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Dynamics of $(e^z - 1)/z$: the Julia set and bifurcation

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Abstract. We describe the dynamical behaviour of the entire transcendental non-critically finite function $f_\lambda(z) = \lambda(e^z - 1)/z$, $\lambda > 0$. Our main result is to obtain a computationally useful characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of points with orbits escaping to infinity under iteration, which in turn is applied to the generation of the pictures of the Julia set of $f_\lambda(z)$. Such a characterization was hitherto known only for critically finite entire transcendental functions [**11**]. We find that bifurcation in the dynamics of $f_\lambda(z)$ occurs at $\lambda = \lambda^*$ (≈ 0.64761) where $\lambda^* = (x^*)^2/(e^{x^*} - 1)$ and x^* is the unique positive real root of the equation $e^x(2 - x) - 2 = 0$.

0. *Introduction*

In complex analytic dynamics, most of the work has centred around the dynamics of rational functions and entire functions. The behaviour of orbits of the critical values and asymptotic values of a function play an important role in determining the dynamics of a function. Devaney and coworkers studied mainly the dynamics of critically finite functions (i.e. having only finitely many critical values and asymptotic values). Exploiting the critical finiteness of the function, Devaney [**2–8**], Devaney and Durkin [**9**], Devaney and Krych [**10**] and Devaney and Tangerman [**11**] analyzed exhaustively the dynamics of some of the most interesting periodic functions like λe^z, $\lambda \sin z$ and $\lambda \cos z$. The dynamical behaviour of critically finite entire transcendental functions shares many of the properties of polynomials and rational functions, e.g. these functions do not have wandering domains [**13, 14**].

The central objects studied in the complex analytic dynamics of a function are its Julia set and Fatou set. The *Julia set* or the set of non-normality of a function f, denoted by $\mathcal{J}(f)$, is defined to be the set of all complex numbers where the family of iterates $\{f^n\}$ of f fails to be normal in the sense of Montel. The *Fatou set* or the set of normality, denoted by $\mathcal{F}(f)$, is the complement of the Julia set of f. The orbits of the points in the Julia set of f are unstable or chaotic in nature. On the other hand, the orbits of the points in the Fatou set of f have a stable behaviour. Therefore, the Julia set is known as the chaotic set (unstable set), while the Fatou set is known as the stable set.

Baker [**1**] obtained the well known characterization for the Julia set of entire functions as the closure of the set of repelling periodic points of f. Devaney and Tangerman [**11**] obtained another characterization that is found to be extremely helpful in computationally generating the Julia sets of critically finite entire transcendental functions. They proved that if f is a critically finite entire transcendental function and has exponential tract D which is hyperbolic with a given asymptotic direction, then the Julia set of f is the closure of the set consisting of all points whose orbits escape to infinity under iteration. Devaney and Tangerman also showed that the entire functions which are critically finite and meet certain growth conditions have Cantor bouquets in their Julia sets.

Devaney and Durkin [**9**], in their description of the Julia set $\mathcal{J}(E_\lambda)$ of $E_\lambda(z) = \lambda e^z$, $\lambda > 0$, illustrated the interesting phenomena of 'explosion' in $\mathcal{J}(E_\lambda)$. The function $E_\lambda(z)$ is periodic, has no critical values and just one asymptotic value, namely 0. For $0 < \lambda < 1/e$, the Julia set $\mathcal{J}(E_\lambda)$ is a nowhere dense subset of the right half-plane. If $\lambda > 1/e$, the orbit of the singular value '0' tends to infinity under iteration. Therefore, by Sullivan's theorem [**9**, Theorem 4.6] $\mathcal{J}(E_\lambda)$ equals the entire complex plane $\mathbb{C}$ for $\lambda > 1/e$. Thus, when the parameter λ crosses the value $1/e$, $\mathcal{J}(E_\lambda)$ suddenly explodes and equals the whole complex plane. This type of bifurcation is also found to occur in other families of functions like $\lambda \sin z$ and $\lambda \cos z$. The characterization of the Julia set of $E_\lambda(z)$ as the closure of the set of all *escaping points* (i.e. the points whose orbits escape to infinity under iteration of the function) gives an efficient algorithm for plotting the Julia sets of $E_\lambda(z)$ [**9**]. Durkin [**12**] analyzed the accuracy of this algorithm for the function λe^z. However, the dynamics of non-critically finite entire transcendental functions has not been explored so far, probably because of non-applicability of Sullivan's theorem to these functions. In the present paper an effort is made in this direction.

