \quad 0]^T$, $24\alpha^2 - 4\alpha + 1$. (b) 0.0833, $(0.166, 0, 0)$.
(c) and (d) See the table below.

k	$\mathbf{x}_k$	$f(\mathbf{x}_k)$	$\mathbf{g}(\mathbf{x}_k)$	α_k	$\mathbf{x}_{k+1}$	d_{k+1}
0	$(0, 0, 0)$	1	$(-2, 0, 0)$	0.083	$(0.166, 0, 0)$	1.64
1	$(0.166, 0, 0)$	0.833	$(0, -1, -0.667)$	0.191	$(0.166, 0.191, 0.127)$	1.45
2	$(0.166, 0.191, 0.127)$	0.695	$(-1.657, 0.020, -0.029)$	0.083	$(0.305, 0.190, 0.130)$	1.38

Chapter 22 (page 188)

1. Global minimum at $x^* = \frac{1}{7}[1 \quad 3 \quad 1 \quad 1]^T$ with $f(x^*) = 0$.
Comment: In the author's test run, Hooke-Jeeves method converged within 10^{-3} in 30

iterations, while steepest descent method took 82 iterations for a tolerance of 10^{-2}.

2. $(1.2, 1.2, 3.4)$; $f^* = -12.4$. [Starting simplex with vertices $\mathbf{0}$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ can be taken.]

3. [Use induction.] $\lim_{m\to\infty} |a_m| = 0$ indicates global convergence, while $\left|\frac{c-1}{c+1}\right| \approx 1$ for large c indicates poor local convergence.

4. $\mathbf{x}_0 = [2 \ \ 2]^T$, $f(\mathbf{x}_0) = 5$; $\mathbf{x}_1 = [1.8 \ \ 3.2]^T$, $f(\mathbf{x}_1) = 0.6416$.
It is a descent step, but in the solution ($\mathbf{x}$) space, the distance from the minimum point $(1, 1)$ actually increases. At this instance, a Cauchy (steepest descent) performs better, as visible in Fig. A.8, both in terms of improvement in function value and approach in the solution space. Newton and Cauchy steps are depicted by $\mathbf{x}_n$ and $\mathbf{x}_c$ respectively in the figure.
Comment: The poor performance of Newton's method here is related to the ill-conditioning of the Hessian.

5. (a) $x = 3.9094$, $y = 0.6969$. (b) $(-0.5652, 0.0570, 1.5652)$.
Comment: Author's test runs of Levenberg-Marquardt program took 12 iterations from $(0, 0)$ in the first case and 15 iterations from $(0, 0, 1)$ in the second case to converge within residual tolerance of 10^{-4}. [The second problem cannot be started from $(0, 0, 0)$.]

6. Attempt 1: For the 5-variable nonlinear least squares, starting from $[1 \ \ 0 \ \ 1 \ \ 1 \ \ -0.5]^T$, after 100 iterations, the result is $[-73.17 \ \ 16.16 \ \ -7.54 \ \ 94.50 \ \ 0.19]^T$, without convergence, with an error value of 6.5957. Still the agreement of data is not too bad. See Fig. A.9(a).
Attempt 2: For the single-variable problem in λ with an embedded 4-variable linear least square problem, starting with bounds $-5 \le \lambda \le 5$, the result is $[91.05 \ \ 5.30 \ \ -1.80 \ \ -71.08 \ \ -0.51]^T$, with error value 0.1961. See the match in Fig. A.9(b). The same result is obtained with the first approach, starting with $[100 \ \ 5 \ \ 0 \ \ -100 \ \ 0]^T$.

Chapter 23 (page 198)

1. Gradient $= \begin{bmatrix} 4x + y - 6 \\ x + 2y + z - 7 \\ y + 2z - 8 \end{bmatrix}$; Hessian $= \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$.
(a) $\mathbf{r}_3 = [1.5 \ \ 2.75 \ \ 2.625]^T$, $\|\mathbf{g}_3\| = 3.8017$, error in position: $[0.3 \ \ 1.55 \ \ -0.775]^T$.
(b) $\mathbf{r}_3 = [1.15 \ \ 1.70 \ \ 3.03]^T$, $\|\mathbf{g}_3\| = 0.7061$, error in position: $[-0.046 \ \ 0.5 \ \ -0.372]^T$.
(c) $\mathbf{r}_3 = [1.2 \ \ 1.2 \ \ 3.4]^T$, $\|\mathbf{g}_3\| = 0$, error in position: $[0 \ \ 0 \ \ 0]^T$.

2. **Comment:** Using restart option, Fletcher-Reeves algorithm converges to a tolerance of 10^{-4} in 19 iterations.

3. $p = 6.7$, $q = 6.9$, $r = 6.91$. [Denoting the points as A, B, C etc, notice that AB and CD are parallel. Then, DE, as extension of BD is the next (conjugate) direction. Next pair of searches are then EF parallel to CD and FG parallel to DE.]

4. Rank-one update: $a_k \mathbf{z}_k \mathbf{z}_k^T = \frac{\mathbf{r}_k \mathbf{r}_k^T}{\mathbf{q}_k^T \mathbf{r}_k}$, where $\mathbf{r}_k = \mathbf{p}_k - \mathbf{B}_k \mathbf{q}_k$. [For the proof, use induction.]
Comment: The result means that, over all previous steps, the current estimate $\mathbf{B}_{k+1}$ operates as the true Hessian inverse, for a convex quadratic problem.

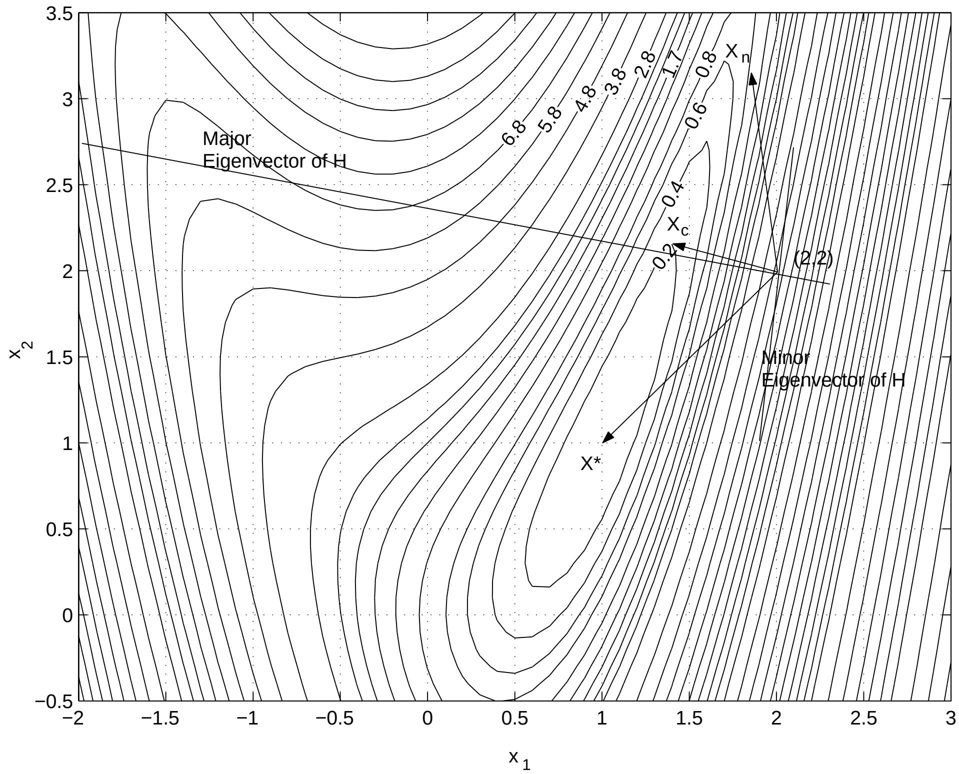
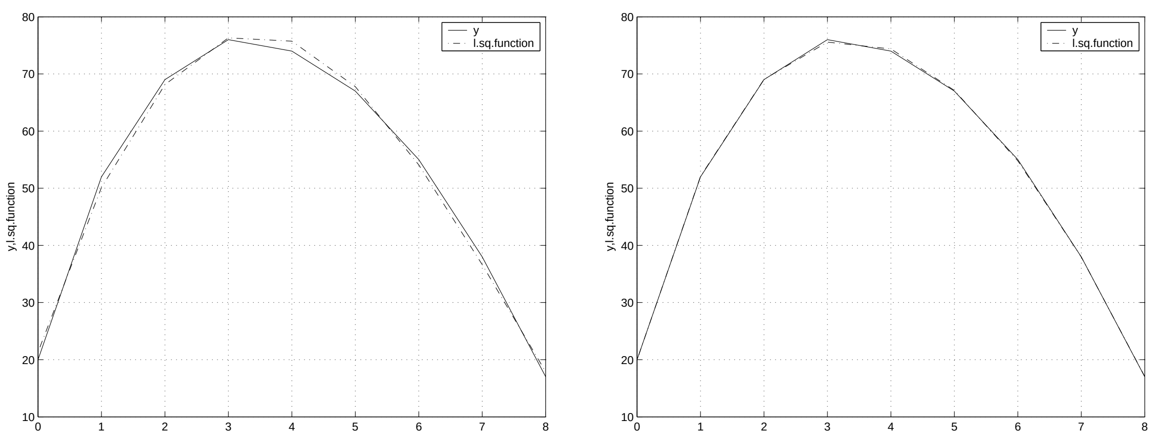


Figure A.8: Analysis of a Newton step



(a) Attempt 1: 5 variables together

(b) Attempt 2: split formulation

Figure A.9: Example of a nonlinear least square problem

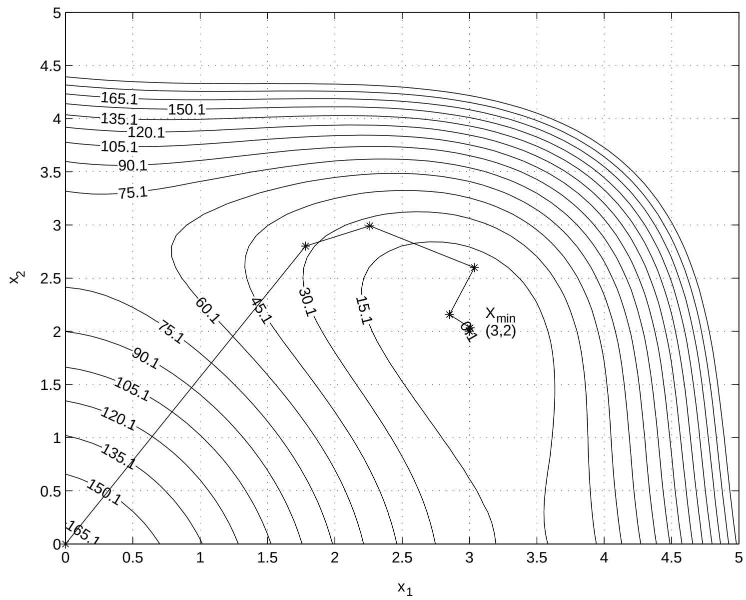


Figure A.10: Progress of DFP method on Himmelblau function

5. $\mathbf{x}^* = [3 \quad 2]^T$, $f(\mathbf{x}^*) = 0$. See Fig. A.10 for progress and contour. [At converged solution, the estimate $\mathbf{B}_k$ also converges and represents true inverse Hessian.]

Chapter 24 (page 208)

1. Denoting a point on the line segment as $\mathbf{s}$, a point on the triangle as $\mathbf{t}$ and the vector connecting them as $\mathbf{d}$, respectively; $\mathbf{s} = \mathbf{a} + x_1(\mathbf{b} - \mathbf{a}), \mathbf{t} = \mathbf{p} + x_2(\mathbf{q} - \mathbf{p}) + x_3(\mathbf{r} - \mathbf{p})$; and

$$\mathbf{d} = \mathbf{s} - \mathbf{t} = \mathbf{a} - \mathbf{p} + [\mathbf{b} - \mathbf{a} \quad \mathbf{p} - \mathbf{q} \quad \mathbf{p} - \mathbf{r}]\ \mathbf{x} \ = \ \mathbf{a} - \mathbf{p} + \mathbf{C}\mathbf{x}.$$
$$\text{Minimize } \ d^2 = \mathbf{x}^T\mathbf{C}^T\mathbf{C}\mathbf{x} + 2(\mathbf{a} - \mathbf{p})^T\mathbf{C}\mathbf{x} + \|\mathbf{a} - \mathbf{p}\|^2$$
$$\text{subject to} \quad x_1 \leq 1, \ x_2 + x_3 \leq 1, \ x_1, x_2, x_3 \geq 0.$$

KKT conditions: $2\mathbf{C}^T\mathbf{C}\mathbf{x} + 2\mathbf{C}^T(\mathbf{a} - \mathbf{p}) + \mu_1 \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T + \mu_2 \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T - \boldsymbol{\nu} = \mathbf{0}$; along with feasibility conditions and $\mu_1(x_1 - 1) = 0$, $\mu_2(x_2 + x_3 - 1) = 0$, $\nu_i x_i = 0$, $\boldsymbol{\mu}, \boldsymbol{\nu} \geq \mathbf{0}$.
A KKT point is a local minimum as well a global minimum. [The problem is convex.]

2. KKT conditions: $x_1 + x_2 - \frac{\pi}{2}x_1x_2\mu_1 - \mu_2 + \mu_3 = 0$, $x_1 - \frac{\pi}{4}x_1^2\mu_1 - \mu_4 + \mu_5 = 0$ and $g_i(\mathbf{x}) \leq 0$, $\mu_i g_i = 0$, $\mu_i \geq 0$ for $i = 1, 2, 3, 4, 5$. The area PQR in Fig. A.11 is the feasible domain. Point $P(12, 13.3)$ is a KKT point, while points $Q(10.3, 18)$ and $R(12, 18)$ are not.

3. Solution: $\mathbf{x}^*(15.81, 1.58)$, $f(\mathbf{x}^*) = 5$, $\mu_1 = 0.2, \mu_2 = 0$. [Through enumeration and verification with the KKT conditions, it is possible to conclude that g_1 is the only active constraint, i.e. $g_1(\mathbf{x}^*) = 0$ and $\mu_2 = 0$. Hessian turns out to be positive semi-definite. Check

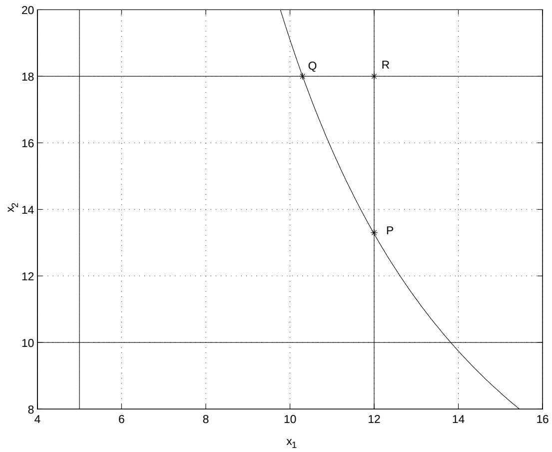


Figure A.11: Example on KKT points

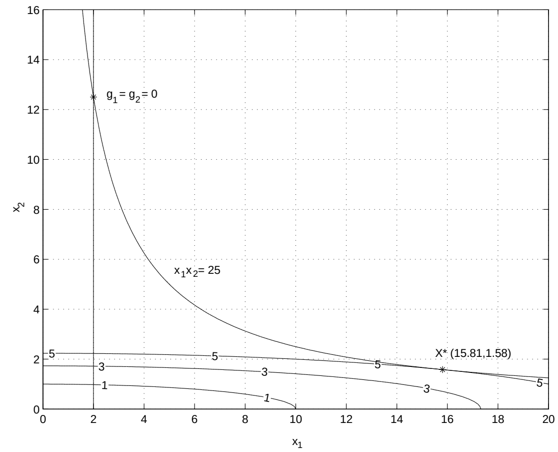


Figure A.12: Example of a convex problem

its restriction on the tangent subspace.]
Comment: Note that it is a convex problem. At $\mathbf{x}^*$, the constraint curve and the contour curve (shown in Fig. A.12) touch each other such that on one side of the common tangent we have feasible directions and on the other side are the descent directions.
(a) New optimal function value $\bar{f}^* = 5.2$.
(b) No change. [The second constraint is inactive.]

4. $\phi(\mu) = -\frac{5}{4}\mu^2 + 7\mu$; $\mu_{max} = 2.8$, $\phi_{max} = 9.8$; $\mathbf{x}_{min} = [0.2 \;\; 1.6]^T$, $f_{min} = 9.8$.
Comment: Note that $f_{min}(\mathbf{x}) = \phi_{max}(\mu)$ in Fig. A.13.

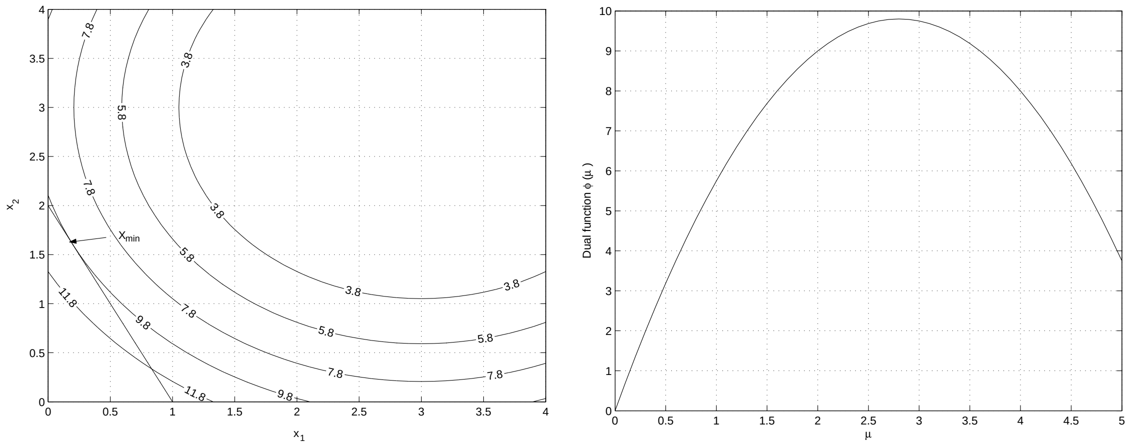
5. $\phi(\boldsymbol{\lambda}) = \lambda_2 L - l\sum_{i=1}^n \sqrt{(c_i+\lambda_1)^2 + \lambda_2^2}$. [It is a separable problem.]
Steepest ascent iteration for the dual: $r_i = \sqrt{(c_i+\lambda_1)^2+\lambda_2^2}$, $y_i = -\frac{l(c_i+\lambda_1)}{r_i}$, $x_i = \frac{l\lambda_2}{r_i}$;
$\nabla\phi(\boldsymbol{\lambda}) = \mathbf{h}(\mathbf{y}(\boldsymbol{\lambda})) = \begin{bmatrix} \sum_{i=1}^n y_i \\ L - \sum_{i=1}^n x_i \end{bmatrix}$; If $\|\nabla\phi\| \le \epsilon_G$, then STOP;
Line search to maximize $\psi(\boldsymbol{\lambda} + \alpha\nabla\phi)$ and update $\boldsymbol{\lambda} \leftarrow \boldsymbol{\lambda} + \alpha\nabla\phi$.
Solution: $\lambda_1 = -10$, $\lambda_2 = 6.76$, $\phi_{max} = -665.5$. [After 10 steepest ascent iterations.]

