SYNOPSIS

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Degree for which submitted: **Ph. D.** Department: **Mathematics**

Thesis Title: **A Study on Coalescence and Spline Fractal Interpolation Functions**

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Month & Year of thesis submission: **December 2004**

Mandelbrot [*Les objects fractals : forme, hasard et dimension, Flammarion, 1975*] introduced *fractals*, to describe objects that were too irregular to fit in realm of Euclidean geometry. Since, then it has turned out to be immensely popular tool for applications in Computer graphics, Physics, Complex Dynamics, Fracture Mechanics, Signal processing, Image processing, Bio-engineering, Financial series, etc. The fractals arising from Iterated Function System (IFS) [*Proc. R. Soc. Lond. A*, **399**, 1985, 243-274], possess many interesting features, most notably self-affinity or non-self-affinity and infinite details regardless of magnification. IFS theory has provided a major impetus in research and led to a closer understanding of many complex phenomena observed in nature and in several seemingly different areas of sciences and engineering. M. F. Barnsley [*Constr. Approx.*, **2**, 1986, 303-329] introduced *Fractal Interpolation Functions* (FIF) to generate and classify a large class of fractals through Hutchinson's operator [*Indiana Univ. Math. J.*, **30**, 1981, 713-747]. FIF are specially suited to describe objects in nature that display some kind of geometrical complexity under magnification. FIF, in general, are self-affine in nature and the Hausdorff-Besicovitch dimensions of their graph are non-integers. To approximate non-self-affine patterns found in nature, the hidden variable FIF are constructed by M. F. Barnsley, et. al [*SIAM. J. Math. Anal.*, **20(5)**, 1989, 1218-1242] employing projection of vector valued FIF for generalized interpolation data.

In practical applications of FIF, the interpolation data might be generated simultaneously from self-affine and non-self-affine functions. Thus, the question that whether it is possible to construct an IFS that gives both self-affine or non-self-affine FIF simultaneously needs to be settled. This is achieved in the present work by the construction of *Coalescence FIF*. Also, since the smoothness analysis of non-self-affine FIF arising from IFS theory is still an open problem, such an analysis is carried out in the present work. The results concerning stability and integral moment theory of Coalescence Affine FIF are needed due to importance of their applications in various science and engineering problems. Such results are found in the present work. Stochastic methods are not generally suitable for constructing fractal surfaces that exactly pass through the prescribed data points. Depending on the nature of an object (self-affine or non-self-affine), we need approximating Fractal Interpolation Surface (FIS) or a hidden variable FIS. In this direction, an attempt has been made only to construct self-affine bivariate FIS. However, the construction of non-self-affine bivariate FIS has not been attempted so far. Also, the question whether it is possible to construct an IFS that gives both self-affine or non-self-affine FIS simultaneously needs to be investigated. These problems are also addressed to in the present work.

The construction of C^r-FIF that generalize classical splines are constructed by taking successive integrals of fractal functions. However, the existing construction method involves complicated matrices so that general boundary conditions for spline FIF, similar to those in the construction of classical splines, can not be specified. In the present work, an attempt is made to construct C^r-FIF with general boundary conditions by introducing a simpler method. Since classical cubic splines have been applied in variety of problems for over last 40 years and Cubic Spline FIF is a natural generalization of classical cubic spline, a need of construction of Cubic Spline FIF through moments and study of convergence results of these are strongly felt. Some convergence results for Cubic Spline FIF with fixed scaling factors, from equidistant interpolation data have been found, if the data generating function is of class C^4. However, existence and methods of construction of Cubic Spline FIF through moments under general boundary conditions, are not known so far. Further, the convergence results of Cubic Spline FIF based on general interpolation data that are generated from a data generating function in the classes C^2, C^3 or C^4 have also remain uninvestigated. The construction of Cubic Spline FIF through moments and their convergence properties are studied in detail in this work. Similar to the classical splines, the construction of spline FIF whose certain derivative is also non-self-affine fractal function might turn out to be of wide use in various science and engineering applications. To this end, Coalescence Spline FIF are introduced in the present work. The construction of these spline FIF through moments and their convergence properties are studied in detail in the present work.

The organization of the thesis is as follows:

Chapter 1, being the introduction, consists a brief review of the history, concepts, basic results and applications concerning IFS, FIF, FIS and spline FIF that are related to the present work.

The existence, construction, smoothness analysis and fractal dimension of Coalescence FIF are studied in *Chapter 2*. In Section 2.1, the existence and method of construction of Coalescence Affine FIF are found for interpolation data in $\mathbb{R}^2$. For this purpose, an IFS is constructed in $\mathbb{R}^3$ from a generalized interpolation data with the introduction of *constrained free variable.* It turns out through the investigations in present work that contrary to the observation of Barnsley [*Fractals Everywhere*, Academic Press, 1988], hidden variable FIF is indeed self-affine under certain conditions. The order of modulus of continuity for a Coalescence FIF is investigated in Section 2.2 by using the operator approximation technique. The smoothness result for a non-self-affine FIF is found for the first time here. It is observed that the deterministic construction of functions having order of modulus of continuity as $O(|t|^{\mu}(\log|t|)^n)$ (n is a non-negative integer and $0 < \mu \leq 1$) is possible through Coalescence FIF. The smoothness results of Gang [*Appl-Math. J. Chinese Univ. Ser. B*, **11**,1996, 409-428] for self-affine FIF follows as a special case of the smoothness results are found in the present section. The bounds on fractal dimension of Coalescence FIF in critical cases are found in Section 2.3. The fractal dimension bounds of Gang for critical self-affine FIF follow as a particular case of the results of this section. In Section 2.4, the construction method for Coalescence FIF and the effects of hidden variables on theirs nature is illustrated through suitable examples.