Let $f_\lambda(z) = \lambda(e^z - 1)/z$ for $z \in \mathbb{C} \setminus \{0\}$, $f_\lambda(0) = \lambda$ and $\lambda > 0$. In §1, we develop some of the basic properties of $f_\lambda(z)$. It is found in this section that $f_\lambda(z)$ has infinitely many critical values in the disk centred at the origin and having radius λ, and $f'_\lambda(z)$ has infinitely many zeros in the left half-plane. Further, we obtain some results concerning $f_\lambda(z)$ needed in subsequent sections. In §2, we describe the dynamics of $f_\lambda(x)$ for x belonging to the real line $\mathbb{R}$ and $\lambda > 0$. In this section, it is shown that there exists a critical parameter value $\lambda^* > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs at $\lambda = \lambda^*(\approx 0.64761)$, i.e. if the parameter value crosses the value λ^*, then a dramatic change in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs. The dynamics of $f_\lambda(z)$, $z \in \mathbb{C}$ for $0 < \lambda < \lambda^*$ is studied in §3. For this case, we prove two different characterizations for the Julia set of $f_\lambda(z)$. The first characterization gives the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, as the closure of the set of escaping points; while the second characterization, under a certain condition, describes it as the complement of the basin of attraction of an attractive real fixed point of $f_\lambda(z)$. Further, in this section, it is found that $\mathcal{J}(f_\lambda)$ is a nowhere dense subset of the right half-plane when $0 < \lambda < \lambda^*$. In §4, the dynamical behaviour of $f_\lambda(z)$ for $\lambda > \lambda^*$ is described. We prove that the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$ contains the entire real line. The characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of escaping points, analogous to the first characterization in §3 is also obtained in this case. In §5, the characterizations of the Julia set of $f_\lambda(z)$, obtained in §§3 and 4, are applied to computationally generate a picture of the Julia set of $f_\lambda(z)$

for different values of λ. Finally, the dynamics of $f_\lambda(z)$ are compared with that of the dynamics of $E_\lambda(z) = \lambda e^z$.

1. *Some basic properties of $f_\lambda(z)$*

In this section some of the basic properties of the function $f_\lambda(z) = \lambda(e^z - 1)/z$, $\lambda > 0$, are developed. In Proposition 1.1, it is shown that $f_\lambda(z)$ maps the left half-plane into an open disk centred at the origin and having radius λ. The critical points play an important role in the dynamics of a function. In Proposition 1.2, it is proved that all the critical points of $f_\lambda(z)$ (i.e. the points where $f'_\lambda(z) = 0$) are contained in the left half-plane. In Proposition 1.3, the function $f_\lambda(z)$ is found to possess infinitely many critical points and critical values. Proposition 1.4 shows that the function $f_\lambda(z)$ is one-to-one in any closed rectangle of the form $R_{a,b,c} = \{z = x + iy : 2 \le a \le x \le b, c \le y \le c + 2\pi\}$. Further, Proposition 1.5 and Proposition 1.6 in this section endeavour to find domains in the right half-plane for which $f_\lambda(z)$ is a homeomorphism. In Proposition 1.7, it is proved that the pre-images of real points are dense in the set of all escaping points.

We begin by proving a mapping property of $f_\lambda(z)$.