6. $x^* = 6$, $f(x^*) = -2$, $g(x^*) = 0$. [Unconstrained minimum at $x = 4$.]
Comment: See the progress of the method in Fig. A.14.

Chapter 25 (page 217)

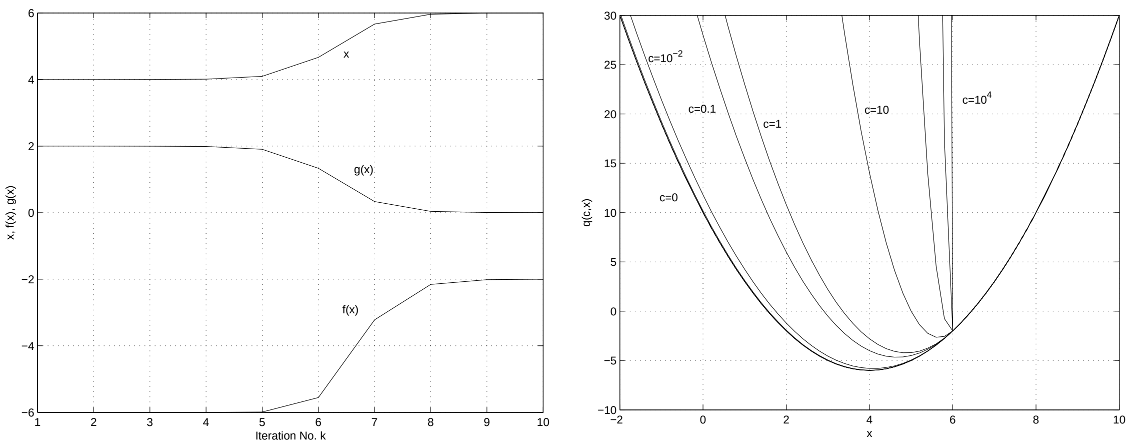
1. Maximize $\sum_{j=1}^n \alpha_j y_j$ subject to $\sum_{j=1}^n \beta_{i,j} y_j \le \log b_i$ for $i = 1, 2, 3, \cdots, m$, $\mathbf{y} \ge \mathbf{0}$, where $y_j = \log x_j$. (Justification: $y = \log x$ is a monotonically increasing function in the domain.)

2. (a) Solution point: $(-15, 1)$; function value: -15. [$x_1 = y_1 - y_2$, $x_2 = y_3 - y_4$, $\mathbf{y} \ge \mathbf{0}$.]
(b) Solution point: $(\frac{3}{4}, 0)$; function value: $\frac{9}{4}$. [Use two-phase method.]



(a) Scenario in **x**-space (b) Scenario in μ-space

Figure A.13: Example on duality: separable problem



(a) Approach through iterations (b) The penalized function

Figure A.14: Example of penalty method

3. (a) Unbounded. [From the intersection point $(-2.7143, -5.5715)$, an open direction is available along which the function can go up to $-\infty$.
(b) No feasible solution. [First phase terminates without eliminating the artificial variable.]

4. [To show that $\mathcal{H}$ is non-singular, show that $\mathcal{H}\begin{bmatrix}\mathbf{x}\\ \boldsymbol{\lambda}\end{bmatrix} = \mathbf{0} \Rightarrow \mathbf{x} = \mathbf{0}, \boldsymbol{\lambda} = \mathbf{0}$. To show indefiniteness, first use an eigenpair of $\mathbf{L}$ to show that $\mathcal{H}$ cannot be negative definite, and then

through a suitable choice of a vector $\begin{bmatrix} \mathbf{x} \\ \boldsymbol{\lambda} \end{bmatrix}$ argue that it cannot be positive definite either.]

5. (i) Point on line segment as well as on triangle: $(2.66, 2.32, 2.66)$, distance zero (intersection).
(ii) Point on line segment: $(1, 3, 8)$, point on triangle: $(0, 0.96, 5.28)$, distance: 3.544.
(iii) Point on line segment: $(6.89, 4.12, 1.33)$, point on triangle: $(6.10, 3.12, 0)$, distance: 1.84.
[Refer to the formulation in the solution of the first exercise in the previous chapter, and use either active set method or Lemke's method to solve the resulting QP problem. Since there are only inequality constraints, Lemke's method is more straightforward to apply in this case.]
Comment: In the third case, the Hessian is singular (positive semi-definite), as the line segment is parallel to the plane of the triangle. As such, infinite solutions are possible, all of which are global minima. A different pair of points, having the same closest distance is valid. See Fig. A.15 for the relative disposition of the triangle and the three line segments.

6. Successive iterates: $\mathbf{x}_0(0,0) \to \mathbf{x}_1(0.2,0) \to \mathbf{x}_2(0.4,0.2) \to \mathbf{x}_3(0.6,0.4)\cdots$, function values: $f_0 = 1$, $f_1 = 0.6544$, $f_2 = 0.3744$, $f_3 = 0.1744$.
Comment: Further iterations should approach the (global) minimum at $(1,1)$, but the speed and precision may suffer due to the ill-conditioning of the function. See Fig. A.16.

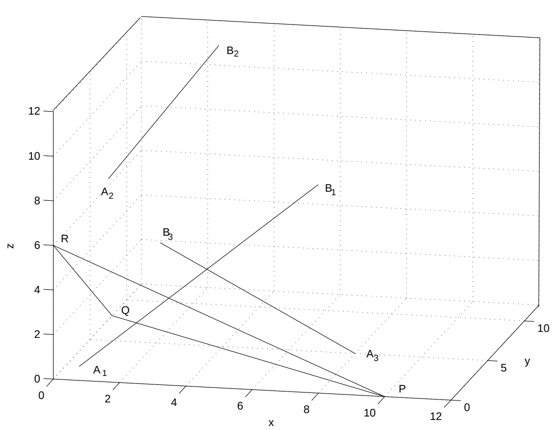


Figure A.15: Triangle and line segments

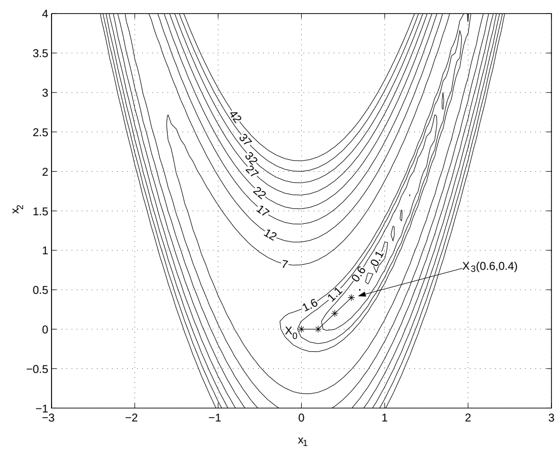
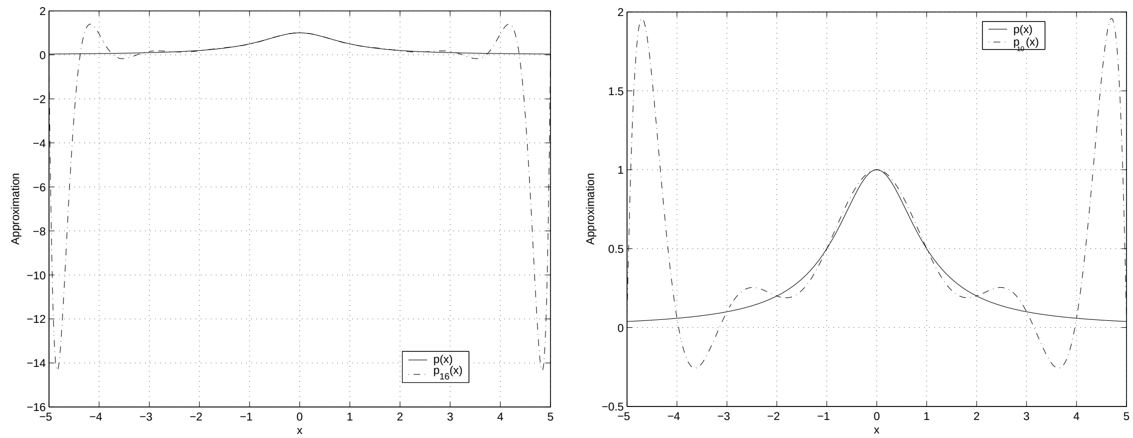


Figure A.16: Progress of trust region method

7. (a) Lagrange multiplier value: $-\frac{3}{2}$. [Show feasibility of $[1 \ \ 0]^T$ and minimality in the feasible directions available at this point.]
(b) $[f(\mathbf{x}_k) = -\cos\theta \approx -1 = f(\mathbf{x}^*).]$
(c) Quadratic program: minimize $\frac{1}{2}\mathbf{d}_k^T\mathbf{H}\mathbf{d}_k + \mathbf{g}_k^T\mathbf{d}_k$ subject to $[\cos\theta \quad \sin\theta]\mathbf{d}_k = \mathbf{0}$, where $\mathbf{H} =$ diag $(4,4)$ and $\mathbf{g}_k = [4\cos\theta - 1 \quad 4\sin\theta]^T$.
Solution: $\mathbf{d}_k = \frac{1}{4}[\sin^2\theta \quad -\sin\theta\cos\theta]^T$, $\lambda_k = \cos\theta - 4$.
(d) $\mathbf{x}_{k+1}$ violates the constraint at the second order. $f(\mathbf{x}_{k+1}) < f(\mathbf{x}_k)$.
(e) In an active set method, the necessity of maintaining the feasibility of iterates may be a costly impediment.

Chapter 26 (page 227)

1. The maximum deviations of Lagrange interpolations [$p_{16}(x)$ and $p_{10}(x)$] with 17 and 11 equally spaced points are found to be 14.39 and 1.92, respectively. (See Fig. A.17).
   Comment: This shows the futility of a very high degree of interpolation. From the shape of the curve, even the 10th degree polynomial approximation $p_{10}(x)$ shown may be unacceptable for many applications. [The same result should be possible to obtain from an interpolation scheme in the form of Eqn. 26.1. But, in the event of ill-conditioning of the resulting coefficient matrix, the result may not be reliable.]



(a) Polynomial of 16th degree (b) Polynomial of 10th degree

Figure A.17: Pathology of Lagrange interpolation

2. $\mathbf{W} = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}.$
   [By using boundary conditions, solve coefficients of Eqn. 26.5 (page 222) and express them as linear combinations of *boundary values*. Alternatively, inserting boundary values in the proposed expression and its derivative, a square system is developed for $\mathbf{W}$.]

3. For each segment, two points (positions) are known: $q_j(x_{j-1}) = f_{j-1}$ and $q_j(x_j) = f_j$ — a total of $2n$ conditions. The continuity of the first derivative at each knot point (where two segments meet) gives one condition, namely $q'_j(x_j) = q'_{j+1}(x_j)$ — a total of $(n-1)$ conditions. One further condition needs to be *artificially* imposed to fix all $3n$ coefficients in the n quadratic segments.
   For the given case, using the extra conditions as $q'_1(3) = 0$, i.e. zero initial slope, $q_1(x) = -0.6667x^2 + 4x - 3.5$ for $3 \leq x \leq 4.5$, $q_2(x) = 1.04x^2 - 11.36x + 31.06$ for $4.5 < x < 7$ and $q_3(x) = -2.1x^2 + 32.6x - 122.8$ for $7 \leq x \leq 9$. See Fig. A.18 for the resulting function

interpolation.
Comment: A different prescription of the free condition will give a different interpolation.

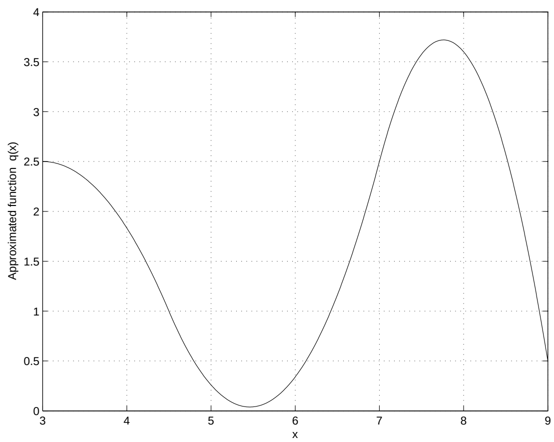


Figure A.18: Piecewise quadratic interpolation

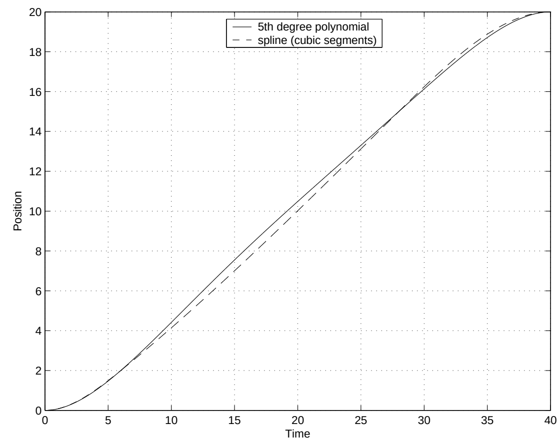


Figure A.19: Position schedule by interpolation

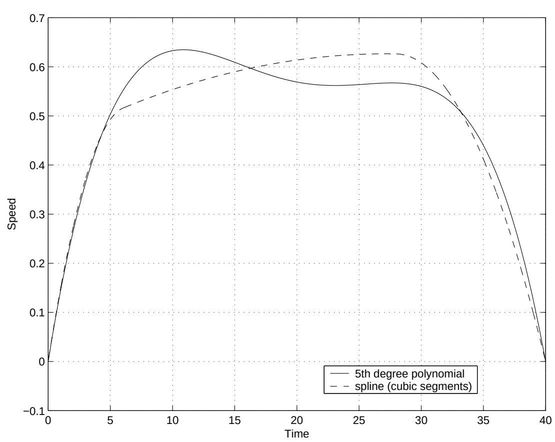


Figure A.20: Speed schedule of the interpolation

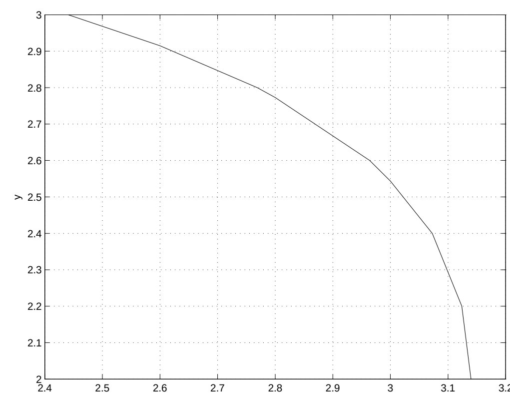


Figure A.21: Piecewise linear curve

4. Single Hermite polynomial: $x(t) = 0.0785t^2 - 0.0045t^3 + 0.000116t^4 - 1.13 \times 10^{-6}t^5$. [Assume a quintic function form and use the values of $x(0)$, $x'(0)$, $x(6)$, $x(28)$, $x(40)$ and $x'(40)$.]
For the cubic spline interpolation, intermediate derivative values: $x'(6) = 0.5157$ and $x'(28) =$

0.6262. Resulting schedule:

$$x(t) = \begin{cases} 0.0807t^2 - 4.19 \times 10^{-3}t^3 & \text{for} \quad 0 \le t \le 6, \\ -0.888 + 0.444t + 6.72 \times 10^{-3}t^2 - 0.0824 \times 10^{-3}t^3 & \text{for} \quad 6 < t < 28, \quad \text{and} \\ 28.9 - 2.745t + 0.1206t^2 - 1.438 \times 10^{-3}t^3 & \text{for} \quad 28 \le t \le 40. \end{cases}$$

Superimposed plots of the two (position) schedules and their rates are shown in Figs. A.19 and A.20, respectively. Both are practically acceptable schedules.

5. [Divide the domain into rectangular patches and work out bilinear interpolations. For a particular patch, if there are function values of different signs at the four vertices, then setting the function value to zero in this patch yields one segment of the curve. Its intersection with the patch boundaries gives the points to be used for the piecewise linear approximation.]
Comment: Actual representation depends on the discretization used. One possible approximation can be obtained by joining the following points. (See Fig. A.21.)

x	3.140	3.124	3.073	3.000	2.964	2.800	2.769	2.600	2.441
y	2.000	2.200	2.400	2.544	2.600	2.773	2.800	2.915	3.000

6. The results of part (b) and part (c) should exactly match.
(d) An affine transformation on the control points is equivalent to the same affine transformation on the entire curve. This is a property of the B-spline basis functions. See the original curve and its transformation along with the control polygon in Fig. A.22.
Comment: This means that the curve becomes 'portable' at low cost. Any positioning and re-orientation of the curve can be accomplished by giving the corresponding transformation to the handful of control points, and the curve can be regenerated at the new location.

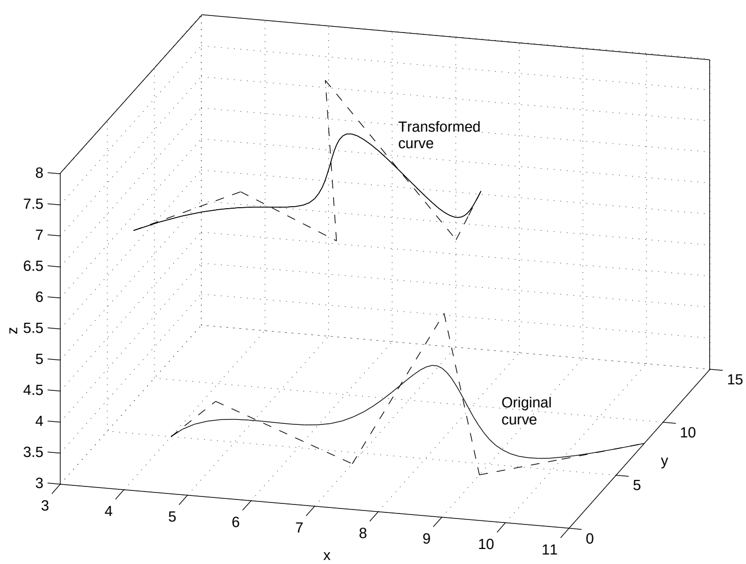


Figure A.22: Affine transformation of B-spline curve

7. (a) $\mathbf{N}_4(u) = [N_{0,4}(u) \quad N_{1,4}(u) \quad \cdots \quad N_{m,4}(u)]^T$; $\quad \mathbf{N}_4(v) = [N_{0,4}(v) \quad N_{1,4}(v) \quad \cdots \quad N_{n,4}(v)]^T$.
(b) $\mathbf{C} = [\mathbf{N}_4(u_0) \quad \mathbf{N}_4(u_1) \quad \cdots \quad \mathbf{N}_4(u_M)]$; $\quad \mathbf{D} = [\mathbf{N}_4(v_0) \quad \mathbf{N}_4(v_1) \quad \cdots \quad \mathbf{N}_4(v_N)]$.
$\mathbf{A}$ is an $(M+1)(N+1) \times (m+1)(n+1)$ matrix arranged in $(N+1) \times (n+1)$ blocks, the i-j block being given by the $(M+1) \times (m+1)$ matrix $d_{ji}\mathbf{C}^T$.
(c) Solution $\mathbf{p} = \mathbf{A}^{\#}\mathbf{q}$ leads to a least square fit of the surface data.