The stability and integral moment theory of Coalescence Affine FIF are studied in *Chapter 3*. An auxiliary smoothness result is derived for a Coalescence Affine FIF for generalized interpolation data in Section 3.1. In Section 3.2, the main stability result for Coalescence Affine FIF is proved by first developing the stability results for co-ordinate wise perturbations in generalized interpolation data. The result in this section generalizes an earlier result of Feng and Xie [*Fractals*, **6(3)**, 1998, 269-273] for equidistant interpolation data. In Section 3.3, the interrelation of integral moment of Coalescence Affine FIF with its lower order moments, IFS parameters and integral moments of self-affine fractals is derived. Finally, an explicit expression of inner product in terms of IFS parameters and integral moments for Coalescence Affine FIF is derived in this section that was hitherto possible only for self-affine FIF [*J. Apprx. Theory*, **71**, 1992, 104-120].

In *Chapter 4*, a method of construction for *Coalescence Bivariate FIS* is developed by defining a suitable vector-valued IFS such that projection of its attractor is the required interpolation surface that may be self-affine or non-self-affine in nature depending on IFS parameters. In Section 4.1, the principle of construction of IFS for bivariate FIS is developed. By introducing the constrained free variables in Section 4.2, the unknowns for IFS are determined. Further, it is shown that the above IFS is such that continuity of the generated FIS is maintained at each point. In Section 4.3, the existence and uniqueness of Coalescence Bivariate FIS are established. Some of the results in this chapter generalize the results of Xie and Sun [*Fractals*, **5(4)**, 1997, 625-634] obtained for self-affine bivariate FIS. The effects of hidden variables on Coalescence Bivariate FIS and its roughness factors are illustrated in Section 4.4 with suitably chosen examples.

A simple method of construction for self-affine spline FIF in *Chapter 5* is introduced. Thus, basic calculus of a C^1-fractal function is reviewed in Section 5.1 and a general method for construction of a C^r-FIF f_2 with all admissible boundary conditions as in classical splines is enunciated by prescribing any combination of r-values of the derivatives $f_2^{(k)}$, $k = 1, 2, \ldots, r$, at boundary points of the interval $[x_0, x_N]$ in Section 5.2. The present construction method, due to functional relations between the values of C^r-FIF involving end points of the interval, is much simpler than that of Barnsley and Harrington [*J. Approx. Theory*, **57**, 1989, 14-34], wherein complicated matrices and particular types of end conditions are employed. Further, the present approach of construction of C^r-FIF takes care of several queries of Barnsley and Harrington. The explicit construction of Cubic Spline FIF $f_{2\Delta}(x)$ through *moments* is developed in Section 5.3 and the convergence of sequence of Cubic Spline FIF $\{f_{2\Delta_k}\}$ to $\Phi \in C^m[x_0, x_N]$,$m = 2, 3$ or 4 is established on two classes of sequences of uniform or non-uniform meshes in Section 5.4.

The present results are derived in a much general set up than that of Navascués and Sebastián [*Fractals*, **11(1)**, 2003, 1-7], who found the convergence of Cubic Spline FIF only with fixed scaling factors from equidistant interpolation data when the data generating function is in class C^4. Finally, in Section 5.5, the results in Section 5.3 are illustrated by generating certain examples of Cubic Spline FIF for a given data and two different sets of vertical scaling factors.

The Coalescence Spline FIF are introduced in *Chapter 6*. In Section 6.1, the calculus of vector valued fractal function f is studied and differentiable Coalescence Fractal Functions are constructed. A general method of construction for Coalescence Spline FIF with all possible boundary conditions is initiated in Section 6.2. The functional relations of Coalescence C^r-FIF at the end points of the interval with join-up conditions and interpolation conditions give a system of equations whose solution determines the coefficients of polynomials in the construction of the non-diagonal IFS. The advantage of such a construction is that, for a prescribed data and given boundary conditions, a choice of Coalescence Spline FIF that are self-affine or non-self-affine can be made according to the need of an experiment by suitably prescribing hidden variables of Coalescence Spline FIF, free variables, constrained free variables and *boundary conditions of the spline fractal function*. In Section 6.3, the construction of Coalescence Cubic Spline FIF $f_{1\Delta}(x)$ on a mesh Δ is developed through *moments* $M_n^* = f_{1\Delta}^{''}(x_n), n = 0, 1, 2, \ldots, N$ with any type of boundary conditions as in classical cubic spline. In Section 6.4, the convergence results of the sequence of Coalescence Cubic Spline FIF $\{f_{1\Delta_k}(x)\}$ to the interpolation data function $\Phi(x)$ on two classes of sequence of meshes are proved when $\Phi^{(r)}(x)$ is continuous on $[x_0, x_N]$ for $r = 2, 3$ or 4. Finally, some of the examples of Coalescence Cubic Spline FIF are generated in Section 6.5 to illustrate the results of Section 6.3 concerning the effect of hidden variables on the nature of Coalescence Cubic Spline FIF.