PROPOSITION 1.1. *$f_\lambda(z)$ maps the left half-plane $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$ into an open disk centred at the origin and having radius λ.*

Proof. For an arbitrarily fixed $z \in H^-$, let $g(z) = e^z$ and γ be the line segment defined by $\gamma(t) = tz$, $t \in [0, 1]$. Then,

$$\int_\gamma g(z)\,dz = \int_0^1 g(\gamma(t))\gamma'(t)\,dt = e^z - 1.$$

Since $M \equiv \max_{t\in[0,1]} |g(\gamma(t))| = \max_{t\in[0,1]} |e^{tz}| < 1$, for $z \in H^-$,

$$|e^z - 1| = \left|\int_\gamma g(z)\,dz\right| \le M\ (\text{length of } \gamma) < |z|.$$

Thus, $|(e^z - 1)/z| < 1$ for all $z \in H^-$. Consequently, $|f_\lambda(z)| = \lambda|(e^z - 1)/z| < \lambda$ for all $z \in H^-$. □

PROPOSITION 1.2. *$f'_\lambda(z)$ has no zeros in the right half-plane $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$.*

Proof. The function $f'_\lambda(z) = \lambda(e^z(z - 1) + 1)/z^2 = 0$ for $z \in H^+$ if and only if $e^z(z - 1) + 1 = 0$ for $z \in H^+$ if and only if $e^{-z} + (z - 1) = 0$ for $z \in H^+$. Define, for all $z \in \mathbb{C}$,

$$\begin{aligned} g_n(z) &= e^{-z} + z - \left(1 + \frac{1}{n}\right), \quad n = 1, 2, 3, \ldots \\ g(z) &= e^{-z} + z - 1. \end{aligned}$$

Then, $g_n(z)$ converges to $g(z)$ uniformly on every compact subset of $\mathbb{C}$. We first show that, for each $n \ge 1$, $g_n(z)$ has only one real zero in the region $\Omega_n(R) = \{z \in \mathbb{C} : |z| \le R, \Re(z) \ge 0, R > 2 + (1/n)\}$.

Let $\phi_n(z) = z - (1 + (1/n))$ and $h(z) = e^{-z}$. For $R > 2 + (1/n)$, choose $\Gamma_n(R)$ to be the boundary curve of $\Omega_n(R)$, i.e.

$$\begin{aligned} \Gamma_n(R) &= \{z \in \mathbb{C} : |z| = R, \Re(z) \geq 0\} \cup \{z \in \mathbb{C} : z = iy, -R \leq y \leq R\} \\ &= C_n(R) \cup D_n(R) \text{ (say)}. \end{aligned}$$

For $z \in C_n(R)$,

$$|\phi_n(z)| \geq |z| - \left(1 + \frac{1}{n}\right) > 1 \geq |e^{-z}| = |e^{-\Re(z)}| = |h(z)|,$$

and, for $z \in D_n(R)$,

$$|\phi_n(iy)| = \left|iy - \left(1 + \frac{1}{n}\right)\right| = \sqrt{y^2 + \left(1 + \frac{1}{n}\right)^2} > 1 = |e^{-iy}| = |h(iy)|.$$

Thus, $|h(z)| < |\phi_n(z)|$ for $z \in \Gamma_n(R)$. Since $\phi_n(z)$ has only one zero in $\Omega_n(R)$, by Rouche's theorem, $g_n(z) = \phi_n(z) + h(z)$ also has only one zero w_n (say) in $\Omega_n(R)$. Since $g_n(0) < 0$ and $g_n(2 + (1/n)) > 0$, the zero w_n of $g_n(z)$ must be real. The sequence $\{w_n\}_{n=1}^{\infty}$ is clearly bounded. Therefore, there exists a subsequence $\{w_{n_k}\}_{k=1}^{\infty}$ such that $w_{n_k} \to w$ as $k \to \infty$, $w \in \Omega_n(R)$. It now follows by the Hurwitz theorem applied to the sequence $\{g_{n_k}(z)\}_{k=1}^{\infty}$ that $w \in \Omega_n(R)$ is the only zero of $g(z)$. It is easily seen that the function $e^{-x} + x - 1$, $x \in \mathbb{R}$, attains the global minimum value 0 at the point $x = 0$. Therefore, it follows that $w = 0$. Consequently, the only zero of $g(z)$ in $\Omega_n(R)$ is at the origin. Since $R > 2 + (1/n)$ is arbitrary, it follows that the only zero of $g(z)$ in $\overline{H^+} = \{z \in \mathbb{C} : \Re(z) \geq 0\}$ is at the origin. Thus, $f_\lambda'(z)$ has no zeros in H^+. □

PROPOSITION 1.3. *$f_\lambda(z)$ possesses infinitely many critical values all lying in the open disk centred at the origin and having radius λ.*