Chapter 27 (page 237)

1. (a) $p(s) = \frac{1}{2}s(s-1)f_0 - (s-1)(s+1)f_1 + \frac{1}{2}s(s+1)f_2$. (b) $\frac{h}{3}[f_0 + 4f_1 + f_2]$.
(c) $-\frac{h^5}{90}f^{iv}(x_1)+$ higher order terms; or $-\frac{h^5}{90}f^{iv}(\xi)$ for some $\xi \in [x_0, x_2]$.

2. (a) $e(h) = [2 - w_0 - w_1 - w_2]f(x_1)h + [w_0 - w_2]f'(x_1)h^2 + [\frac{1}{3} - \frac{w_0}{2} - \frac{w_2}{2}]f''(x_1)h^3 + \frac{1}{6}[w_0 - w_2]f'''(x_1)h^4 + [\frac{1}{60} - \frac{w_0}{24} - \frac{w_2}{24}]f^{iv}(x_1)h^5 + \cdots$.
(b) $w_0 = w_2 = \frac{1}{3}$, $w_1 = \frac{4}{3}$. (c) Zero and $-\frac{f^{iv}(x_1)}{90}$.

3. 2.495151. [May use $h = 0.005,\ 0.001,\ 0.0002$.]
Comment: With $h = 0.0002$, raw formula gives an accuracy of 6 places of decimal, but cannot detect on its own the accuracy that has been achieved. Richardson extrapolation gives the same accuracy with the first two of these values, and an accuracy of 10 places with the third. Analytical value: 2.4951508355.

4. (a) $I_4 = 0.697023810$, $I_8 = 0.694121850$, $I_{16} = 0.693391202$; $I_R = 0.693147194$ correct to seven places of decimal, as the analytically available value is $\ln 2 = 0.693147180559945$. [Results of intermediate extrapolations: 0.693154531 (between I_4 and I_8) and 0.693147653 (between I_8 and I_{16}).]
Comment: With a miserly implementation starting with a single sub-interval (the entire interval), the same 17 function values lead to the Romberg integral value 0.693147181916 that is correct to 8 places of decimal, through four stages of Richardson extrapolation.
(b) $I_4 = 0.944513522$, $I_8 = 0.945690864$, $I_{16} = 0.945985030$; $I_R = 0.946083070351$ correct to ten places of decimal, as the sine-integral function $\mathrm{Si}\,(t) = \int_0^t \frac{\sin x}{x}dx$ gives $\mathrm{Si}\,(1) = 0.946083070367183$. [$\frac{\sin x}{x}$ needs to be specially defined as 1, in the limit.]
Comment: Starting from a single interval, four stages of Richardson extrapolation uses the same 17 function values to give the Romberg integral as 0.946083070367182 that is correct to 14 places of decimal.

5. Integral: $\frac{hk}{9}[(f_{0,0} + f_{2,0} + f_{0,2} + f_{2,2}) + 4(f_{1,0} + f_{1,2} + f_{0,1} + f_{2,1}) + 16f_{1,1}]$;
Error: $-\frac{hk}{45}\left[h^4\frac{\partial^4 f}{\partial x^4}(\xi_1, \eta_1) + k^4\frac{\partial^4 f}{\partial y^4}(\xi_2, \eta_2)\right]$ for some $\xi_1, \xi_2 \in [x_0, x_2], \eta_1, \eta_2 \in [y_0, y_2]$.

Chapter 28 (page 244)

1. $\frac{1}{135}f^{iv}(0)$.

2. $\int_{-1}^1 f(x)dx = \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right)$.

3. Simpson's one-third rule: integral estimate $I_s = 0.4119$, error $e_s = -0.0440$.
Two-point Gauss-Legendre rule: integral estimate $I_2 = 0.4876$, error $e_2 = 0.0317$.

Three-point Gauss-Legendre rule: integral estimate $I_3 = 0.4532$, error $e_3 = -0.0027$.
[Analytical value $I = 0.4559$, from $\left[\frac{-e^{-x}(\sin 2x+2\cos 2x)}{5}\right]_0^2$.]

4. Such a formula is not possible. Condition of exact integration of x^4 and y^4 conflicts with that of x^2y^2. An attempt to impose the necessity of exact integration of x^4 and y^4 leads to the five-point formula

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)dxdy = w_0 f(0,0) + w[f(-a,-a)+f(-a,a)+f(a,-a)+f(a,a)],$$

where $w_0 = 16/9$, $w = 5/9$ and $a = \sqrt{3/5}$; but it compromises the condition of exact integration of x^2y^2.

5. 0.2207 (exact solution: 0.2197). [Quadrature points: $\frac{1}{2} - \frac{1}{2}\sqrt{\frac{3}{5}}$, $\frac{1}{2}$ and $\frac{1}{2} + \frac{1}{2}\sqrt{\frac{3}{5}}$.]
Comment: The integral can be evaluated as $\int_0^1 \frac{e^{x^2}-e^{x^3}}{x}dx$ and in the limit expanded in the convergent series $\frac{1}{6}\sum_{n=1}^{\infty}\frac{1}{n!n}$.

6. 213 cubic units. [Apply Monte Carlo method with 10^4, 10^5 and 10^6 points. Using symmetry, better result can be obtained by working on only one octant first, e.g. $V = \{(x,y,z):\ x \in [0,5],\ y \in [0,4],\ z \in [0,2]\}$.]

Chapter 29 (page 252)

1. See solution in Fig. A.23.
Comment: The first order (Euler) method splits early from the analytical solution. The second order (Euler) method splits much later when the solution rises sharply. (See hazy fringe near $x = 2$.) The fourth order (Runge-Kutta) method tracks the solution so closely (in this interval) that splitting is not visible.

2. $y(5) = -0.428$, $y'(5) = 0.512$. See Fig. A.24 for a plot.
Comment: Marching further in t, you get the trajectory in phase plane (i.e. plane of y and y') as in Fig. A.25, that suggests a limit cycle inside the loop of the trajectory. (See Chap. 38 to know more about limit cycles.)

3. By the choice of $w_2 = 3/4$, $w_1 = 1/4$, $\alpha = \beta = 2/3$ (refer page 247), three terms of the third order error get eliminated and only $-\frac{h^3}{6}(f_x f_y + f f_y^2)$ remains.

4. [Compare with the true trajectory in the form $q(x) = q_0 + \sum_{j=1}^{\infty} b_j (x-x_0)^j$ up to the cubic term.]

5. See solution in Fig. A.26.
Comment: With the necessity to ensure the accuracy, the method needs to adapt to the appropriate step sizes, different in different segments of the domain. (Note the unequal density of nodes in the figure.)

6. Adams-Bashforth formula: $y_{n+1} = y_n + \frac{h}{12}[23f(x_n,y_n) - 16f(x_{n-1},y_{n-1}) + 5f(x_{n-2},y_{n-2})]$.
Adams-Moulton formula: $y_{n+1} = y_n + \frac{h}{12}[5f(x_{n+1},y_{n+1}) + 8f(x_n,y_n) - f(x_{n-1},y_{n-1})]$.

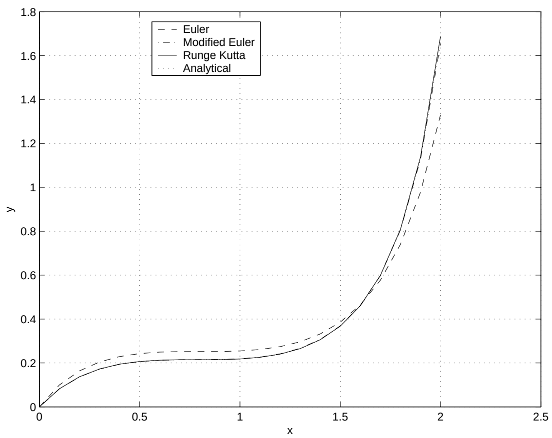


Figure A.23: Explicit methods on 1st order ODE

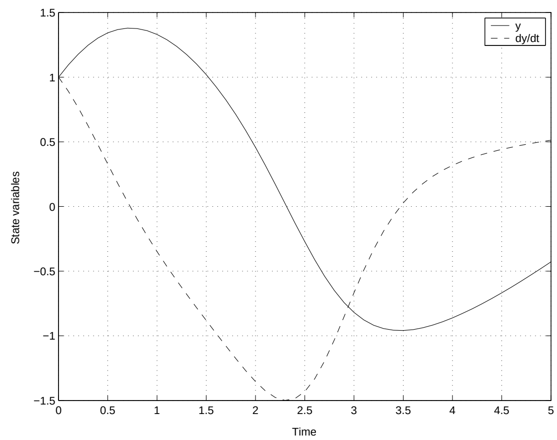


Figure A.24: R-K method on 2nd order ODE

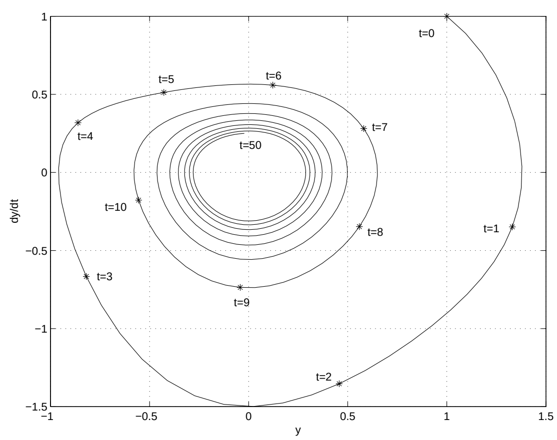


Figure A.25: Trajectory in phase plane

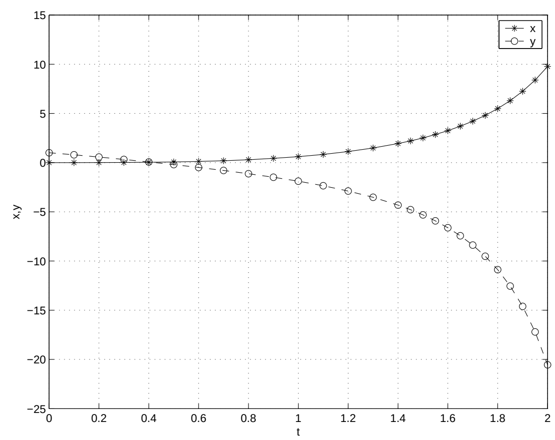


Figure A.26: Adaptive Runge-Kutta method

7. (a) $\dot{\mathbf{z}} = \left[z_3 \quad z_4 \quad z_1 z_4^2 - \frac{k}{z_1^2} \quad -\frac{2z_3z_4}{z_1}\right]^T, \quad z_1(0) = 1,\ z_2(0) = 0,\ z_1(1) = 1.5,\ z_2(1) = \frac{3\pi}{4}.$
   (b) $\Delta r = 0.0234$, $\Delta\theta = -0.1109$. [Approximate initial conditions: $z_3(0) = 0$, $z_4(0) = \frac{365\pi}{360}$.]
   (c) $\mathbf{e} = [\Delta r \quad \Delta\theta]^T = \mathbf{f}(\mathbf{v}),\ \ \mathbf{v} = [z_3(0) \quad z_4(0)]^T.$ Solve $\mathbf{f}(\mathbf{v}) = \mathbf{0}$.
   (d) Jacobian $\mathbf{J} = \left[\frac{\mathbf{e}_{\mathrm{I}} - \mathbf{e}}{\delta} \quad \frac{\mathbf{e}_{\mathrm{II}} - \mathbf{e}}{\delta}\right]$.
   (e) $\mathbf{e}_{new} = [0.0233 \quad -0.1108]^T$. [$\mathbf{v}_{new} = \mathbf{v} - \mathbf{J}^{-1}\mathbf{e}$.]
   Comment: Progress of the shooting method in this case is quite slow.

Chapter 30 (page 262)

1. $\left|1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}\right|<1.$ [Compare $\Delta_{n+1}=y_{n+1}-y(x_{n+1})$ against Δ_n.]

2. See solution in Fig. A.27.
Comment: The solution is with a step size of $h=2.4$. Eigenvalues of the Jacobian are ± 1 and $\pm 100i$. (This is a stiff equation.) Due to the eigenvalue $+1$, there is a step size limitation of $h>2$, below which the method may be unstable.

3. $f''(0)=0.4696.$ [Solve $\phi(v)=0$ if $\phi(v)=\bar{f}(\eta_{free})$ where $\bar{f}(\eta)$ is the solution of the ODE with initial conditions $f(0)=0, f'(0)=0, f''(0)=v$.]
In Fig. A.28, the solution is shown up to $\eta=5$ only, for the sake of clarity.
Comment: The result shows that taking $\eta_{free}=5$ is enough, because the solution has already reached the free-stream condition at $\eta=5$.

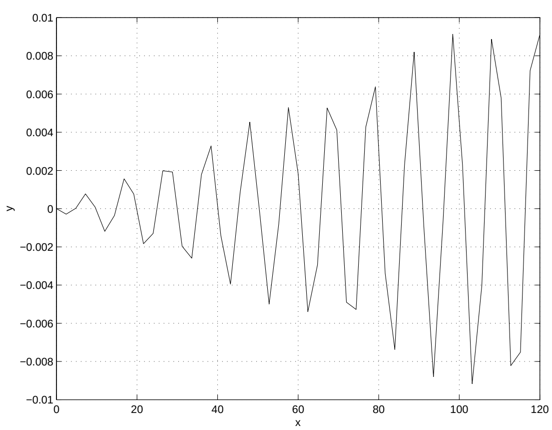


Figure A.27: Backward method on stiff ODE

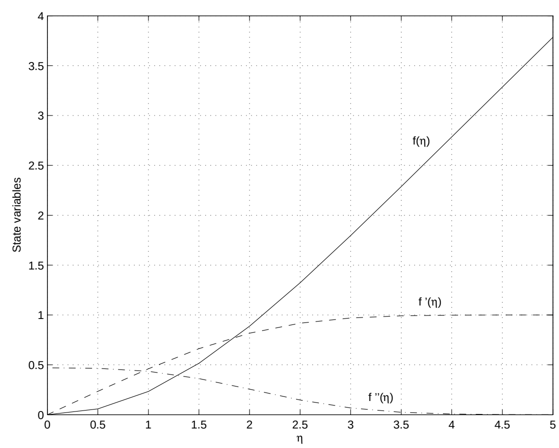


Figure A.28: Blasius solution by shooting

4. [$y(x)=(1-w)y^{(0)}(x)+wy^{(1)}(x)$ is the solution, where $y^{(0)}(x)$ and $y^{(1)}(x)$ are two solutions of the ODE with $y(a)=p_1$ and arbitrary values of $y'(a)$, and w is determined from $(1-w)y^{(0)}(b)+wy^{(1)}(b)=p_2$.]
Generalization:

   - Take vectors $\mathbf{q}_j\in R^{n_2}$, $j=0,1,2,\cdots,n_2$ such that vectors $\mathbf{q}_i-\mathbf{q}_0$ are linearly independent for $i=1,2,\cdots,n_2$.
   - With each $\mathbf{q}_j$, complete the initial conditions and define n_2+1 IVP's.
   - Solve these IVP's and denote the terminal conditions as $\mathbf{r}_j=\mathbf{y}_2^{(j)}(t_f)$.
   - Solve the linear system $\sum_{j=0}^{n_2}w_j\mathbf{r}_j=\mathbf{b}$, $\sum_{j=0}^{n_2}w_j=1$ for weights w_j.
   - Compose the solution as $\mathbf{y}(t)=\sum_{j=0}^{n_2}w_j\mathbf{y}^{(j)}(t)$. Show its validity.

5. $y(1) = 0.20$, $y(1.6) = 1.18$ and $y(2) = 2.81$. [From the ODE $ky^{iv} + 2k'y''' + k''y'' = w(x)$, develop state space equation and hence FDE's. With central difference formulation, Richardson extrapolation will give $y = (4y^{(2)} - y^{(1)})/3$.]

6. (a) $\mathbf{Pf} = \mathbf{PM\ddot{q}}$, where $\mathbf{P} = \mathbf{I} - \mathbf{J}^T(\mathbf{JJ}^T)^{-1}\mathbf{J}$. (See exercise 9 of Chap. 4.)
It implies that the component of the driving force in the tangent plane of constraints is active in producing acceleration.
(b) $\mathbf{J(q)\dot{q}} = \mathbf{0}$ and $\mathbf{J(q)\ddot{q}} + \frac{\partial}{\partial \mathbf{q}}[\mathbf{J(q)\dot{q}}]\mathbf{\dot{q}} = \mathbf{0}$.
(c) With state vector $\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{q} \\ \mathbf{\dot{q}} \end{bmatrix}$, state space equation:

$$\mathbf{\dot{z}} = \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{M}^{-1}[\mathbf{f} - \mathbf{J}^T(\mathbf{JM}^{-1}\mathbf{J}^T)^{-1}\{\mathbf{JM}^{-1}\mathbf{f} + \frac{\partial}{\partial \mathbf{z}_1}(\mathbf{Jz}_2)\mathbf{z}_2\}] \end{bmatrix},$$

where $\mathbf{M(q)} \equiv \mathbf{M}(\mathbf{z}_1)$, $\mathbf{J(q)} \equiv \mathbf{J}(\mathbf{z}_1)$ and $\mathbf{f}(\mathbf{q}, \mathbf{\dot{q}}) \equiv \mathbf{f}(\mathbf{z}_1, \mathbf{z}_2)$.
Comment: Solution of the state space equation as IVP in this form satisfies the constraints, as the constraints turn out to be the *invariants*.

Chapter 31 (page 270)

1. (a) Solution does not exist. (b) Solution $y = cx$ is not unique; Lipschitz condition is not satisfied at $(0, 0)$. (c) Solutions are $y = x^2/4$ and $y = 0$, not unique. Lipschitz condition is not satisfied, solution is sensitive to initial condition. (d) Lipschitz condition is satisfied, IVP is well-posed; but $\frac{\partial f}{\partial y}$ does not exist for $y = 0$. (e) $\frac{\partial f}{\partial y} = -P(x)$ is continuous and bounded. As a cosequence, Lipschitz condition is satisfied and the IVP is well-posed.

