Pre-requisite Problem Sets

Problem set 1

1. Find the point on the line $y = 3x + 1$ that is equidistant from $(0, 0)$ and $(-3, 4)$.

2. Express the coordinates of a point Q, lying in the first quadrant and on the parabola $y = x^2$, as functions of the angle of inclination of the line joining Q to the origin.

3. Sketch the graph of the equation $y = 1 + \sin 2(x + \pi/4)$.

4. Observers at points A and B, which are 2 km apart, simultaneously measure the angle of elevation of a helicopter to be $40°$ and $80°$, respectively. If the helicopter is directly above a point on the *line segment* AB, then find its height.

5. Plot the function

$$g(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x > 1. \end{cases}$$

   Discuss the limits, one-sided limits, continuity and one-sided continuity of g at each of the points $x = -1, 0$ and 1. Identify removable discontinuities, if any.

6. If $f(x)$ and $g(x)$ are defined for all x and $\lim_{x \to c} f(x) = -7$ and $\lim_{x \to c} g(x) = 0$, then find out the limits of the following functions as $x \to c$:

$$(i)\ \frac{f(x)}{g(x) - 7}, \qquad\qquad (ii)\ f(x) \cdot g(x).$$

7. Evaluate the limit of $g(x)$ as $x \to \sqrt{5}$, such that $\lim_{x \to \sqrt{5}} \frac{1}{x + g(x)} = 2$.

8. Does the limit $\lim_{x \to 1} \frac{1 - \sqrt{x}}{1 - x}$ exist? If so, then evaluate it.

9. Plot the function $f(x) = x(x^2 - 1)/|x^2 - 1|$. Is it possible to extend it so as to make it continuous at $x = 1$ or -1?

10. Let $f(x) = x^3 - x - 1$.

    (a) Show that $f(x)$ has a zero in the interval $[-1, 2]$.

    (b) Find out an approximate value of the zero by graphical method.

    (c) Show that the exact value of the zero is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3}.$$

    Find out the numerical value of this expression and compare it with the graphical solution.

Problem set 2

1. Differentiate $y = x^{-2}\sin^2(x^3)$.

2. Find dy/dx if $x^3 + 4xy - 3y^{4/3} = 2x$.

3. If $x^3 + y^3 = 8$, find d^2y/dx^2 by implicit differentiation, or directly.

4. Determine dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

5. If

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

   then

   (a) plot the function, and

   (b) investigate its continuity and differentiability at $x = 0$.

6. Show that the tangents to the curve $y = (\pi \sin x)/x$ at points $x = \pi$ and $x = -\pi$ intersect at right angles.

7. Find out the lines that are tangent and normal to the curve $(y - x)^2 = 2x + 4$ at $(6, 2)$.

8. A particle moves along the curve $y = x^{5/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 14 units per second. Find dx/dt when $x = 2$.

Problem set 3

1. Find out values of a and b so that the function $f(x) = \frac{ax+b}{x^2-1}$ has a local extremum of 1 at $x = 3$. Classify this extremum as maximum or minimum?

2. Show that the solution of the equation $x^4 + 2x^2 - 2 = 0$ on the interval $[0, 1]$ is unique. Find the solution.

3. If $y' = 4x^2 - x^4$, examine $y(x)$ for local maxima, minima or inflection points? Sketch the general shape of the graph.

4. Find the height and the radius of the largest right circular cylinder that can be enclosed inside a sphere of radius $\sqrt{3}$.

5. Develop a formula to estimate the change in the volume of a right circular cone if the radius changes from r_0 to $r_0 + dr$ and the height remains unaltered.

6. Evaluate the total area of the region bounded by $f(x) = 1 - \sqrt{x},\ 0 \le x \le 4$ and the x-axis.

7. Solve the initial value problem: $\frac{d^3r}{dt^3} = -\cos t$; $r''(0) = r'(0) = 0,\ r(0) = -1$.

8. Express y in terms of a quadrature (integral) if $\frac{dy}{dx} = \frac{\sin x}{x}$ and $y(5) = -3$.

9. Evaluate $\int \left(\frac{1}{\sqrt{2\theta - \pi}} + 2\sec^2(2\theta - \pi)\right) d\theta$.

10. Evaluate $\int_{-\pi/2}^{\pi/2} 15 \sin^4 3x \ \cos 3x \ dx$.

Problem set 4

1. Find the area of the region enclosed by the curve $y^2 = 4x$, and line $y = 4x - 2$.

2. Find out the volume of the solid generated by revolving the area between the x-axis and the curve $y = x^2 - 2x$ about (a) the x- axis; (b) the line $x = 2$.

3. Find the length of the curve $x = (y^3/12) + (1/y)$ for $1 \leq y \leq 2$.

4. Find the centre of mass of a thin, flat lamina covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

5. A force of 100 N stretches a garage door spring 0.4 m beyond its unstressed length. How far will a 300 N force stretch the spring? How much work does it take to stretch the spring this far?

6. If the velocity of a particle moving along a coordinate line is $v = t^3 - 3t^2 + 2t$ (m/sec), then find

   (a) the total distance the particle travels during the time interval $0 \leq t \leq 2$, and

   (b) its displacement during the same time interval.