Proof. Let $f_\lambda(z) = \lambda\Psi(z)$ where $\Psi(z) = (e^z - 1)/z$ for $z \neq 0$ and $\Psi(0) = 1$. Then, for $z \neq 0$, $f_\lambda'(z) = \lambda\Psi'(z) = \lambda(e^z(z-1)+1)/z^2 = \lambda(e^z - \Psi(z))/z$ and $f_\lambda'(0) = \lambda/2$ so that the critical points of $f_\lambda(z)$ are non-zero roots of the equation $\Psi(z) = e^z$. If z^* is any critical point then the corresponding critical value is given by $f_\lambda(z^*) = \lambda\Psi(z^*) = \lambda\exp(z^*)$.

Now,

$$\begin{aligned} f_\lambda'(z) = 0 \quad &\Leftrightarrow \quad e^z(z-1) + 1 = 0 \text{ and } z = x + iy \neq 0 \\ &\Leftrightarrow \quad e^x((x-1)\cos y - y \sin y) + 1 = 0 \quad \text{and} \\ &\qquad\; e^x((x-1)\sin y + y\cos y) = 0 \\ &\Leftrightarrow \quad \frac{y}{\sin y} - \exp(y\cot y - 1) = 0 \text{ and } x = 1 - y\cot y. \end{aligned} \tag{1.1}$$

Define,

$$g(y) = \frac{y}{\sin y} - e^{(y\cot y - 1)}, \quad y \in \mathbb{R} \setminus \{n\pi : n = 0, \pm 1, \pm 2, \ldots\}. \tag{1.2}$$

If $y \in [(2n+(1/2))\pi, (2n+1)\pi)$, $n = 0, 1, 2, \ldots$, then $(y/\sin y) \geq y$ and $y\cot y - 1 \leq 0$. Therefore, the function $g(y)$ is positive in the interval $[(2n + (1/2))\pi, (2n+1)\pi)$. The function $g(y)$ is obviously negative in the interval $((2n+1)\pi, 2(n+1)\pi)$. Further, $g((2n+(1/4))\pi) < 0$ and $g((2n+(1/2))\pi) > 0$ for all integers $n \geq 1$. Since $g(y)$ is continuous in $[(2n+(1/4))\pi, (2n+(1/2))\pi)$, there exist points $y_n \in [(2n+(1/4))\pi, (2n+(1/2))\pi)$, for $n = 1, 2, \ldots$, such that $g(y_n) = 0$. In view of $g(-y) = g(y)$, it follows that there exists a point $y_{-n} \in [-(2n+(1/2))\pi, -(2n+(1/4))\pi)$, $n = 1, 2, \ldots$, such that $y_{-n} = -y_n$ and $g(y_{-n}) = 0$. Thus, there exists a sequence $\{y_n\}_{n=-\infty, n\neq 0}^{n=\infty}$ such that $g(y_n) = 0$. Set $x_n = 1 - y_n \cot(y_n)$ for $n = \pm 1, \pm 2, \ldots$. In view of (1.1), it follows that $f_\lambda'(z_n) = 0$ so that the points $z_n = x_n + iy_n$, $n = \pm 1, \pm 2, \ldots$, are critical points for $f_\lambda(z)$.

Define the critical points set $\mathcal{P}$ and the critical values set $\mathcal{V}$ as follows:

$$\begin{aligned} \mathcal{P} &= \{z_n : f_\lambda'(z_n) = 0, z_n = x_n + iy_n, n = \pm 1, \pm 2, \ldots\} \\ \mathcal{V} &= \{f_\lambda(z_n) = \lambda \exp(z_n) : z_n \in \mathcal{P}\}. \end{aligned}$$

Let, if possible, $f_\lambda(z)$ have finitely many critical values. Then, $\mathcal{V}$ is a finite set. This implies the following.

(i) $\mathcal{X} = \{x_n : z_n = x_n + iy_n \in \mathcal{P}\}$ must be a finite set, since if $\mathcal{X}$ is an infinite set then $\mathcal{V}$ is also an infinite set.