2. (a) $y_1(x) = 1 + x + \frac{1}{2}x^2$. (b) $y_2(x) = 1 + x + \frac{3}{2}x^2 + \frac{2}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{20}x^5$.
(c) $y_3(x) = 1+x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{13}{12}x^4+\frac{49}{60}x^5+\cdots$. $y_4(x) = 1+x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{17}{12}x^4+\frac{17}{12}x^5+\cdots$.
(d) Convergence of the sequence of functions $\{y_i(x)\}$ implies the existence of a solution of the given IVP.
Comment: In this particular case, the sequence does converge to the series of the actual solution, which is

$$y(x) = 1 + x + \frac{3}{2}x^2 + \frac{4}{3}x^3 + \frac{17}{12}x^4 + \frac{31}{20}x^5 + \cdots.$$

(e) **Picard's method:** Assume $y_0(x) = y_0$ and develop the sequence of functions

$$\begin{aligned}
y_1(x) &= y_0 + \int_{x_0}^{x} f[t, y_0(t)]dt, \\
y_2(x) &= y_0 + \int_{x_0}^{x} f[t, y_1(t)]dt, \\
y_3(x) &= y_0 + \int_{x_0}^{x} f[t, y_2(t)]dt, \\
\cdots &\quad \cdots \quad \cdots \quad \cdots, \\
y_{k+1}(x) &= y_0 + \int_{x_0}^{x} f[t, y_k(t)]dt,
\end{aligned}$$

If the sequence converges to $y^*(x)$, then it is a solution of the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f[t, y(t)]dt,$$

and hence of the original initial value problem (Eqn. 31.1).

Chapter 32 (page 280)

1. (a) $x^2 \cos y + x^4 = c$. (b) $x^2 - 3xy - y^2 + 2x - 5y + c = 0$.
   (c) $3\ln(14x + 21y + 22) = -7x + 14y + k$. (d) $y^2 = \frac{x}{2}e^x + cxe^{-x}$.
   (e) $x^y = e^{-x^3/3}$. (f) $e^x \sin^2 y = \sin y - y\cos y + c$.
   (g) $e^y \cos(x + c) = k$. (h) $y = \frac{A}{1-e^{Ax+B}}$.

2. Differential equation: $y = xy' - e^{y'}$. [Clairaut's form.]
   Complete solution: general solution $y = mx - e^m$, singular solution $y = x\ln x - x$.

3. $x^2 + 2y^2 = c^2$. [Family of ellipses.] $x^2 + ny^2 = c$. [Family of central conics.]
   Steepness of the curves of the original family is directly related to the eccentricity of the curves in the orthogonal family. [Make interpretations for $n > 2$, $1 \le n \le 2$, $0 < n < 1$ and $n < 0$.]
   Comment: Interpretation may exclude negative coordinate values in the case of fractional powers.

4. It is valid, but no extra generality is to be gained by the alternative.

5. (a) $\frac{dA}{dt} = D + rA$. (b) $\frac{D}{r}(e^{rt} - 1)$. (c) Approximately Rs. 59,861.
   [Continuously: approximately Rs 7 every hour.]
   Comment: Even without the ODE of part (a), the result of part (b) can be arrived at, as the limit of a sum.

6. $y = \frac{x^{1-s/v} - x^{1+s/v}}{2}$ with $(1, 0)$ as the starting point and the tower at the origin ($x = 1$ and $x = 0$ being the banks). [From the swimmer's general position, work out his attempted velocity vector towards the tower, and hence the resultant velocity. Eliminate time from the expressions for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ to develop the differential equation of the trajectory.]
   Comment: By varying the speed ratio $r = s/v$, one can determine several trajectories. (The case of $r = 1$ is indeterminate.) Fig. A.29 depicts a few of them.

7. (a) Bernoulli's equation: $z' + [I + s(t)]z = Iz^2$, where $z = 1 - y$ is the fraction of people who **do not** use cellphone.
   Solution: $y = 1 - \frac{(1-y_0)R(t)}{R(0)-I(1-y_0)S(t)}$, where $y_0 = y(0)$ is the initial condition and $P(t) = -[I + s(t)]$, $Q(t) = \int P(t)dt$, $R(t) = e^{Q(t)}$ and $S(t) = \int_0^t R(\tau)d\tau$.
   (b) 32.3 %. [$I = 0.05$, $y_0 = 0.2$, $a = Iy_0 = 0.01$, $b = c = 0.001$.]
   Comment: This model predicts almost complete spread within 20 years! See Fig. A.30.

Chapter 33 (page 289)

1. [Simply try to complete the analysis of page 284, in the manner suggested in the text.]

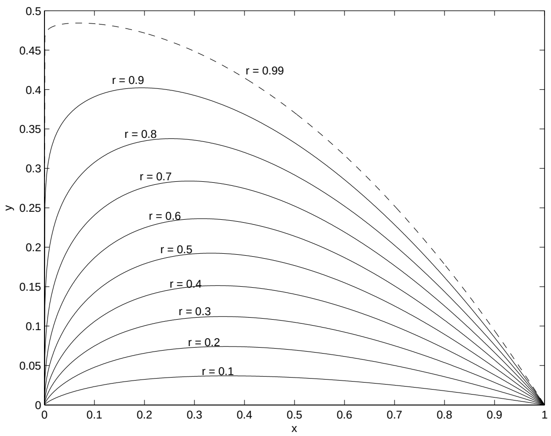


Figure A.29: Swimmer's trajectories

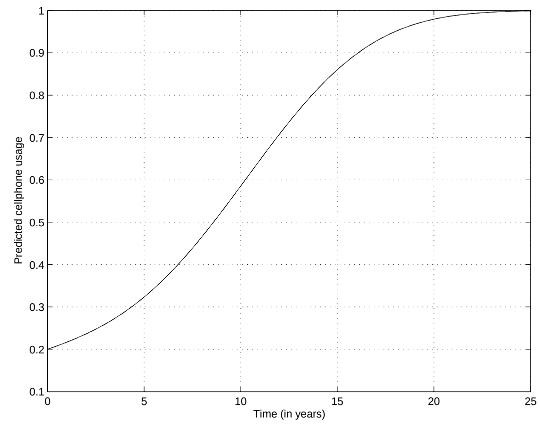


Figure A.30: Fraction of cellphone users

2. (a) $y = e^{-x}(c_1 \cos x + c_2 \sin x)$ or $Ae^{-x}\cos(x-\alpha)$. (b) $y = e^{-x/2}$. (c) $y = 3x^2 \ln x$.

3. $P(x) = -\frac{y_1 y_2'' - y_2 y_1''}{W(y_1,y_2)}$, $Q(x) = \frac{y_1' y_2'' - y_2' y_1''}{W(y_1,y_2)}$. [Substitute y_1 and y_2 into the ODE, and simply solve P and Q from the resulting equations.]
Comment: It is important to note that the Wrosnkian is non-zero.

4. [*Without loss of generality*, consider them as *consecutive* zeros. (Why?) Then, from sign change of y_1' and sign invariance of the Wronskian, argue out the sign change of y_2.
Alternatively, assuming function $\bar{y}(x) = \frac{y_1(x)}{y_2(x)}$ to be continuous and differentiable, and applying Rolle's theorem on it leads to a contradiction, implying the zero-crossing of $y_2(x)$.]
Comment: Some further arguments in this line leads to the **Sturm separation theorem:** "If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the equation $y'' + P(x)y' + Q(x)$, then zeros of $y_1(x)$ and $y_2(x)$ occur alternately."

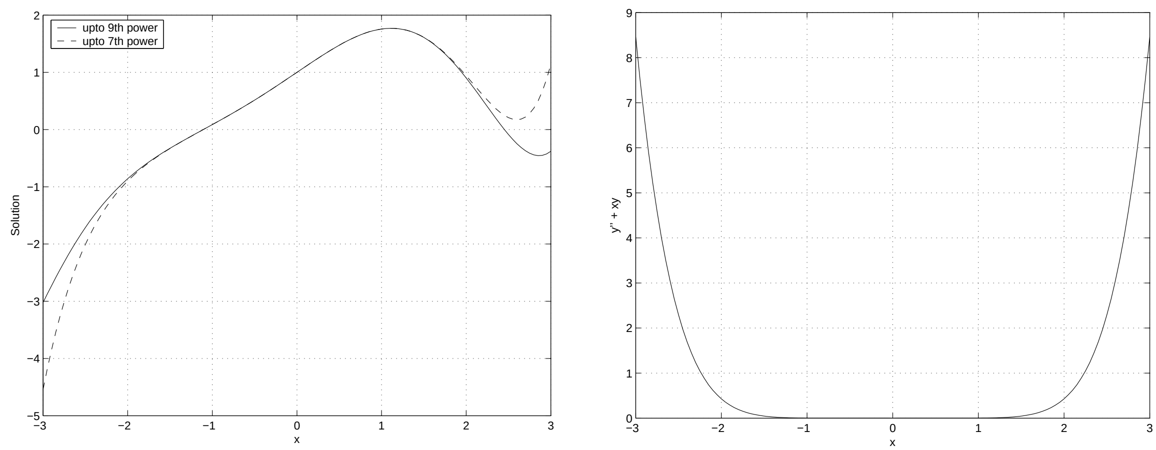
5. $y(t) = [y_0 + (y_1 + ay_0)t]e^{-at}$. [Start with the model $\ddot{y} + c\dot{y} + a^2 y$ and show $c = 2a$ for critical damping.]
(a) At time $\bar{t} = -\frac{y_0}{y_1 + ay_0}$, under the condition that y_1 has the opposite sign of y_0 and $|y_1| > a|y_0|$.
Comment: Initial velocity is of compensating nature and is large enough to compensate for the initial positional disturbance.
(b) At time $\bar{t} = \frac{y_1}{a(y_1 + ay_0)}$, under the condition that either y_0 and y_1 are of the same sign or they are of opposite signs with $|y_1| > a|y_0|$.
Comment: In the first case, there is an extremum of the same sense as the initial condition, while in the second case the extremum comes after a zero-crossing.

6. Condition: $\frac{Q' + 2PQ}{Q^{3/2}}$ is constant. Corresponding substitution: $z(x) = \int \sqrt{Q(x)/K}\,dx$.
[Find $y' = z' \frac{dy}{dz}$ etc, substitute in the ODE and simplify to the standard form.]

Comment: In the case of Euler-Cauchy equation, $P = \frac{a}{x}$, $Q = \frac{b}{x^2}$ and hence $\frac{Q'+2PQ}{Q^{3/2}} = \frac{2(a-1)}{\sqrt{b}}$, which is constant. Therefore, the substitution $z(x) = \int \sqrt{Q(x)/b}dx = \ln x$ leads to an ODE with constant coefficients.

7. (a) $y_1(x) = x$. (b) $x(x+1)(2x+1)u'' + 2(3x^2+3x+1)u' = 0$. (c) $u = \frac{1}{x} - \frac{1}{x+1}$. (d) $y(x) = c_1 x + \frac{c_2}{x+1}$. Not valid for interval $[-5, 5]$. [Guaranteed to be valid for intervals not containing -1 and $-1/2$, roots of the coefficient function $2x^2 + 3x + 1$.]

8. $y(x) = 1 + x - \frac{1}{(2)(3)}x^3 - \frac{1}{(3)(4)}x^4 + \frac{1}{(2)(3)(5)(6)}x^6 + \frac{1}{(3)(4)(6)(7)}x^7 - \frac{1}{(2)(3)(5)(6)(8)(9)}x^9 + \cdots$. [Find $y''(0)$ from the given equation and values of higher derivatives by differentiating it.] **Comment:** Fig. A.31(a) shows how closely the series up to x^7 and up to x^9 match about $x = 0$. This suggests the validity of this solution in a close neighbourhood, say $[-2, 2]$. The validity of this truncated series solution gets further reinforced by the study of the residual $y'' + xy$ shown in Fig. A.31(b).



(a) Series estimates of the solution (b) Residual of the solution

Figure A.31: Solution of IVP of Airy's equation

Chapter 34 (page 297)

1. (a) $y = x^4 + x$. (b) $y = c_1 \cos 2x + c_2 \sin 2x + 4x + \frac{8}{5}e^x$. (c) $y = c_1 x + c_2 x^2 - x \cos x$.

2. $u(x) = \int \left\{ \frac{1}{f(x)} \int f(x)dx \right\} dx$, where $f(x) = y_0^2(x)e^{\int P(x)dx}$.
Comment: This analysis goes behind the modification rule in the *method of undetermined coefficients*. Implement this formulation on the example in the text of the chapter (see page 293) to see how $u(x)$ turns out as x and $\frac{1}{2}x^2$ in cases (b) and (c), respectively.

3. $y(x) = -\frac{37x}{24} + \frac{23}{12(x+1)} + \frac{x^2(4x+3)}{6(x+1)}$.

4. [Frame the linear equations in the undetermined coefficients and note that the coefficient matrix is triangular.]
Comment: For zero coefficient of y in the ODE, the constant term in the candidate solution becomes arbitrary, the system of equations then yields a unique solution for the other coefficients.

5. $y(t) = \frac{Ate^{-ct}}{2\omega} \sin \omega t$.
With the given coefficients, damped frequency of the system (ω) remains same as the driving frequency, and the system follows the driver with a constant phase difference of $\pi/2$. Amplitude increases till $ct < 1$ and then decreases. Maximum amplitude $= \frac{A}{2ec\omega}$ at $t = 1/c$: smaller the value of c, higher is the extent of *resonance*.

6. There are three distinct cases.

Case I: $\nu = 0$.
(a) $y = x - \sin x$, (b) $y = \frac{4}{\pi}\left(1 + \frac{1}{\sqrt{2}}\right) x - \sin x$, (c) $y = -\sin x$, (d) $y = \frac{x}{2\pi} - \sin x$ (unique for all the sets of conditions).

Case II: $\nu = 1$ (which includes the case of $\nu = -1$).
(a) $y = \frac{1}{2}(\sin x - x\cos x)$ (unique), (b) $y = \left(\sqrt{2} + \frac{\pi}{8}\right) \sin x - \frac{1}{2} x \cos x$ (unique), (c) and (d) no solution exists.

Case III: $\nu \neq 0, \nu \neq 1, \nu \neq -1$.
(a) $y = \frac{1}{\nu^2-1}\left(\sin x - \frac{\sin \nu x}{\nu}\right)$ (unique), (b) no solution exists for $\nu = \pm 4, \pm 8, \pm 12, \cdots$, otherwise $y = \csc \frac{\nu\pi}{4}\left[1 - \frac{1}{\sqrt{2}(\nu^2-1)}\right] \sin \nu x + \frac{\sin x}{\nu^2-1}$ (unique), (c) infinite solutions $y = B \sin \nu x + \frac{\sin x}{\nu^2-1}$ with arbitrary B for $\nu = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots$, otherwise $y = \frac{\sin x}{\nu^2-1}$ (unique), (d) no solution exists for $\nu = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots$, otherwise $y = \csc 2\nu\pi \sin \nu x + \frac{\sin x}{\nu^2-1}$ (unique).

Chapter 35 (page 304)

1. (a) $y(x) = (c_1 + c_2 x + c_3 x^2 + \frac{8}{105} x^{7/2}) e^{2x}$.
Comment: The author's advice is to use the method of variation of parameters. Nevertheless, the reader should note how economically the operational method, $y_p(x) = \frac{1}{(D-2)^3}\sqrt{x}e^{2x}$, works in this particular case.
(b) $y(x) = \sin x + \sin 3x + 2\sinh x$.

2. (a) [If all basis members have extrema at the same point, then (show that) at that point the Wronskian vanishes: contradiction.]
(b) [Argue that members of the new set are solutions to begin with, and then show that Wronskians of the two sets are related as $W_z(x) = W_y(x)\det(A)$.]

3. (a) $z(x, y) = ye^{-kx}$.
(b) [Define $\bar{y}(x)$ such that $[f(D)]\bar{y} = x^n e^{ax} \cos \omega x$, compose $y^* = \bar{y} + iy$ and arrive at $[f(D)]y^* = x^n e^{(a+i\omega)x}$. Applying the above formulation, solve the ODE for $y^*(x)$, simplify and collect the imaginary terms.
(c) $y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{4x}$
$+ \frac{e^x}{4\times 26^3}\left[(5 \times 26^2 x^2 + 42 \times 26x - 3565)\cos 2x + (26^2 x^2 - 236 \times 26x - 401)\sin 2x\right]$.

4. (a) With $p = 1$, $v_x = -v_y$; with $p = 6$, $v_x = 4v_y$. Here, $[v_x \quad v_y]^T$ is an eigenvector of the coefficient matrix, with p as the eigenvalue.
(b) The y-eliminant ODE: $\frac{d^2x}{dt^2} - 7\frac{dx}{dt} + 6x = 0$. $x = c_1 e^t + c_2 e^{6t}$, $y = -c_1 e^t + \frac{1}{4} c_2 e^{6t}$.

5. $y_0(-L/2) = y_0(L/2) = \frac{9}{4480} kL^4$, $y_0'(-L/2) = -\frac{kL^3}{112}$, $y_0'(L/2) = \frac{kL^3}{112}$.
[Function norm (see Chap. 15) minimization of the general solution $C_0 + C_1 x + C_2 x^2 + C_3 x^3 - kx^4/24$ leads to a system of linear equations in the coefficients. From the solution of this system, the boundary conditions are determined. See the resulting deflection in Fig. A.32.]
Comment: This exercise continues further with the function space perspective offered at the end of Chap. 33. Here, from a four-dimensional subspace of the function space, defined by the differential equation, we chose our desired function in a least square sense, the manner of which is analogous to the solution of under-determined linear algebraic systems of the form $\mathbf{Ax} = \mathbf{b}$.

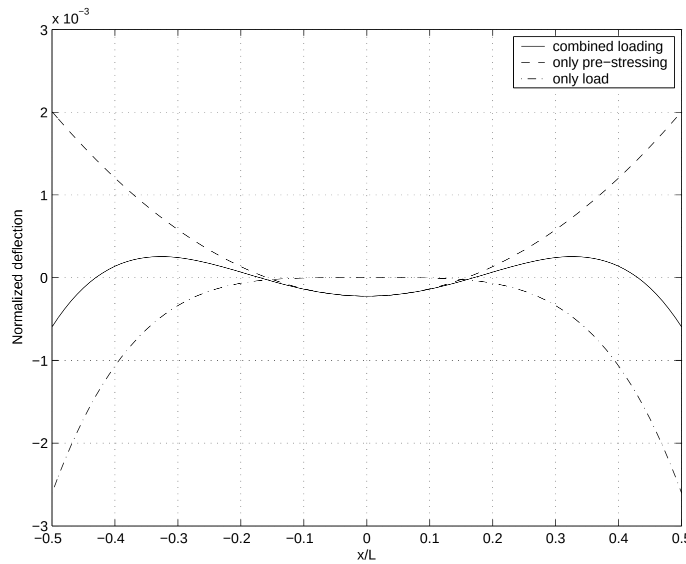


Figure A.32: Deflection of a beam under pre-stressing and combined loading

Chapter 36 (page 313)

1. (a) $y(t) = \frac{1}{5} e^{-t} + \frac{4}{5} e^{-2t} \cos 3t + \frac{14}{15} e^{-2t} \sin 3t$. See the resulting form of solution in Fig. A.33.
(b) $y(t) = \begin{cases} e^{-2t}(\cos 3t + \sin 3t), & \text{for } 0 \le t < 3; \text{ and} \\ \frac{1}{13}\left[1 - e^{-2(t-3)}\{\cos 3(t-3) + \frac{2}{3}\sin 3(t-3)\}\right] + e^{-2t}(\cos 3t + \sin 3t), & \text{for } t \ge 3. \end{cases}$
Comment: Verify by subsitution, and note that $y(3^-) = y(3^+)$ and $y'(3^-) = y'(3^+)$, but $y''(3^-) = y''(3^+) - 1$. See Fig. A.34.