7. Find dy/du if $y = \sin^{-1}\sqrt{1-u^2}$, $0 < u < 1$.

8. Using logarithmic differentiation (or otherwise), find dy/du if $y = \frac{2u2^u}{\sqrt{u^2+1}}$.

9. Evaluate $\int_1^8 \frac{\log_4 \theta}{\theta}\, d\theta$.

10. Evaluate $\int \frac{dt}{(t+1)\sqrt{t^2+2t-8}}$.

11. Evaluate $\lim_{x\to 0} \frac{5-5\cos x}{e^x - x - 1}$.

12. A girl is sliding down a curved slide whose equation is $y = 9e^{-x/3}$. Her altitude is changing at the rate $dy/dt = (-1/4)\sqrt{9-y}$ units. Find out the approximate value of $\frac{dx}{dt}$ when she reaches the bottom of the slide at $x = 9$ units? (Take $e^3 = 20$ and round-off your answer to the nearest integer value).

13. Solve the initial value problem (IVP) $x\frac{dy}{dx} + 2y = x^2 + 1$, $x > 0$, $y(1) = 1$.

Problem set 5

1. Integrate

   (a) $\int z^{-1/5}(1 + z^{4/5})^{-1/2}\, dz$,

   (b) $\int \frac{dv}{v \log v}$,

   (c) $\int \frac{\cot x}{\cot x + \csc x}\, dx$,

   (d) $\int e^{ax} \sin bx\, dx$,

   (e) $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\, dx$.

2. Integrate $\int \frac{x}{\sqrt{4+x^2}}\, dx$

   (a) without using trigonometric substitution,

   (b) using trigonometric substitution.

3. Find a vector of magnitude 5 units in the direction opposite to $\mathbf{A} = (3/5)\mathbf{i} + (4/5)\mathbf{k}$.

4. If $\mathbf{A} = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{B} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}$, Find $|\mathbf{A}|$, $|\mathbf{B}|$, $\mathbf{A} \cdot \mathbf{B}$, $\mathbf{B} \cdot \mathbf{A}$, $\mathbf{A} \times \mathbf{B}$, $\mathbf{B} \times \mathbf{A}$, $|\mathbf{A} \times \mathbf{B}|$, the angle between the directions of $\mathbf{A}$ and $\mathbf{B}$, the (scalar) component of $\mathbf{B}$ in the direction of $\mathbf{A}$, and the vector projection of $\mathbf{B}$ onto $\mathbf{A}$.

5. Write $\mathbf{B}$ as the sum of two vectors, one parallel to $\mathbf{A}$ and the other orthogonal to it, if $\mathbf{A} = 2\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

6. Let ABC be the triangle determined by vectors $\mathbf{u}$ and $\mathbf{v}$ as two of its sides, away from the common vertex.

   (a) Express the area of ΔABC in terms of $\mathbf{u}$ and $\mathbf{v}$.

   (b) Express the triangle's altitude h, from the third side, in terms of $\mathbf{u}$ and $\mathbf{v}$.

   (c) Evaluate both area and altitude if $\mathbf{u} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{k}$.

7. Find an equation for the plane through $A(-2, 0, -3)$ and $B(1, -2, 1)$ that lies parallel to the line through $C(-2, 5, 5)$ and $D(5, 5, -2)$.

Problem set 6

1. Find the domain and range of the function $f(x, y) = 9x^2 + 4y^2$ and sketch its contours.

2. Find the partial derivative of the function $f(r, l, T, w) = \frac{1}{2rl}\sqrt{\frac{T}{\pi w}}$ with respect to each of the variables.

3. Find all the second order partial derivatives of the function $f(x, y) = x + xy - 5x^3 + \ln(x^2 + 1)$.

4. Around the point $(1, 2)$, is the function $f(x, y) = x^2 - xy + y^2 - 3$ more sensitive to changes in x, or to changes in y?

5. Find dw/dt at $t = 0$ if $w = \sin(xy + \pi)$, $x = e^t$ and $y = \ln(t + 1)$.

6. Sketch the domain of integration $\int_0^1 \int_{\sqrt{y}}^{2-\sqrt{y}} xy\, dx\, dy$ and evaluate the double integral.

7. Evaluate $\int_0^8 \int_{x^{1/3}}^2 \frac{dy dx}{y^4+1}$.

8. Determine the volume under the parabolic cylinder $z = x^2$ and above the planar region, which is enclosed by the parabola $y = 6 - x^2$ and the line $y = x$ in the xy -plane.

9. Find the mass and the first moments about the coordinate axes of a thin square plate, the boundaries of which are given by lines $x = \pm 1$, $y = \pm 1$ if the density is $\rho(x, y) = x^2 + y^2 + 1$.

10. Evaluate the integral by changing to polar coordinates:

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\, dx\, dy.$$

11. Convert $\int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} 3\, dz\, r\, dr\, d\theta$, $r \geq 0$ to

    (a) rectangular coordinates with the order of integration $dz\, dx\, dy$, and

    (b) spherical coordinates;

    (c) then, evaluate one of the integrals.