(ii) There exist positive integers M and N, a subsequence $\{m_k\}_{k=0}^{\infty}$ of non-negative integers with $m_0 = 0$ such that $m_k \to \infty$ as $k \to \infty$ and a subsequence $\{y_{n_k}\}_{k=0}^{\infty}$ of $\{y_n\}_{n=-\infty, n\neq 0}^{n=\infty}$ such that

$$\begin{aligned} y_{n_k} &= 2m_k\pi + y_N \\ x_{n_k} &= x_M. \end{aligned}$$

Let $z_{n_k} = x_M + iy_{n_k} = x_M + i(y_N + 2m_k\pi)$. Then, since $f_\lambda'(z_{n_k}) = 0$, by (1.1) and (1.2),

$$g(y_{n_k}) = 0, \quad \text{for } k = 0, 1, 2, \ldots. \tag{1.3}$$

Now, by (1.2), for any positive integer s,

$$g(y_N + 2s\pi) = \frac{y_N + 2s\pi}{\sin(y_N + 2s\pi)} - \exp\{(y_N + 2s\pi)\cot(y_N + 2s\pi) - 1\}.$$

Since $g(y_N) = g(y_{n_0}) = 0$,

$$\begin{aligned} g(y_N + 2s\pi) &= \frac{2s\pi}{\sin(y_N)} - [\exp(y_N \cot(y_N) - 1)][\exp(2s\pi \cot(y_N)) - 1] \\ &< \frac{2s\pi}{\sin(y_N)} - [\exp(2s\pi \cot(y_N)) - 1]. \end{aligned} \tag{1.4}$$

However, (1.4) gives that, for all sufficiently large values of s, $g(y_N + 2s\pi) < 0$, which is a contradiction to (1.3). Thus $f_\lambda(z)$ has infinitely many critical values. The assertion that all critical values lie in the open disk centred at the origin and having radius λ follows easily by Propositions 1.1 and 1.2. □

From Proposition 1.2, it follows that $f_\lambda(z)$ is locally one-to-one in the right half-plane $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$. However, $f_\lambda(z)$ is not one-to-one in H^+, since two distinct points z_1 and z_2 in H^+ with $f_\lambda(z_1) = f_\lambda(z_2)$ may be constructed as follows: for a fixed

$x > 0$, let $l_k : l_k(t) = (1-t)A_k + tB_k; 0 \le t \le 1$ be the line segment joining the points $A_k = x + i(2k\pi)$ and $B_k = x + i((2k+1)\pi)$, where k is any positive integer. Since $\Im(f_\lambda(A_k)) < 0$ and $\Im(f_\lambda(B_k)) > 0$, due to continuity of $\Im(f_\lambda(z))$, there exists a point z_k (say) on l_k such that $\Im(f_\lambda(z_k)) = 0$. Thus, $f_\lambda(z_k) = r_k$ (say) is a real number. In view of $f_\lambda(\overline{z}) = \overline{f_\lambda(z)}$, it follows that $f_\lambda(z_k) = f_\lambda(\overline{z_k}) = r_k$, where z_k and $\overline{z_k}$ are in H^+.

The following proposition shows that $f_\lambda(z)$ is one-to-one in any closed rectangle of the form $R_{a,b,c} = \{z = x+iy : a \le x \le b, c \le y \le c + 2\pi\}$ contained in $H_2 = \{z : \Re(z) \ge 2\}$. In particular, $f_\lambda(z)$ is one-to-one in any closed disk $B_\pi(z_0) \subseteq H_2$ having centre at z_0 and radius π.

PROPOSITION 1.4. *Let* $H_2 = \{z : \Re(z) \ge 2\}$.

(a) *For any vertical line segment* Γ_1, *contained in* H_2 *and having length* 2π, $f_\lambda(\Gamma_1)$ *is a starlike curve with respect to the origin (i.e. with parametric equation* $\Gamma_1 : z(t)$, $0 \le t \le 1$, $\arg(f_\lambda(z(t)))$ *is a non-decreasing function of* t, *for* $t \in [0,1]$*).*

(b) *For any horizontal line segment* $\Gamma_2 = \{x + iy_0 : a \le x \le b, a, b \in \mathbb{R}$ *and fixed* $y_0 \in \mathbb{R}\}$ *contained in* H_2, $|f_\lambda(x+iy_0)|$ *is an increasing function of* x.