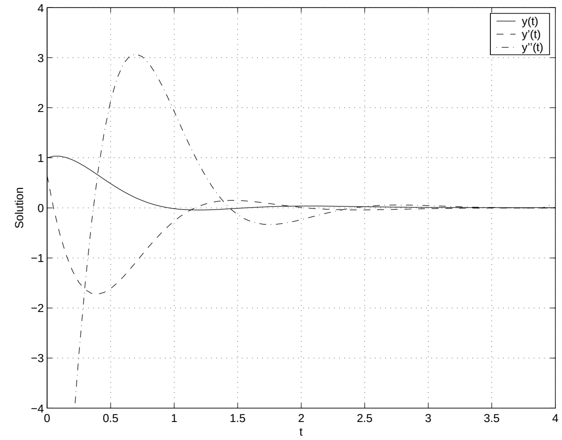


Figure A.33: Solution of $y'' + 4y' + 13y = 2e^{-t}$

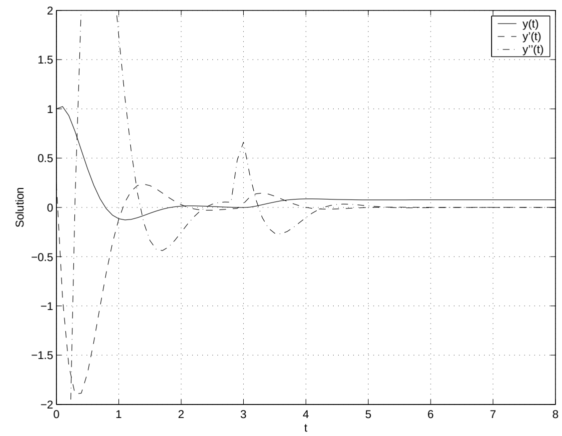


Figure A.34: Solution of $y'' + 4y' + 13y = u(t-3)$

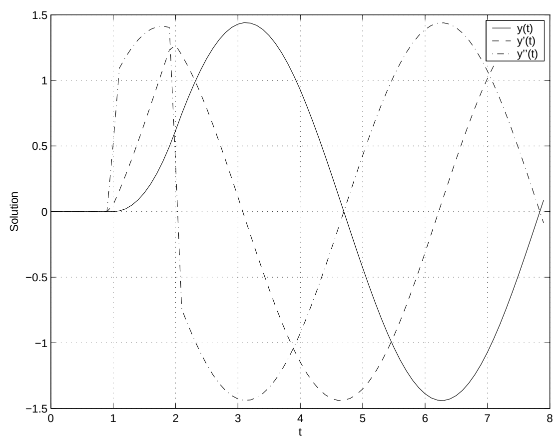


Figure A.35: Solution of $y'' + y = t[u(t-1) - u(t-2)]$

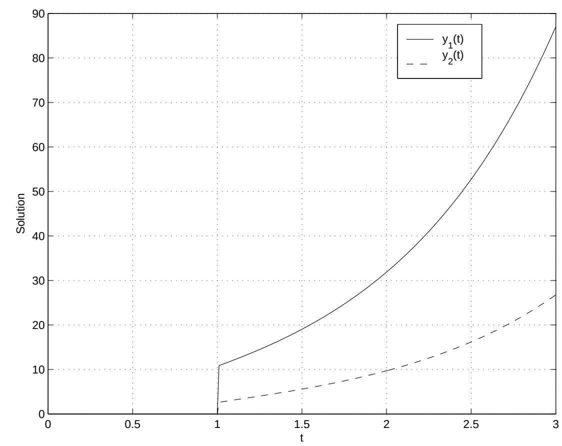


Figure A.36: Example of an ODE system

2. (a) $y(t) = \begin{cases} 0, & \text{for } 0 \le t < 1; \\ t - \cos(t-1) - \sin(t-1), & \text{for } 1 \le t < 2; \text{ and} \\ 2\cos(t-2) + \sin(t-2) - \cos(t-1) - \sin(t-1), & \text{for } t \ge 2. \end{cases}$

Comment: Sketch the input function $r(t)$, then plot the response $y(t)$ and compare with Fig. A.35. Note the discontinuity of $y''(t)$ at $t = 1$ and $t = 2$.

(b) [First, frame the system of linear equations in $Y_1(s)$ and $Y_2(s)$, solve it and then take inverse Laplace transforms.]

$$
\begin{aligned}
y_1(t) &= \frac{e}{3}u(t-1)\{e^{t-1}-e^{-2(t-1)}\}+4e^t-e^{-2t},\\
y_2(t) &= \frac{e}{3}u(t-1)\{e^{t-1}-e^{-2(t-1)}\}+e^t-e^{-2t}.
\end{aligned}
$$

Comment: By substitution, verify continuity and satisfaction of the given equation. Plot the solution and compare with Fig. A.36.

3. $y(t)=\sinh t$. [By taking e^t inside the integral, you can put it as the convolution of $y(t)$ and e^t. Use convolution theorem, while taking the Laplace transforms.]

4. $y(x)=\frac{M}{6EI}\left[(x-a)^3u(x-a)-\frac{L-a}{L}\{x^3-a(2L-a)x\}\right]$.
$\theta_0=\frac{Ma(L-a)(2L-a)}{6EIL}$, $F_0=-\frac{M}{EI}\frac{L-a}{L}$, $y(a)=\frac{Ma^2(L-a)^2}{3EIL}$.
[After finding the solution in terms of θ_0 and F_0, the boundary conditions (at $x=L$) are imposed to find out these unknowns.]
Comment: From the solution, sketch the shear force diagram (SFD), bending moment diagram (BMD) and shape of the beam; and satisfy yourself that they make intuitive sense. See rough sketches in Fig A.37.

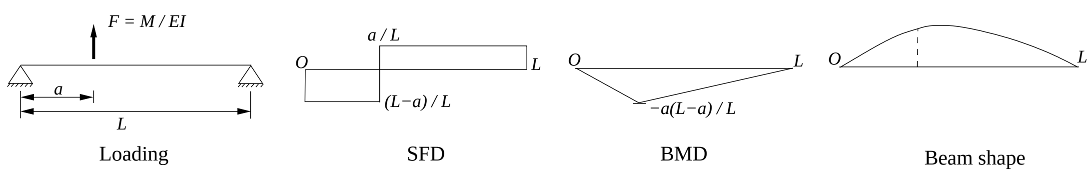


Figure A.37: Simply supported beam under concentrated load

5. $u(x,t)=kc\left[\left\{H(t-\frac{l-x}{c})-H(t-\frac{3l-x}{c})+H(t-\frac{5l-x}{c})-\cdots\right\}\right.$
$\left.\quad-\left\{H(t-\frac{l+x}{c})-H(t-\frac{3l+x}{c})+H(t-\frac{5l+x}{c})-\cdots\right\}\right]$.
(The symbol H, for Heaviside function, has been used for unit step function, because the usual symbol u has a different meaning here.)
Or, $u(x,t)=\begin{cases} kc, & \text{if } (4n+1)\frac{l}{c}-\frac{x}{c}\le t\le(4n+1)\frac{l}{c}+\frac{x}{c};\\ -kc, & \text{if } (4n+3)\frac{l}{c}-\frac{x}{c}\le t\le(4n+3)\frac{l}{c}+\frac{x}{c};\\ 0, & \text{otherwise};\end{cases}$ for $n=0,1,2,3,\cdots$.
[Steps: Take Laplace transform to arrive at the ODE $\frac{d^2}{dx^2}U(x,s)-\frac{s^2}{c^2}U(x,s)=0$. Solve it and use boundary conditions to get $U(x,s)=\frac{kc}{s}\frac{e^{sx/c}-e^{-sx/c}}{e^{sl/c}+e^{-sl/c}}$. Use binomial expansion to obtain an infinite series and then take inverse Laplace transform term by term.]
At $x=l$, $u(l,t)=\begin{cases} kc, & \text{if } 0<t<2l/c, \text{ and}\\ -kc, & \text{if } 2l/c<t<4l/c;\end{cases}$ repeated with a period of $4l/c$.
See the response at the two end-points and at a general (interior) point in Fig. A.38.

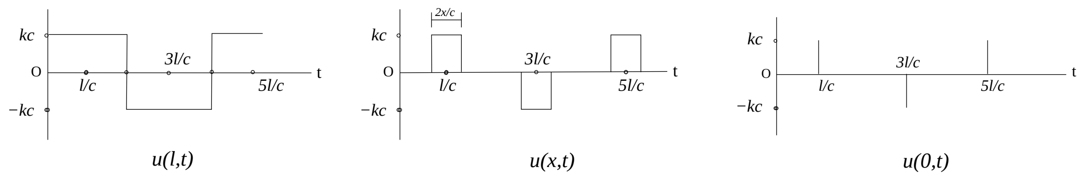


Figure A.38: Response of an elastic rod under shock input

Chapter 37 (page 318)

1. (a) $\mathbf{x}(t) = \begin{bmatrix} 2e^t + 3e^{-t} - \sin t + 2 \\ 6e^t + 3e^{-t} - \cos t - 2\sin t + 3 \end{bmatrix}$. (b) $\mathbf{x}(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t \tan t$.

   (c) $\mathbf{x}(t) = \begin{bmatrix} 1 & t-1 & t^2/2 - t \\ -1 & -t & -t^2/2 \\ 2 & 2t-1 & t^2 - t + 1 \end{bmatrix} \begin{bmatrix} c_1 e^t + \frac{1}{2}t^2 + t + 1 - \left(\frac{1}{4}t^2 + \frac{3}{4}t - \frac{1}{8}\right) e^{-t} \\ c_2 e^t - t - 1 + \left(\frac{1}{2}t + \frac{3}{4}\right) e^{-t} \\ c_3 e^t + 1 - \frac{1}{2}e^{-t} \end{bmatrix}$.

   [Coefficient matrix deficient, use generalized eigenvectors.]

2. Jacobian $= \begin{bmatrix} -\alpha & \alpha & 0 \\ \beta - y_3 & -1 & -y_1 \\ y_2 & y_1 & -\gamma \end{bmatrix}$, $\mathbf{y}_c = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} \pm\sqrt{\gamma(\beta-1)} \\ \pm\sqrt{\gamma(\beta-1)} \\ \beta - 1 \end{bmatrix}$.

   Around origin, (a) $\bar{\mathbf{f}}(\mathbf{y}) = \begin{bmatrix} \alpha(y_2 - y_1) & \beta y_1 - y_2 & -\gamma y_3 \end{bmatrix}^T$; (b) For $\beta > 1$, all trajectories converge to a single line (eigenvector corresponding to positive eigenvalue of Jacobian), along which they tend to infinity. For $0 < \beta < 1$, all eigenvalues are negative, all trajectories converge to origin, and the origin is what is known as a *stable critical point* (see next chapter).

   Around critical point $\mathbf{y}_c = \begin{bmatrix} \pm\sqrt{\gamma(\beta-1)} \\ \pm\sqrt{\gamma(\beta-1)} \\ \beta - 1 \end{bmatrix}$, (a) $\bar{\mathbf{f}}(\mathbf{y}_c + \mathbf{z}) = \begin{bmatrix} \alpha(z_2 - z_1) \\ z_1 - z_2 \mp \sqrt{\gamma(\beta-1)}z_3 \\ \pm\sqrt{\gamma(\beta-1)}(z_1 + z_2) - \gamma z_3 \end{bmatrix}$;

   (b) In one direction, there is fast exponential convergence to $\mathbf{y}_c$ (real negative eigenvalue), while in the complementary subspace trajectories *spiral in* towards it (complex pair of eigenvalues with negative real part).

Chapter 38 (page 327)

1. (a) $\mathbf{x}' = \begin{bmatrix} \phi' \\ \phi'' \\ I'_a \end{bmatrix} = \begin{bmatrix} x_2 \\ (k_t x_3 - \mu x_2)/J \\ (V - R_a x_3 - k_b x_2)/L_a \end{bmatrix}$.

   (b) $\mathbf{x}'$ is independent of x_1 or ϕ, first column of Jacobian is zero and one complete line in state space is in equilibrium. Physically, the motor shaft's *angular position* is irrelevant in any practical sense, and hence cannot contribute in describing the system state. Therefore, in the revised model, state vector $\mathbf{y} = [\omega \;\; I_a]^T$, with $\omega = \phi'$, and

$$\mathbf{y}' = \begin{bmatrix} \omega' \\ I'_a \end{bmatrix} = \begin{bmatrix} -\mu/J & k_t/J \\ -k_b/L_a & -R_a/L_a \end{bmatrix} \mathbf{y} + \begin{bmatrix} 0 \\ V/L_a \end{bmatrix}$$

(c) Steady state: $\mathbf{y}_0 = [k_t \quad \mu]^T V/\Delta$, where $\Delta = k_b k_t + \mu R_a$.
(d) $\mathbf{z} = \mathbf{y} - \mathbf{y}_0$, $\mathbf{z}' = \mathbf{B}\mathbf{z}$, where $\mathbf{B} = \begin{bmatrix} -\mu/J & k_t/J \\ -k_b/L_a & -R_a/L_a \end{bmatrix}$.
(e) Stable critical point (necessary for the motor to continue working at the operating point), either spiral (underdamped) or node (overdamped, also critically damped).

2. Origin is a centre (stable), and there are four saddle points (unstable) at $(\pm 1, \pm 1)$.
Comment: In the phase portrait, check the consistency of neighbouring trajectories.

3. Critical points: origin with eigenvalues $-0.5 \pm 0.866i$ is a stable spiral, and points $(\pm 1, 0)$ with eigenvalues $-2, 1$ are (unstable) saddle points. (See the phase portrait in Fig. A.39.)

4. Two critical points. (i) Origin: saddle point (rabbits grow, foxes decay). (ii) $(d/c, a/b)$: centre (stable, but cyclic oscillations continue).

5. (a) Assumption: in the absence of cooperation, neither species survives. Uncoupled rates of decay are indicated by a and m, while b and n signify the sustaining effect of cooperation.
(b) Equilibrium points: origin is a stable node, while $P(m/n, a/b)$ is a saddle point.
(c) The key elements of the phase portrait are the two trajectories intersecting at the saddle point $P(m/n, a/b)$. All trajectories are destined to eventually merge with the outward trajectory (locally the eigenvector with positive eigenvalue) from P, either towards origin (extinction of both species) or towards infinity (runaway growth of both). The curve of the inward trajectory at P (locally the eigenvector with negative eigenvalue) separates the phase plane into two parts: trajectories starting below this curve converge, while those above diverge.
(d) Below the separating curve, the combined population is not enough to sustain the cooperation and both species eventually decay. Above this threshold, the cooperation succeeds and leads to a population explosion in both the species.
Comment: Interestingly, if at least one species is populous enough to locate the initial state above this curve, then even a miniscule non-zero population of the other is enormously boosted by the majority 'friend' species towards eventual growth for both! (Work out the mechanism of the dynamics in common sense terms and check with the sketch of Fig. A.40.)

6. [First, construct the *rate* of the candidate Lyapunov function by arranging the cancellation of positive terms. Then, develop the function itself. One such Lyapunov function is $x_1^4 + 2x_2^2$.]
Comment: The linearized system being stable does imply the original system to be stable, possibly with a finite domain of attraction. But, the converse is not true.

Chapter 39 (page 339)

1. The formal power series $x + x^2 + 2!x^3 + 3!x^4 + 4!x^5 + \cdots$ satisfies the ODE, but does not constitute a solution, since it is divergent everywhere!

2. (a) $y(x) = a_0\left[1 + \frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{3}{6!}x^6 + \cdots\right] + a_1 x + \left[\frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{3}x^4 + \frac{7}{40}x^5 + \frac{19}{240}x^6 + \cdots\right]$.
(b) $y(x) = c_1 y_1(x) + c_2 y_2(x)$, $y_1(x) = \frac{1}{x^2}\sum_{k=0}^{\infty}\frac{(-x)^k}{(k!)^2}$, $y_2(x) = y_1(x)\ln x + \frac{2}{x} - \frac{3}{4} + \frac{11}{108}x + \cdots$.

3. [Substitute $x = u^2$. See exercise 4 in Chap. 16.]

4. **Comment:** These results are useful in deriving properties of Legendre polynomials, besides making their evaluation simple.

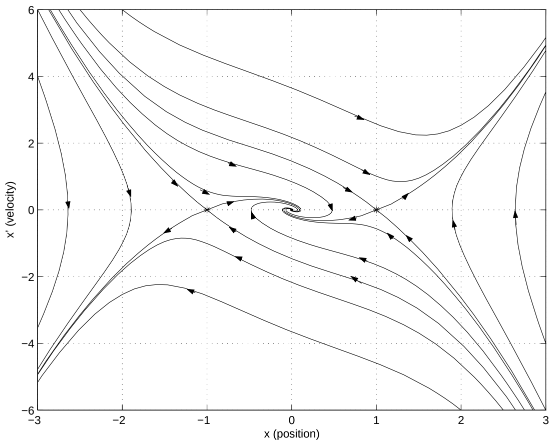


Figure A.39: Damped nonlinear spring

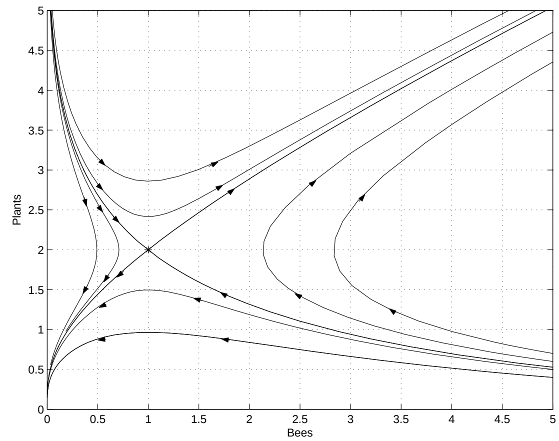


Figure A.40: Cooperating species model

5. $y_1(x) = 1 + \frac{(-k^2)}{2!}x^2 + \frac{(2^2-k^2)(-k^2)}{4!}x^4 + \frac{(4^2-k^2)(2^2-k^2)(-k^2)}{6!}x^6 + \frac{(6^2-k^2)(4^2-k^2)(2^2-k^2)(-k^2)}{8!}x^8 + \cdots$;
$y_2(x) = x + \frac{(1-k^2)}{3!}x^3 + \frac{(3^2-k^2)(1-k^2)}{5!}x^5 + \frac{(5^2-k^2)(3^2-k^2)(1-k^2)}{7!}x^7 + \frac{(7^2-k^2)(5^2-k^2)(2^2-k^2)(1-k^2)}{9!}x^9 + \cdots$. For integer k, one of the solutions is a polynomial in the form
$a_k\left[x^k - \frac{k}{1!2^2}x^{k-2} + \frac{k(k-3)}{2!2^4}x^{k-4} - \frac{k(k-4)(k-5)}{3!2^6}x^{k-6} + \cdots\right]$.

6. $\sqrt{\frac{2}{\pi x}}(c_1 \cos x + c_2 \sin x)$.

Chapter 40 (page 349)

1. $p(x) = \frac{d(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}$; $q(x) = \frac{c(x)}{a(x)}e^{\int \frac{b(x)}{a(x)}dx}$ and $r(x) = e^{\int \frac{b(x)}{a(x)}dx}$ in Eqn. 40.5.

2. $p(x) = e^{-x}$. [Convert to self-adjoint form and show singularity of the S-L problem.]

3. (a) [Substitute $J_k(t)$ in Bessel's equation and use $t = \lambda x$.]
(b) [Use $u_m(x) = J_k(\lambda_m x)$ and $u_n(x) = J_k(\lambda_n x)$ in the ODE and follow the procedure used to prove the Sturm-Liouville theorem in the text.]
(c) If λ_m and λ_n are distinct roots of $J_k(x)$, then $\int_0^1 xJ_k(\lambda_m x)J_k(\lambda_n x)dx = 0$, i.e. $J_k(\lambda_m x)$ and $J_k(\lambda_n x)$ are orthogonal over the interval $[0,1]$ with respect to the weight function $p(x) = x$.

4. (a) $y = \cos(n\cos^{-1}x)$, $y = \sin(n\cos^{-1}x)$.
(b) $T_n(x) = \cos(n\cos^{-1}x)$: $T_0(x) = 1$, $T_1(x) = x$, $T_{k+1} = 2xT_k(x) - T_{k-1}(x)$; examples: $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$ etc.
(c) Weight function: $\frac{1}{\sqrt{1-x^2}}$. $\int_{-1}^1 \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = 0$ for $m \neq n$.
(d) [You can use the substitution $x = \cos\theta$ for evaluating the integral.]
(e) $\|T_0(x)\| = \sqrt{\pi}$ and $\|T_n(x)\| = \sqrt{\frac{\pi}{2}}$ for $n = 1, 2, 3, \cdots$.
(f) Fig. A.41 shows that $T_n(x)$ has $(n+1)$ extrema of value ± 1, all same in magnitude and alternating in sign, distributed uniformly over θ.

5. (a) Eigenvalues: $\lambda = \frac{n^2\pi^2}{16}$, for $n = 1, 2, 3, \cdots$. Normalized eigenfunctions: $\phi_n(x) = \frac{1}{\sqrt{2}} \sin \frac{n\pi x}{4}$.
   $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$, with $c_{4k} = c_{4k+1} = c_{4k+3} = 0$ and $c_{4k+2} = \frac{(-1)^k}{(2k+1)^2} \frac{8\sqrt{2}}{\pi^2}$.
   (b) $e_4(x) = f(x) - f_4(x) = f(x) - \sum_{n=1}^{4} c_n \phi_n(x) = f(x) - \frac{8}{\pi^2} \sin \frac{\pi x}{2}$.
   (c) $\|f\|^2 = 4/3$. Convergence of $\sum_i c_i^2$:

Number of terms	4	8	12	16	20	24
$\sum_i c_i^2$	1.3140	1.3303	1.3324	1.3329	1.3331	1.3332

   $\sum_i c_i^2 \leq \|f\|^2$ and $\sum_{i=1}^{\infty} c_i^2 = \|f\|^2$.
   Comment: The particular symmetry is behind the zero coefficients. Strong resemblance with sine function explains the dominance of the first coefficient.

6. (a) $\|P_n(x)\| = \sqrt{\frac{2}{2n+1}}$.
   (b) $|x| = \frac{1}{2} P_0(x) + \frac{5}{8} P_2 x - \frac{3}{16} P_4(x) + \frac{13}{128} P_6(x) + \cdots$.
   Comment: Verify Bessel's inequality. Fig. A.42 shows how additional terms tune the approximation.

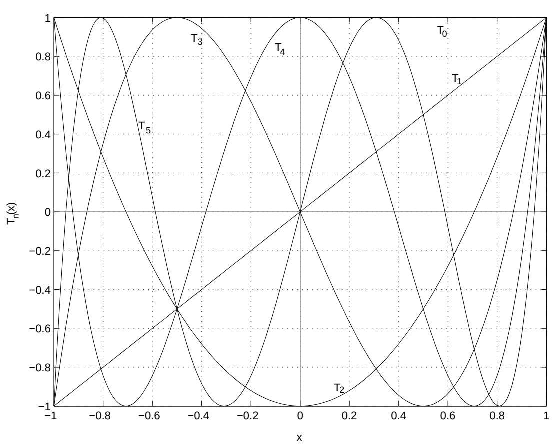


Figure A.41: Chebyshev polynomials

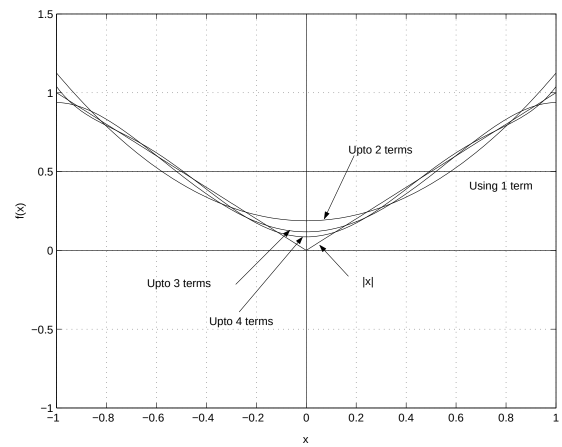


Figure A.42: Fourier-Legendre series of $|x|$

7. $\psi_3(x) = 6x^2 - 6x + 1$.
   Comment: This is Gram-Schmidt orthogonalization of functions. The functions could be normalized to develop the unique orthonormal basis.

Chapter 41 (page 357)

1. [Show that $(1, \cos nx) = (1, \sin nx) = (\cos mx, \sin nx) = 0$. Further, for $m \neq n$, also show that $(\cos mx, \cos nx) = (\sin mx, \sin nx) = 0$.]

2. (a) $\frac{4}{\pi} \sum_{1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{n\pi x}{2}$.
   (b) See Fig. A.43 for the approximations with 2, 4, 8, 16 and 24 terms. Notice how the

overshoots, as expected by the Gibb's phenomenon, are approaching a definite value.
(c) At $x = 0$ and at $x = \pm 1$ (for that matter, at every interior point), the derivative of the series oscillates between 2 and 0 with odd and even terms of the series. The actual derivative of $g(x)$ is obviously 1. At $x = \pm 2$, the differentiated series, $g'(x)$ diverges, as there are discontinuities of the periodic function at those points.

3. Cosine series: $f_c(x) = 0.857 - 0.798\cos\frac{\pi x}{2} - 0.345\cos\pi x + 0.351\cos\frac{3\pi x}{2} + 0.053\cos 2\pi x - 0.154\cos\frac{5\pi x}{2} - 0.052\cos 3\pi x + 0.124\cos\frac{7\pi x}{2} + 0.019\cos 4\pi x + \cdots$.
$f_c(0) \approx 0.0566$, $f_c(1) \approx 1.3266$ and $f_c(2) \approx 1.0089$.
Sine series: $f_s(x) = 1.235\sin\frac{\pi x}{2} - 0.843\sin\pi x + 0.078\sin\frac{3\pi x}{2} + 0.056\sin 2\pi x + 0.180\sin\frac{5\pi x}{2} - 0.245\sin 3\pi x + 0.058\sin\frac{7\pi x}{2} + 0.020\sin 4\pi x + \cdots$.
$f_s(0) \approx 0$, $f_s(1) \approx 1.2785$ and $f_s(2) \approx 0$.
The cosine and sine series are shown superimposed with the original function in Fig. A.44.
[Numerical integration can be used in the evaluation of the coefficients.]
Comment: Note how the function is approximated by the average of limits at points of discontinuity and values $f_c(2)$ and $f_s(2)$ differ widely. The even periodic extension is continuous at $x = 2$, while the odd periodic extension is discontinuous. The function being studied is the same as used in the illustration of half-range expansions in Fig. 41.1.

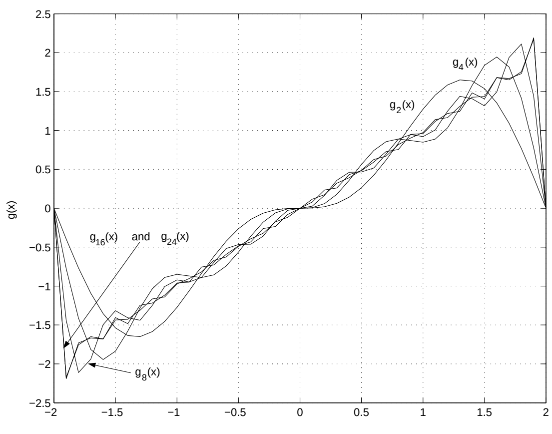


Figure A.43: Fourier series of $g(x) = x$

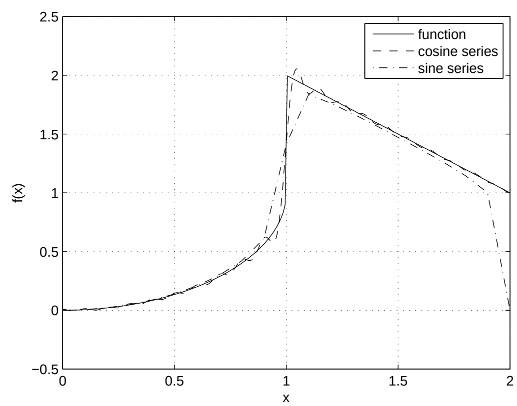


Figure A.44: Example on cosine and sine series

4. $f(t) \approx 13.58 - 0.55\cos\frac{\pi t}{6} - 4.71\sin\frac{\pi t}{6} + 4.58\cos\frac{\pi t}{3} - 3.90\sin\frac{\pi t}{3} + 0.33\cos\frac{\pi t}{2} + 0.83\sin\frac{\pi t}{2} + \cdots$.
[Set $f(12) = f(0)$ and perform the required integrations numerically.]

5. $I(t) = \sum_{k=1,k\text{ odd}}^{\infty} \frac{80}{k^2\pi[(10-k^2)^2+(10k)^2]}\left[(10-k^2)\cos kt + 10k\sin kt\right]$.
[Differentiating the circuit equation $LI' + RI + \frac{1}{C}\int_0^t I dt = E(t)$, obtain the differential equation $I'' + 10I' + 10I = \phi(t)$, where $\phi(t) = 10(\pi - 2|t|)$ for $-\pi \le t \le \pi$ and $\phi(t+2\pi) = \phi(t)$. Express $\phi(t)$ as a Fourier series, solve the ODE for each component and compose the solutions.]

6. [In $f(x) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n\sin\frac{n\pi x}{L}\right)$, use the formulae for cosine and sine functions in terms of exponential ones to obtain the complex form and work out C_0, C_n and C_{-n} from

A_0, A_n and B_n.]

7. $\frac{1}{\pi}\int_0^\infty \frac{\sin p(t-1)-\sin p(t-2)}{p}dp.$

8. [Replace the cosine term with an exponential function to get $I = \frac{1}{2}e^{-\left(\frac{b}{a}\right)^2}\int_{-\infty}^{\infty} e^{-a^2\left(p-\frac{ib}{a^2}\right)^2}dp.$]
Comment: Look in the exercises of Chap. 47 for a consolidation of this problem.

Chapter 42 (page 365)

1. (a) $\frac{1}{\sqrt{2\pi}}\frac{1}{a+iw}$. (b) $\sqrt{2\pi}\delta(w-w_0)$.

2. $\frac{2a\sin wk}{w\sqrt{2\pi}}$; $\frac{1}{\sqrt{2\pi}}$.
Comment: Thus, we get the Fourier transform of the δ function.

3. $\mathcal{F}(e^{-t^2}) = \frac{e^{-w^2/4}}{\sqrt{2}}$ and $\mathcal{F}(te^{-t^2}) = -\frac{iwe^{-w^2/4}}{2\sqrt{2}}$. [For the second case, one may use the time derivative rule, frequency derivative rule or a substitution $\tau = t + \frac{iw}{2}$.]

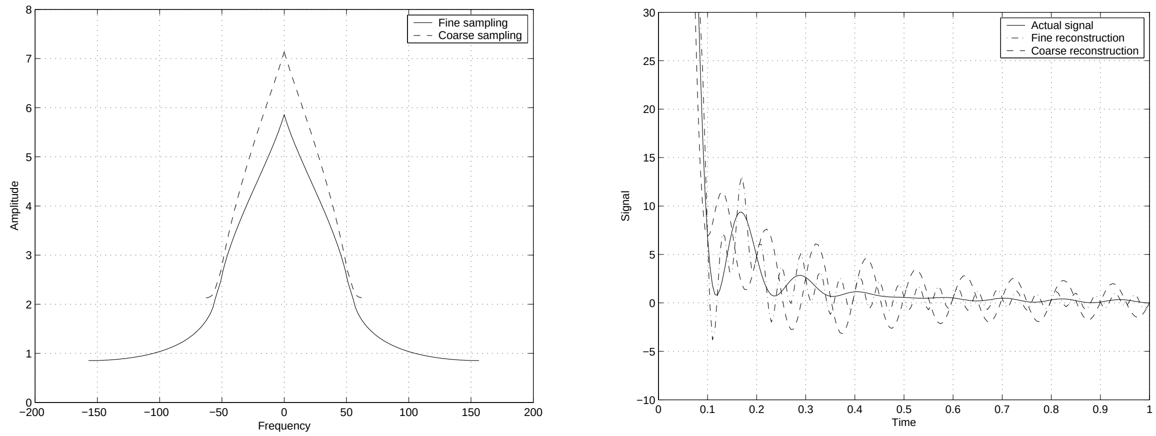
4. (a) $\hat{f}(w) = \frac{\sqrt{2\pi}}{2L}\sum_{n=-\infty}^{\infty} f\left(-\frac{n\pi}{L}\right)e^{in\pi w/L}$. [Since $f(t) = \frac{1}{\sqrt{2\pi}}\int_{-L}^{L}\hat{f}(w)e^{iwt}dw$.]
(b) [Use the expression for $\hat{f}(w)$ in the expression for $f(t)$.]
Comment: For a signal with band limits $\pm L$, complete reconstruction requires sampling only at discrete time steps of $\frac{n\pi}{L}$ and the sum converges pretty fast with the factor $\frac{\sin(Lt-n\pi)}{Lt-n\pi}$.

5. (a) See the superimposed Fourier transform estimates in Fig. A.45(a).
[$\hat{\phi}(w) \approx \frac{\Delta_t}{\sqrt{2\pi}}\sum_{k=0}^{N-1} e^{(-iw\Delta_t)^k}f(t_k)$ with $N = \frac{10}{\Delta_t}$.]
Comment: The roughly sampled data estimates the amplitudes of a smaller band of frequencies and shows a higher level of aliasing.
(b) Study the extent of signal reconstruction (for first one second) in Fig. A.45(b).
[$\phi(t) \approx \frac{\Delta_w}{\sqrt{2\pi}}\sum_{j=-N_w}^{N_w-1} e^{(i\Delta_w t)^j}\hat{\phi}(w_j)$ with $w_c = \frac{\pi}{\Delta_t}$ and $\Delta_w = \frac{2w_c}{N}$.]
Comment: The reconstruction is closer in the case of the finer sampling.

6. $\hat{f}(w) = \sqrt{\frac{2}{\pi}}\frac{1-\cos w}{w^2}$. (See Fig. A.46(a).)
$g(t) = \frac{1}{\pi}\left[(t+1)\mathrm{Si}\frac{(t+1)\pi}{2} + (t-1)\mathrm{Si}\frac{(t-1)\pi}{2} - 2t\mathrm{Si}\frac{t\pi}{2}\right] - \frac{4}{\pi^2}\cos\frac{t\pi}{2}$. In the form of an infinite series, $g(t) = \sum_{j=0}^{\infty} A_j t^{2j}$, where $A_j = \frac{(-1)^j(\frac{\pi}{2})^{2j}}{(2j)!}\sum_{k=0}^{\infty}\frac{(-1)^k(\frac{\pi}{2})^{2k}}{(2j+2k+1)(2k+2)!}$. Evaluation of the coefficients gives $g(t) = 0.4674 - 0.1817t^2 + 0.0231t^4 - 0.0013t^6 + 0.00005t^8 + \cdots$. The filtered signal is shown in comparison with the raw one in Fig. A.46(b).

Chapter 43 (page 372)

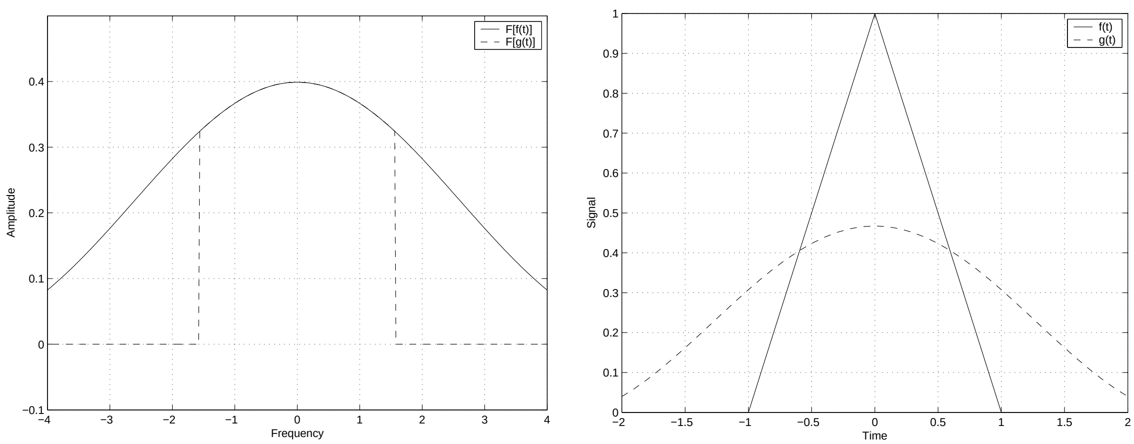
1. $a_0' = \frac{1}{2}a_2' + a_1$ and $a_{n-1}' = a_{n+1}' + 2na_n$ for $n > 1$.
[With $x = \cos\theta$, $\frac{dT_k}{dx} = \frac{k\sin k\theta}{\sin\theta}$. Evaluate the right-hand-side and simplify to prove the identity.]