(c) $f_\lambda(z)$ *is one-to-one on any closed rectangle*

$$R_{a,b,c} = \{z = x + iy : 2 \le a \le x \le b, c \le y \le c + 2\pi\}.$$

Proof. (a) Let Γ_1 be the vertical line segment in H_2, joining the points $x_0 + i\gamma_0$ and $x_0 + i(\gamma_0 + 2\pi)$. Then, the parametric equation of Γ_1 is given by

$$\Gamma_1 : z \equiv z(t) = x_0 + i(\gamma_0 + 2\pi t), t \in [0,1].$$

It is known [**15**, vol. 1, p. 110] that the image of Γ_1 under $f_\lambda(z)$ is a starlike curve with respect to the origin if and only if

$$\Im\left\{\frac{f'_\lambda(z(t))}{f_\lambda(z(t))} z'(t)\right\} \ge 0 \quad \text{for } t \in [0,1].$$

Since $z'(t) = 2\pi i$, $f_\lambda(\Gamma_1)$ is starlike with respect to origin, if and only if

$$\Re\left\{\frac{f'_\lambda(z(t))}{f_\lambda(z(t))}\right\} \ge 0 \quad \text{for } t \in [0,1]. \tag{1.5}$$

Now, for any $z = x + iy \in H_2$,

$$\begin{aligned}\Re\left\{\frac{f'_\lambda(z)}{f_\lambda(z)}\right\} &= \Re\left(\frac{e^z}{e^z - 1}\right) - \Re\left(\frac{1}{z}\right)\\ &= \frac{(1 - e^{-x}\cos y)}{(1 - e^{-x}\cos y) + (e^{-2x} - e^{-x}\cos y)} - \frac{x}{x^2 + y^2}.\end{aligned}$$

Thus, in view of (1.5), $f_\lambda(\Gamma_1)$ is starlike with respect to the origin if and only if for $x_0 \ge 2$ and $\gamma_0 \le y \le \gamma_0 + 2\pi$,

$$1 + \frac{e^{-2x_0} - e^{-x_0}\cos y}{1 - e^{-x_0}\cos y} \le x_0 + \frac{y^2}{x_0}. \tag{1.6}$$

Now, for $x_0 \ge 2$, the inequalities $|1 - e^{-x_0}\cos y| \ge |1 - e^{-2}|$ and $|e^{-x_0}(e^{-x_0} - \cos y)| \le e^{-2}(e^{-2} + 1)$ hold for any $y \in \mathbb{R}$ and so it follows that

$$\left|\frac{e^{-2x_0} - e^{-x_0}\cos y}{1 - e^{-x_0}\cos y}\right| \le \frac{e^{-2}(e^{-2}+1)}{1 - e^{-2}} < 0.5 < x_0 - 1 + \frac{y^2}{x_0}.$$

This proves (a).

(b) Let $\Gamma_2 = \{x + iy_0 : a \le x \le b, a, b \in \mathbb{R}$ and fixed $y_0 \in \mathbb{R}\}$ be any horizontal line segment contained in H_2.

Define, for $x + iy_0 \in \Gamma_2$,

$$A_{y_0}(x) = |f_\lambda(x + iy_0)|^2 = \frac{\lambda^2(1 + e^{2x} - 2e^x \cos y_0)}{x^2 + y_0^2}.$$

Since

$$A'_{y_0}(x) = \frac{\lambda^2[(2e^{2x} - 2e^x \cos y_0)(x^2 + y_0^2) - 2x(1 + e^{2x} - 2e^x \cos y_0)]}{(x^2 + y_0^2)^2},$$

it follows that for $x \in [a, b]$,

$$A'_{y_0}(x) > 0 \Leftrightarrow e^x(x^2 + y_0^2) \ge ((x^2 + y_0^2)\cos y_0 + xe^{-x} + xe^x - 2x \cos y_0). \qquad (1.7)$$

Since, for $x \ge 2$,

$$\begin{aligned} |((x^2 + y_0^2)\cos y_0 + xe^{-x} + xe^x - 2x\cos y_0)| &\le x^2 + y_0^2 + xe^{-x} + xe^x + 2x \\ &\le y_0^2 + xe^x(xe^{-x} + 1 + e^{-2x} + 2e^{-x}) \\ &\le y_0^2 + xe^x\left(\frac{2}{e^2} + 1 + e^{-4} + 2e^{-2}\right) \\ &\le y_0^2 + 2xe^x \\ &\le (y_0^2 + x^2)e^x \end{aligned}$$

the inequality in (1.7) follows. Thus, $A_{y_0}(x)$ is an increasing function of x, for $x + iy_0 \in \Gamma_2$. Consequently, $|f_\lambda(x + iy_0)|$ is also an increasing function of x, for $x + iy_0 \in \Gamma_2$, completing the proof of (b).

(c) Let $R_{a,b,c} = \{z = x + iy : 2 \le a \le x \le b, c \le y \le c + 2\pi\}$ be the closed rectangle contained in H_2. Let $\Gamma_{1,x}$ be the vertical line segment in $R_{a,b,c}$ joining the points $x + ic$ and $x + i(c + 2\pi)$.

It follows from (a) that $f_\lambda(\Gamma_{1,a})$ and $f_\lambda(\Gamma_{1,b})$ are starlike curves with respect to the origin. It is easily seen that

$$\phi(x, c, \lambda) \le |f_\lambda(\Gamma_{1,x})| \le \psi(x, c, \lambda) \qquad (1.8)$$

where

$$\phi(x, c, \lambda) = \left(\frac{\lambda^2(e^{2x} + 1 - 2e^x)}{(c + 2\pi)^2 + x^2}\right)^{1/2} \quad \text{and} \quad \psi(x, c, \lambda) = \left(\frac{\lambda^2(e^{2x} + 1 + 2e^x)}{c^2 + x^2}\right)^{1/2}.$$

Let $a \ge 2$ be arbitrarily chosen. Since $\phi(x, c, \lambda) \to \infty$ as $x \to \infty$, there exists a number $b_0 \equiv b_0(a) > a$ such that $\psi(a, c, \lambda) < \phi(b, c, \lambda)$ for $b \ge b_0$. It now follows from (1.8) that $f_\lambda(\Gamma_{1,a})$ and $f_\lambda(\Gamma_{1,b})$ do not intersect each other for $b \ge b_0$. Thus, $f_\lambda(z)$ is one-to-one on the vertical line segments $\Gamma_{1,a}$ and $\Gamma_{1,b}$ for $b \ge b_0$ (see Figure 1).

Let $\Gamma_{2,y}$ be the horizontal line segment in $R_{a,b,c}$ joining the points $a + iy$ and $b + iy$. We show that $f_\lambda(z)$ is one-to-one also on the horizontal boundary line segments $\Gamma_{2,c}$ and

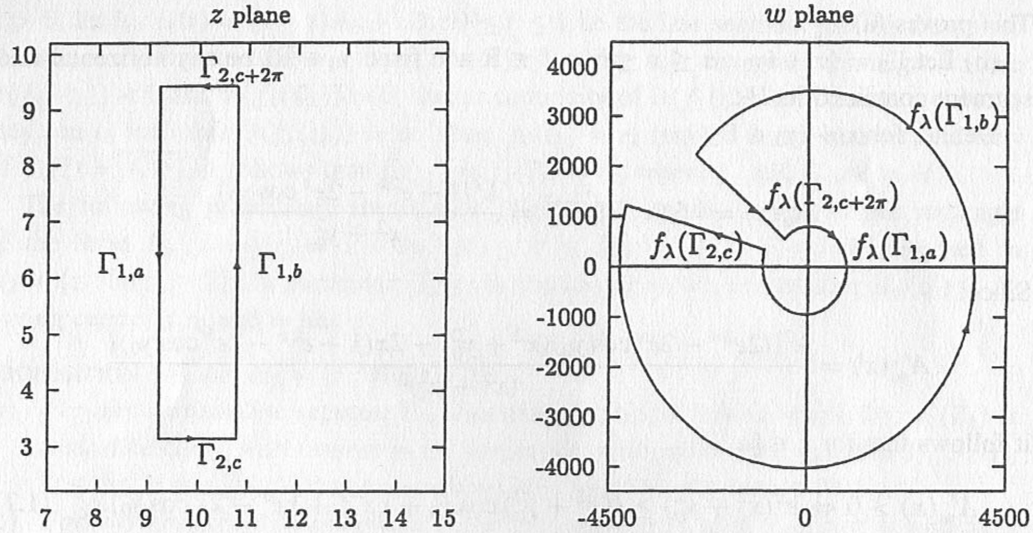


FIGURE 1. Image of the rectangle $R_{a,b,c}$, under the mapping $w = f_\lambda(z)$.