(a) DFT of sampled signal

(b) Original and reconstructed signals

Figure A.45: Example on reconstruction of sampled signal



(a) Original and filtered spectra

(b) Raw and filtered signals

Figure A.46: Example of lowpass filtering

2. Taylor's series approximation: $e_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$, maximum deviation 0.0516 at $x = 1$.
Legendre series approximation: $e_P(x) = 1.1752 + 1.1036P_1(x) + 0.3578P_2(x) + 0.0705P_3(x)$, maximum deviation 0.0112 at $x = 1$.
Chebyshev series approximation: $e_T(x) = 1.2660 + 1.1302T_1(x) + 0.2714T_2(x) + 0.0442T_3(x)$, maximum deviation 0.0064 at $x = 1$. [Special care is needed while integrating near the

limits.]
Comment: Further, compare deviations $e_3(x) - e^x$, $e_P(x) - e^x$ and $e_T(x) - e^x$ over $[-1, 1]$ from Fig. A.47. Among the three, Chebyshev series representation is closest to the minimax approximation. As expected, Taylor series representation is at the other extreme with its highly accurate approximation near the centre and large deviation away from it.

3. $1 = T_0(x)$, $x = T_1(x)$, $x^2 = \frac{1}{2}T_2(x) + \frac{1}{2}T_0(x)$, $x^3 = \frac{1}{4}T_3(x) + \frac{3}{4}T_1(x)$, $x^4 = \frac{1}{8}T_4(x) + \frac{1}{2}T_2(x) + \frac{3}{8}T_0(x)$.

4. Chebyshev accuracy points: $x_j = 5\cos\left(\frac{2j+1}{18}\pi\right)$ for $j = 0, 1, 2, \cdots, 8$. See Fig. A.48 for comparison of the two approximations.
Comment: Chebyshev-Lagrange approximation is exact at these accuracy points, while nine-point Lagrange approximation with equal spacing (see also exercise in Chap. 26) is exact at -5.00, -3.75, -2.50, $\cdots$, 5.00.

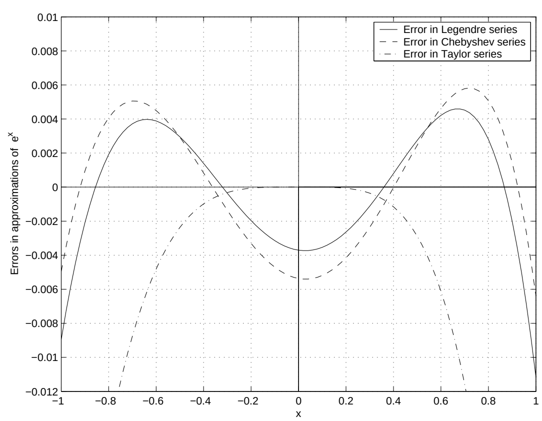


Figure A.47: Deviations in Taylor, Legendre and Chebyshev series approximations

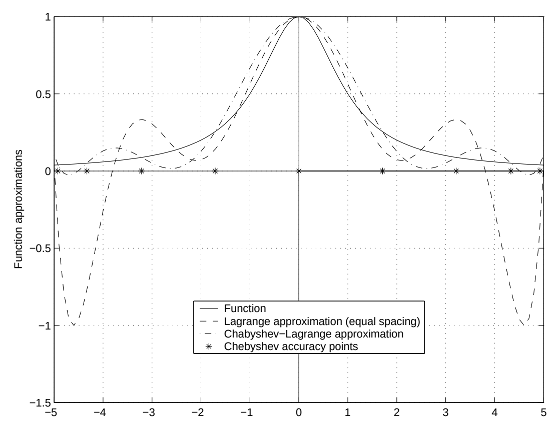
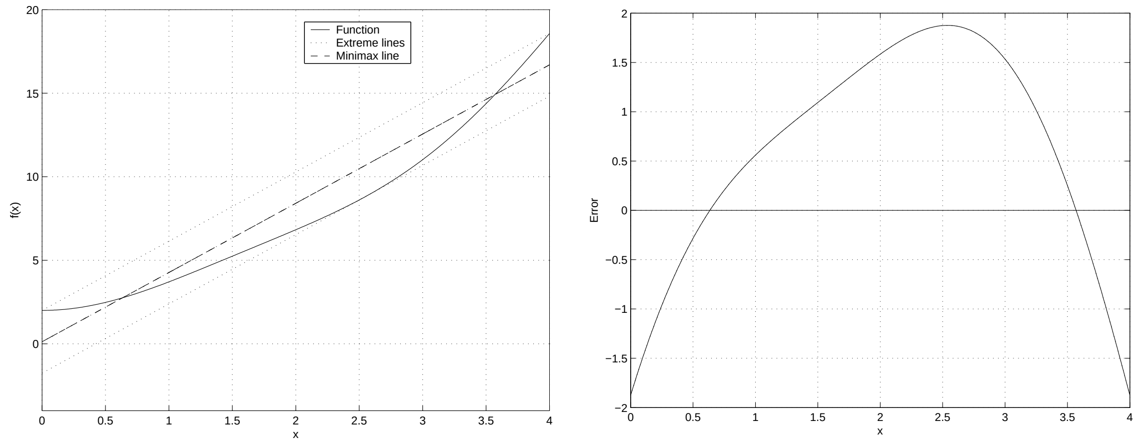


Figure A.48: Chebyshev-Lagrange approximation and Lagrange interpolation

5. Linear minimax approximation: $0.124 + 4.143x$. For the error function, the extrema are $e(0) = -1.876$, $e(2.54) = 1.876$ and $e(4) = -1.876$, while its zeros are 0.635 and 3.569.
[Graphical approach: Plot $f(x)$ in the interval. Draw a pair of parallel lines, tightly enclosing the plot, with some suitable slope, so as to keep their vertical distance minimum. Pick the parallel line mid-way between them. Denoting this line as $A + Bx$, plot the error $e(x) = A + Bx - f(x)$ to verify. Fig. A.49 depicts the construction and verification for the present problem.]
Comment: Note that the applicability and efficiency of the graphical approach is quite limited. It may be difficult to decide the slope if the end-points of the domain are not at the extrema of the error.



(a) Construction of the minimax line

(b) Error function

Figure A.49: Example of linear minimax approximation

Chapter 44 (page 387)

1. $u(x,y) = \sum_i U_i x^{k_i} e^{-y^2/k_i}$.
   Comment: k_i and U_i may depend on initial/boundary conditions.

2. $u(x,y) = f_1(y-x) + f_2(y-2x) + (y-x)(y-2x)^2 - \frac{3}{2}(y-x)^2(y-2x)$,
   or, $u(x,y) = f_1(y-x) + f_2(y-2x) - \frac{1}{2}(y^3 - 2xy^2 - x^2y + 2x^3)$.
   [Characteristic equation: $U_{\xi\eta} = 2\eta - 3\xi$ with $\xi = y - x$ and $\eta = y - 2x$.]

3. $y(x,t) = \frac{5}{3\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(\pi\epsilon)}{n^2-\epsilon^2} \sin(nx)\sin(1.2nt)$. See the string shapes in Fig. A.50.
   Comment: Note how the kink of discontinuity due to the initial shock input travels back and forth along the string.

4. (a) $u(x,t) = \int_0^\infty [A(p)\cos px + B(p)\sin px]\cos cpt\, dp$, such that $e^{-|x|} = \int_0^\infty [A(p)\cos px + B(p)\sin px] dp$.
   (b) $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{\cos px \cos cpt}{1+p^2} dp = \frac{1}{\pi}\int_0^\infty \frac{\cos p(x+ct) + \cos p(x-ct)}{1+p^2} dp$.
   (c) $\frac{d^2\hat{u}}{dt^2} + c^2w^2\hat{u} = 0$, $\hat{u}(w,0) = \sqrt{\frac{2}{\pi}}\frac{1}{1+w^2}$, $\frac{d\hat{u}}{dt}(w,0) = 0$.
   (d) $\hat{u}(w,t) = \sqrt{\frac{2}{\pi}}\frac{\cos cwt}{1+w^2}$ and hence $u(x,t)$ results as above.
   Comment: See page 488 (exercise 9 of Chap. 47) for the final solution.

5. (a) [Consider heat rate balance as $\Delta x \sigma \rho \frac{\partial u}{\partial t} = K\left(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\right)$, with the usual notation of symbols.]
   (b) $u(x,t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{L}\, e^{-\frac{c^2n^2\pi^2}{L^2}t}$, $A_0 = \frac{1}{L}\int_0^L f(x)dx$ and $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$, for $n = 1, 2, 3, \cdots$.
   (c) As $t \to \infty$, $u(x,t) \to A_0$, constant temperature along the entire bar, equal to the initial

mean temperature.
Comment: With homogeneous Neumann conditions at both the boundaries, heat cannot flow out of the system, and hence gets uniformly distributed over the bar.

6. $u = \frac{1}{\sinh(2\pi)} \sinh \pi x \sin \pi y + \frac{1}{\sinh \pi} \sin \pi x \sinh \pi y + \sum_{n=1, n\neq 2}^{\infty} \frac{16n[(-1)^n - 1]}{\pi^2 (n-2)^2 (n+2)^2 \sinh \frac{n\pi}{2}} \sin \frac{n\pi x}{2} \sinh \frac{n\pi y}{2}$.
[Split the problem into two BVP's with $u(x,y) = v(x,y) + w(x,y)$, where $v(2,y) = 0,\ v(x,1) = x \sin \pi x$ and $w(2,y) = \sin \pi y,\ w(x,1) = 0$.]

7. $u(x,y) = \frac{\gamma}{2\eta}(y^2 - b^2) + \frac{16\gamma b^2}{\eta \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \frac{\cosh \frac{(2n-1)\pi x}{2b}}{\cosh \frac{(2n-1)\pi a}{2b}} \cos \frac{(2n-1)\pi y}{2b}$.
[The first term is the particular solution u_p, depending on y only and satisfying the boundary conditions in y, can be solved from $\frac{d^2 u_p}{dy^2} = \frac{\gamma}{n}$. The second term, in the form of a Fourier series, is the solution of the corresponding homogeneous equation $\nabla^2 u_h = 0$ with matching (compensating) boundaries conditions. Symmetrically, an alternative form of the solution is given by $u(x,y) = \frac{\gamma}{2\eta}(x^2 - a^2) + \frac{16\gamma a^2}{\eta \pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} \frac{\cosh \frac{(2n-1)\pi y}{2a}}{\cosh \frac{(2n-1)\pi b}{2a}} \cos \frac{(2n-1)\pi x}{2a}$.]
Comment: The two forms of the solution look quite different, but if you examine the contours of $u(x,y)$ in both forms, including enough terms in the series, you find them to be identical.

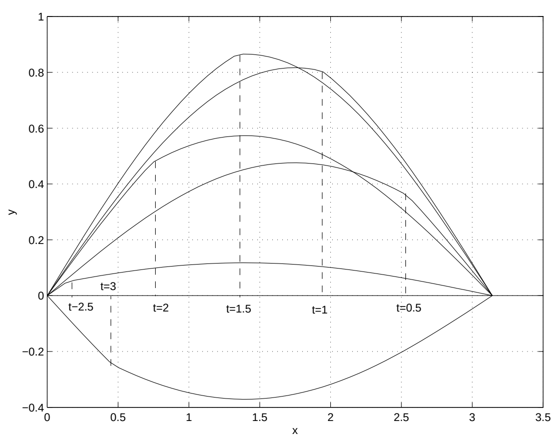


Figure A.50: String shapes under shock input

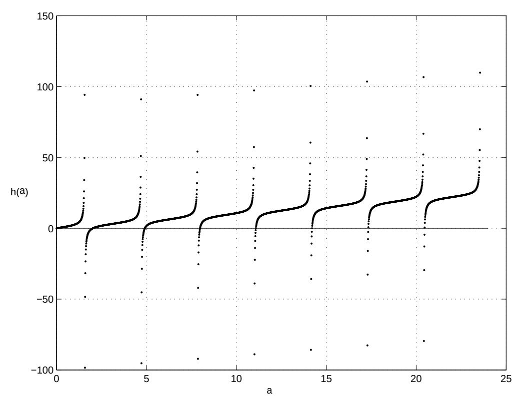


Figure A.51: Typical eigenvalue distribution in heat transfer with radiating end

8. $u(x,t) = \sum_{n=1}^{\infty} A_n \sin \alpha_n x \, e^{-c^2 \alpha_n^2 t}$, where $A_n = \frac{\int_0^L f(x) \sin \alpha_n x \, dx}{\int_0^L \sin^2 \alpha_n x \, dx}$, with α_n being the roots of $\alpha + A \tan \alpha L$.
[BVP on $X(x)$ is $X'' + \lambda X = 0,\ X(0) = 0,\ X'(L) + AX(L) = 0$.]
Comment: As α_n's are not at regular intervals, the solution is not in the form of a Fourier sine series, but a generalized Fourier series nevertheless. Solution of the eigenvalues $\lambda_n = \alpha_n^2$ requires the numerical solution of the equation $\alpha + A \tan \alpha L = 0$ for consecutive solutions. For the specific parameter values satisfying $AL = 1$, denoting $\alpha L = a$, we get $h(a) = a + \tan a = 0$.

See the plot of $h(a)$ in Fig. A.51. From the (non-uniformly distributed) roots $a_1 = 2.03$, $a_2 = 4.91$, $a_3 = 7.98$, $a_4 = 11.09$, $a_5 = 14.21, \cdots$, eigenvalues can be obtained.
Physically, the BC at $x = L$ means that the rod radiates heat to the environment (at zero temperature), the temperature gradient being proportional to the temperature difference between the radiating end and the environment. This is a case of the modelling of a fin.

9. $u(x, y, t) = k \sin \pi x \sin \pi y \cos(\sqrt{2}\pi t)$.
[In the double Fourier (sine) series $f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin m\pi x \sin n\pi y$, defining the simple Fourier (sine) series $C_m(y) = \sum_{n=1}^{\infty} A_{mn} \sin n\pi y$ acting as coefficients in the other simple Fourier (sine) series $f(x) = \sum_{m=1}^{\infty} C_m(y) \sin m\pi x$, we have
$$C_m(y) = 2\int_0^1 f(x, y) \sin m\pi x\, dx \quad \text{and} \quad A_{mn} = 2\int_0^1 C_m(y) \sin n\pi y\, dy,$$
leading to the matrix of coefficients $A_{mn} = 4\int_0^1 \int_0^1 f(x, y) \sin m\pi x \sin n\pi y\, dx\, dy$.]

10. (a) $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$.
(b) With $\frac{\partial}{\partial z} \equiv 0 \equiv \frac{\partial}{\partial \theta}$, BVP of the circular membrane: $u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r)$, $u(a, t) = 0$, $u(r, 0) = f(r)$, $u_t(r, 0) = g(r)$. Eigenvalues $k = \frac{\alpha_n}{a}$, where α_n, for $n = 1, 2, 3, \cdots$, are zeros of $J_0(x)$.
[Separation of variable as $u(r, t) = R(r)T(t)$ leads to $R'' + \frac{1}{r}R' + k^2 R = 0$ and $T'' + c^2k^2T = 0$. With change of variable $s = kr$, the Bessel's equation of order zero is obtained. Consider only bounded solution.]
(c) $u(r, t) = \sum_{n=1}^{\infty} [A_n \cos \frac{c\alpha_n t}{a} + B_n \sin \frac{c\alpha_n t}{a}] J_0\left(\frac{\alpha_n r}{a}\right)$, where coefficients are obtained from the Fourier-Bessel series $f(r) = \sum_{n=1}^{\infty} A_n J_0\left(\frac{\alpha_n r}{a}\right)$ and $g(r) = \sum_{n=1}^{\infty} \frac{c\alpha_n}{a} B_n J_0\left(\frac{\alpha_n r}{a}\right)$, i.e.
$$A_n = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a r f(r) J_0\left(\frac{\alpha_n r}{a}\right) dr \quad \text{and} \quad B_n = \frac{2}{ca\alpha_n J_1^2(\alpha_n)} \int_0^a r g(r) J_0\left(\frac{\alpha_n r}{a}\right) dr.$$

11. (a) With $u(r, \phi) = F(r)G(\phi)$, the separated equations are $(1 - x^2)\frac{d^2G}{dx^2} - 2x\frac{dG}{dx} + \lambda G = 0$ and $r^2F'' + 2rF' - \lambda F = 0$, where $x = \cos\phi$.
(b) $u(r, \phi) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\phi)$, the coefficients being evaluated from the Fourier-Legendre series $f(\phi) = \sum_{n=0}^{\infty} (a_n R^n) P_n(\cos\phi)$ as $a_n = \frac{1}{R^n}\frac{2n+1}{2}\int_{-1}^{1} f(\cos^{-1} x) P_n(x) dx$.

Chapter 45 (page 396)

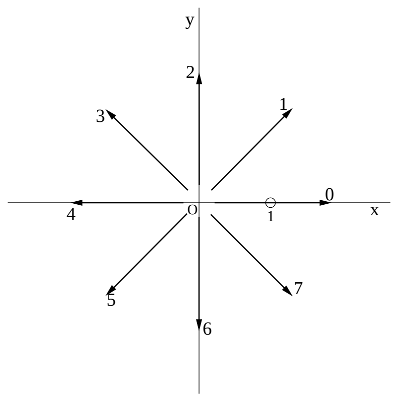
1. $\frac{d}{dz}(\cos z) = -\sin z$ and $\frac{d}{dz}(\sin z) = \cos z$. [First establish that $\cos z = \cos x \cosh y - i \sin x \sinh y$, $\sin z = \sin x \cosh y + i \cos x \sinh y$, and show analyticity.]
Comment: Though the formula is the same as in ordinary calculus, the derivation is non-trivial. Formulations should not be borrowed from ordinary calculus into complex analysis as obvious facts. For example, note that, with complex z, the simple result, $|\sin z| \leq 1$, of real functions is no more true.

2. (a) [$e^{z+2\pi i} = e^z$. Period $= 2\pi i$.]
Comment: The strip $\{(x, y) : -\infty < x < \infty, 0 \leq y \leq 2\pi\}$ maps to the entire w-plane. And, so does any strip $\{(x, y) : -\infty < x < \infty, \theta \leq y \leq \theta + 2\pi\}$.

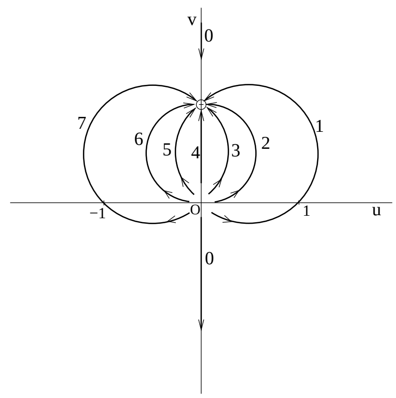
(b) $\text{Ln}\, z = \ln r + i\theta$, $-\pi < \theta \le \pi$. Derivative $\frac{d}{dz}[\text{Ln}\, z] = \frac{1}{z}$ exists everywhere except the negative real axis (i.e. $\theta = \pm\pi$), including the origin ($r = 0$).

3. $v(x, y) = e^{-x}(x \cos y + y \sin y) + c$. $w(z) = i(ze^{-z} + c)$.

4. (a) $w(z) = \frac{iz}{z-1}$.
Comment: Mapping of three independent points is enough to define a linear fractional transformation.
(b) $\theta = k\frac{\pi}{4}$ maps to $\tan k\frac{\pi}{4} = \frac{u}{u^2+v(v-1)}$, which are circular arcs. See Fig. A.52 for the conformal mapping, in which the rays (in z-plane) and their images (in w-plane) are annotated with the values (0 to 7) of k. [Inverse transformation is given by $z(w) = \frac{w}{w-i} = \frac{u^2+v(v-1)+iu}{u^2+(v-1)^2}$.]
Comment: Note that rays in w-plane are in order, at angles of $\frac{\pi}{4}$, as in z-plane, since the mapping is conformal. Each image starts at origin and terminates at i, which is the image of the point at infinity. Even at $w = i$, the angle-preservation is evident, since the mapping is conformal at $z = \infty$. From the way all rays converge at a single point through the mapping with all nice properties intact, it becomes clear why it makes good sense to consider the so-called point at infinity as a single point. Note also the typical mapping of the ray for $k = 0$ passing through the point $z = 0$, which is mapped to the point at infinity in the w-plane.



(a) The z-plane



(b) The w-plane

Figure A.52: Example of linear fractional transformation

5. Potential in the sector is $V(z) = V_2 + \frac{2}{\pi}(V_1 - V_2)\text{Re}(\sin^{-1} z^2)$ which can be further reduced to $V(z) = V_2 + \frac{2}{\pi}(V_1 - V_2)\sin^{-1}\left[\frac{1}{2}\left(\sqrt{r^4 + 2r^2\cos 2\theta + 1} - \sqrt{r^4 - 2r^2\cos 2\theta + 1}\right)\right]$.
[Successive conformal mappings with $w = z^2$ and $\sin\frac{\pi\zeta}{2} = w$ accomplish the required mapping of the given domain in the z-plane to a semi-infinite strip in the ζ-plane, as shown in Fig. A.53.]

6. $u(r, \theta) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\frac{2r\sin\theta}{1-r^2}$. [Use Poisson's integral formula.]

7. [Suppose that $\Phi(w) = \phi(u, v) + \psi(u, v)$ defines a potential flow in the w-plane, related to the z-plane through a conformal mapping $w(z) = u(x, y) + iv(x, y)$. Then, ϕ and ψ are harmonic over u, v, while u and v are harmonic over x, y. Use this along with the Cauchy-Riemann conditions to show that $\phi(x, y)$ and $\psi(x, y)$ are harmonic functions satisfying Cauchy-Riemann conditions in z-plane as well.]

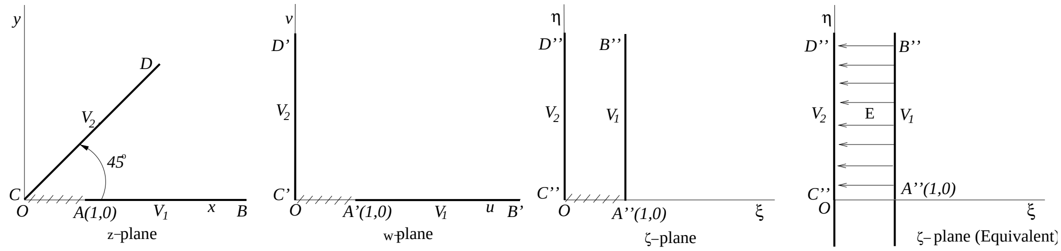


Figure A.53: Example of conformal mappings in a Dirichlet problem

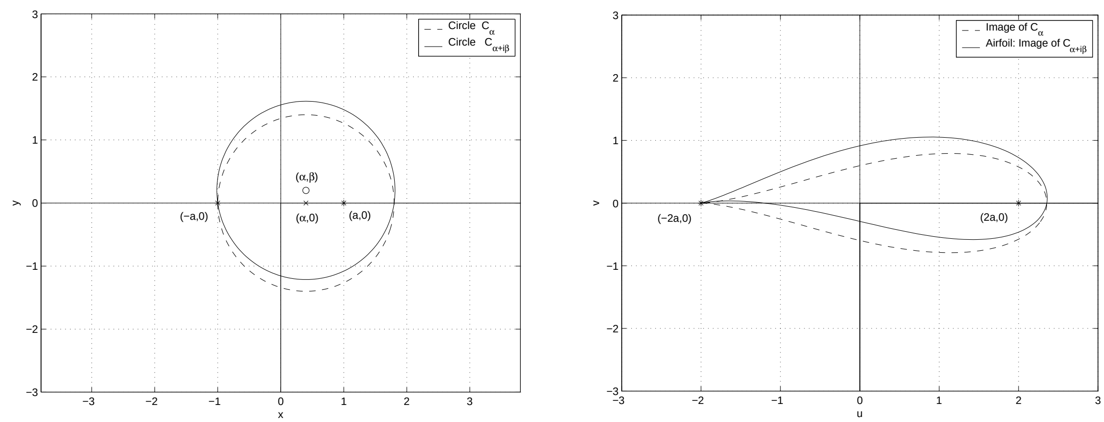
8. (a) [Potential function: $\phi(x,y) = Ux(x^2 - 3y^2)$. Stream function: $\psi(x,y) = Uy(3x^2 - y^2)$. In particular, $\psi(x,y) = 0$ along $y = 0$ and $y = \sqrt{3}x$, enclosing a sector of interior angle $\frac{\pi}{3}$.]
   (b) Flow around a sharp corner. [As $w = z^6 = r^6 e^{i(6\theta)}$, the sector angle becomes 6 times $\frac{\pi}{3}$, i.e. 2π. Alternatively, in the w-plane, complex potential function is $\tilde{\Phi}(w) = Uw^{1/2}$, from which velocity components can be determined and the same conclusion is arrived at independently.]

9. (a) Potential function $\phi(r,\theta) = \left(r + \frac{a^2}{r}\right)\cos\theta$, potential curves are $(r^2 + a^2)\cos\theta = cr$.
   Stream function $\psi(r,\theta) = \left(r - \frac{a^2}{r}\right)\sin\theta$ streamlines are $(r^2 - a^2)\sin\theta = kr$.
   Streamline $k = 0$ constitutes of three branches $\theta = 0$, $\theta = \pi$ and $r = a$, meeting at stagnation points $z = \pm a$, signifying $r = a$ as a circular obstacle.
   For $r >> a$ (far away) and on the line $\theta = \frac{\pi}{2}$, the flow is horizontal.
   In summary, Joukowski transformation models the flow past a circular cylinder.
   (b) Image: $w = 2a\cos\theta$, line segment from $-2a$ to $2a$ along real axis (covered twice).
   Interpretation: Flow past a circular cylinder in z-plane is equivalent to uniform flow in w-plane.
   (c) See the circle C_α given by $z = \alpha + (a+\alpha)e^{i\theta}$ in Fig. A.54(a) and its image in Fig. A.54(b), both with dashed lines.
   (d) For the circle $C_{\alpha+i\beta}$ given by $z = \alpha + i\beta + \sqrt{(a+\alpha)^2 + \beta^2}e^{i\theta}$, as shown in Fig. A.54(a), the image is as shown with solid line in Fig. A.54(b), an airfoil profile.
   Comment: Flow past an airfoil in the w-plane is equivalent to flow past a circular cylinder in the z-plane. In aerodynamics, Joukowski transformation is vital in the study of flow around airfoil profiles, and hence also in the design of airfoils.

Chapter 46 (page 404)

1. $2\pi i \rho^2$.

2. $\int z^n dz = \frac{z^{n+1}}{n+1}$ except for $n = -1$. For integer n, other than -1, it is single-valued, hence along a closed curve the integral is zero. For rational n, it is multi-valued, but the same result appears as we stay consistent with a single branch, mostly the one of the principal value. For irrational n, the only interpretation is through logarithm. For $n = -1$, we have $\int \frac{1}{z}dz = \ln z$. With one cycle around the origin, $\ln z$ just changes its branch once, from whichever point we start. Therefore, we get $\int \frac{1}{z}dz = 2\pi i$, the difference of $\ln z$ between two adjacent branches or the period of e^z.



(a) Offset circles in the z-plane

(b) Images in the w-plane

Figure A.54: Airfoil profile through Joukowski transformation

3. [Continuity of $f(z)$ and the property $\oint_C f(z)dz = 0$ are enough to establish the indefinite integral $F(z) = \int f(z)dz$. Then, use Cauchy's integral formula for derivative.]

4. Zero. [The function is analytic and the ellipse is a closed path.]

5. (a) πe^{2i}. (b) $\pi(8i-5)$.

6. [Connect C_1 and C_2 with a line segment L. Consider the integral along the contour consisting of C_1 (counterclockwise), L, C_2 (clockwise) and L (reverse).]

7. (a) $\oint \frac{f(z)}{z-w}dz = 2\pi i f(w)$ and $\oint \frac{f(z)}{z-\bar{w}}dz = 0$. [$\bar{w}$ is outside the contour of integration.]
   (b) $f(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta f(x)}{(x-\xi)^2+\eta^2}dx$. [Difference of the two integrals above gives $2\pi i f(w) = \oint \frac{2i\eta f(z)}{(z-\xi)^2+\eta^2}dz$. Over the semi-circle, $\left|\int_S \frac{\eta f(z)}{(z-\xi)^2+\eta^2}dz\right| < \frac{\pi|\eta|M}{R^{k-1}(R-\xi)^2}$.]

Chapter 47 (page 412)

1. $\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \frac{z^9}{9} - \frac{z^{11}}{11} + \cdots$, radius of convergence = 1.
   [Integrate $\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + z^8 - z^{10} + \cdots$.]

2. There are two possible Laurent's series, valid in different regions of convergence:
   (a) $\frac{z^2+1}{z^3-4z} = -\frac{1}{4z} - \frac{5z}{4^2} - \frac{5z^3}{4^3} - \frac{5z^5}{4^4} - \frac{5z^7}{4^5} - \cdots$ for $0 < |z| < 2$, and
   (b) $\frac{z^2+1}{z^3-4z} = \frac{1}{z} + \frac{5}{z^3} + \frac{5\cdot 4}{z^5} + \frac{5\cdot 4^2}{z^7} + \cdots$ for $2 < |z| < \infty$.

3. [For the construction, proceed as the first part of the construction of Laurent's series (see page 407). Show that the remainder term T_n is bounded as

$$|T_n| < \left|\frac{z-z_0}{w-z_0}\right|^n \frac{M}{1-\frac{|z-z_0|}{r}} \to 0,$$

as $n \to \infty$, where M is the bound on $f(z)$ on C and r is the closest distance of the contour from z_0.]

4. Zeros of $\sin z$: $n\pi$ for $n = 0, \pm 1, \pm 2, \cdots$, all real and isolated.
Zeros of $\sin \frac{1}{z}$: $\frac{1}{n\pi}$ for $n = \pm 1, \pm 2, \cdots$, all real and isolated, apart from $z = 0$.
Comment: The origin is an accumulation point of the zeros of $\sin \frac{1}{z}$. For arbitrarily small ϵ, it has a zero $\frac{1}{n\pi} < \epsilon$ for a value of $n > \frac{1}{\pi\epsilon}$.

5. At the origin, $e^{1/z}$ has an isolated essential singularity, and no zero in its neighbourhood. Arbitrarily close to this point, the function can assume any given (non-zero) value. For arbitrary complex number $Ae^{i\phi}$, $z = \frac{1}{\sqrt{(\ln A)^2 + (\phi)^2}} e^{i \tan^{-1}(-\phi/\ln A)}$ satisfies $e^{1/z} = Ae^{i\phi}$. To make $|z|$ arbitrarily small, one can add $2k\pi$ to ϕ.

6. (a) $2\pi i n$. $[f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic with no zero inside C.]
(b) $-2\pi i p$. $[f(z) = \frac{h(z)}{(z-z_0)^p}$, where $h(z)$ is analytic with no zero inside C.]
(c) [Enclose individual zeros or poles in disjoint (small) circles and evaluate $\oint_C \frac{f'(z)}{f(z)} dz$ with Cauchy's integral theorem for multiply connected domain, using the above results.]

7. $\frac{p(z_0)}{q'(z_0)}$. [Since $q(z_0) = 0$, we have $q(z) = q'(z_0)(z - z_0) + \frac{1}{2} q''(z_0)(z - z_0)^2 + \cdots$.]

8. $2\pi i$. [In whatever orientation you consider the ellipse, both the poles lie inside.]

9. $u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos p(x+ct) + \cos p(x-ct)}{1+p^2} dp = \frac{e^{-|x+ct|} + e^{-|x-ct|}}{2}$. See Fig. A.55.
[Using complementary sine terms, you can reduce the integral into two equivalent forms
$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(x+ct)} + e^{ip(x-ct)}}{1+p^2} dp \quad \text{or} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ip(x+ct)} + e^{-ip(x-ct)}}{1+p^2} dp.$$
After establishing $\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\alpha p}}{1+p^2} dp = \pi e^{-\alpha}$ for $\alpha \geq 0$, use the first form with $\alpha = x + ct$ and $\alpha = x - ct$ for $x \geq ct$ and with $\alpha = -x - ct$ and $\alpha = -x + ct$ for $x \leq -ct$. For $-ct < x < ct$, use the second form with $\alpha = x + ct$ and $\alpha = -x + ct$.]

10. [For evaluating $I = \frac{1}{2} e^{-\left(\frac{b}{a}\right)^2} \int_{-\infty}^{\infty} e^{-a^2\left(p - \frac{ib}{a^2}\right)^2} dp = \frac{1}{2} e^{-\left(\frac{b}{a}\right)^2} J$, consider the rectangular contour C in the plane of $z = p + iq$, with vertices $-d$, d, $d + \frac{ib}{a^2}$ and $-d + \frac{ib}{a^2}$. Formulate the contour integral and let $d \to \infty$.]

Chapter 48 (page 426)

1. (a) $s(\alpha, \beta) = \frac{m}{6}(\alpha^2 + \beta^2)T^3 + \frac{m(a^2+b^2)}{2T} + mg\left(\frac{\beta T^3}{6} - \frac{bT}{2}\right)$.
(b) $\alpha = 0$, $\beta = -g/2$. Consequently, $x(t) = at/T$, $y(t) = -\frac{g}{2}t(t - T) + bt/T$; $\dot{x}(t) = a/T$, $\dot{y}(t) = -\frac{g}{2}(2t - T) + b/T$; $\ddot{x}(t) = 0$, $\ddot{y}(t) = -g$.
Comment: Application of Hamilton's principle of least action led to this direct consequence of Newton's second law. Using the principle appropriately, one can *derive* the second law of motion.
(c) [Use boundary conditions $\delta x = \delta y = 0$ to evaluate $\delta s = \int_0^T (m\dot{x}\delta\dot{x} + m\dot{y}\delta\dot{y} - mg\delta y) dt$.]

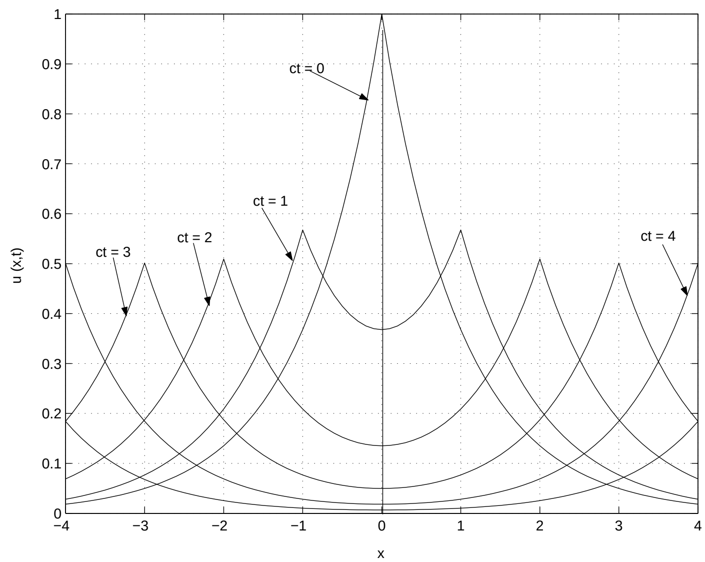


Figure A.55: Response of an infinite string in a Cauchy problem

2. (a) [Using Euler's equation, establish the (total) derivative of $y'\frac{\partial f}{\partial y'}$ with respect to x as $\frac{df}{dx}$, and then integrate both sides.]
(b) $y(1+y'^2) = \frac{1}{2gC^2} = k$.
(c) The cycloid $x = \frac{k}{2}(\theta - \sin\theta) + a, y = \frac{k}{2}(1 - \cos\theta)$.

3. You can appreciate this formulation in two alternative ways.

- As in the constrained optimization of ordinary functions, convert the problem to an unconstrained one, for the functional $\bar{I}[y(x), \lambda] = I + \lambda(J - J_0)$.
- Introducing function $z(x) = \int_a^x g dx$, with $z(a) = 0$, $z(b) = J_0$ $z'(x) = g(x, y, y')$, get the constraint in the form $\phi(x, y, y') = g(x, y, y') - z'(x)$, as discussed in the text of the chapter.

(a) $[r(x)y']' + [q(x) + \lambda p(x)]y = 0$.
Comment: This is the variational formulation of the Sturm-Liouville (eigenvalue) problem.
(b) $\inf I[y] = \min_{[a,b]}\left[-\frac{q(x)}{p(x)}\right]$. [To obtain the least eigenvalue, multiply the above equation with y and integrate by parts.]

4. $\frac{\partial L}{\partial \phi} + \nabla \cdot \frac{\partial L}{\partial \mathbf{E}} = 0$. [Total energy is $I[\phi(\mathbf{r})] = \int\int_T\int L(\mathbf{r}, \phi, \mathbf{E})dv$. In the first variation $\delta I[\phi] = \int\int_T\int\left[\frac{\partial L}{\partial \phi} + \left(\frac{\partial L}{\partial \mathbf{E}}\right)^T \delta\mathbf{E}\right] dv$, expand the second term in the integrand as

$$\left(\frac{\partial L}{\partial \mathbf{E}}\right)^T \delta(-\nabla\phi) = -\frac{\partial L}{\partial \mathbf{E}} \cdot \nabla(\delta\phi) = -\nabla \cdot \left[\delta\phi\frac{\partial L}{\partial \mathbf{E}}\right] + \delta\phi\nabla \cdot \frac{\partial L}{\partial \mathbf{E}}.$$

The integral of the first term in this expression vanishes by the application of Gauss's divergence theorem.]

5. $f(x) \approx 0.2348x^2 - 0.0041x^3 + 0.0089x^4 - 0.0089x^5 + 0.0024x^6 - 0.0003x^7 + 0.00001x^8$.
[Solve the equations (in coefficients) in the form $\int_0^5 (f''' + ff'')x^i dx$ along with the boundary condition $f'(5) = 1$ as a nonlinear least square. The function evaluation in the least square problem involves the evaluation of these integrals, which can be performed numerically.]
Comment: The reader should verify that the solution matches with that obtained with shooting method in Chap. 30. (See Fig. A.28.)