$\Gamma_{2,c+2\pi}$ of the rectangle $R_{a,b,c}$. Let $z_0 = x_0 + i(c+2\pi)$ be any arbitrarily fixed point on $\Gamma_{2,c+2\pi}$ and $z = x + ic$ be any point on $\Gamma_{2,c}$. Then,

$$\begin{aligned} |f_\lambda(z) - f_\lambda(z_0)| = \lambda \left| \frac{e^z - 1}{z} - \frac{e^{z_0} - 1}{z_0} \right| &\geq \lambda \left| \frac{|e^z - 1|}{|z|} - \frac{|e^{z_0} - 1|}{|z_0|} \right| \\ &\geq \lambda \left| \frac{|z_0||e^z - 1| - |z||e^{z_0} - 1|}{|z_0||z|} \right| \\ &\geq K(x, x_0) \left| \frac{|e^z - 1| - |e^{z_0} - 1|}{|z_0||z|} \right| \end{aligned} \tag{1.9}$$

where $K(x, x_0) = \lambda^* \min\{|z|, |z_0|\} > 0$. Now, if $a \le x < x_0$, $|e^z - 1| < |e^{z_0} - 1|$ and if $x_0 < x \le b$, $|e^{z_0} - 1| < |e^z - 1|$. Thus $||e^{z_0} - 1| - |e^z - 1|| > 0$ if $\Re(z) \in [a, x_0) \cup (x_0, b]$. Further, $|f_\lambda(z) - f_\lambda(z_0)| > 0$ for $\Re(z) = x_0 = \Re(z_0)$. Consequently, by (1.9), $|f_\lambda(z) - f_\lambda(z_0)| > 0$ for any $z \in \Gamma_{2,c}$. Since $z_0 \in \Gamma_{2,c+2\pi}$ is arbitrary, it follows that $f_\lambda(\Gamma_{2,c})$ and $f_\lambda(\Gamma_{2,c+2\pi})$ do not intersect. Further, by Proposition 1.4(b), $|f_\lambda(x+ic)|$ and $|f_\lambda(x + i(c+2\pi))|$ are increasing functions of x, for $x \ge 2$. Thus, $f_\lambda(z)$ is one-to-one on $\Gamma_{2,c} \bigcup \Gamma_{2,c+2\pi}$ (see Figure 1).

Let $\Delta_{\Gamma_{1,\mu}} \arg(f_\lambda(z_\mu(t))) = \arg(f_\lambda(z_\mu(1))) - \arg(f_\lambda(z_\mu(0)))$, $\mu = a, b$, be the change in the argument of $f_\lambda(\Gamma_{1,\mu})$ as $z_\mu(t)$ traverses on $\Gamma_{1,\mu}$ from $(\mu + ic)$ to $(\mu + i(c+2\pi))$, where $z_\mu(t)$, $0 \le t \le 1$, is the parametric equation of $\Gamma_{1,\mu}$. For any non-zero $z \in \mathbb{C}$, $\arg(f_\lambda(z)) = \arg(e^z - 1) - \arg(z)$. Since $\Delta_{\Gamma_{1,\mu}} \arg(e^{z_\mu(t)} - 1) = 2\pi$ and $0 < \Delta_{\Gamma_{1,\mu}} \arg(z_\mu(t)) < \pi$, it follows that $\pi \le \Delta_{\Gamma_{1,\mu}} \arg(f_\lambda(z_\mu(t))) < 2\pi$, for $\mu = a, b$. Further, by Proposition 1.4(b), $|f_\lambda(x+iy)|$ for $y = c, c+2\pi$, is an increasing function for $x \in [a, b]$ and, by Proposition 1.4(a), $f_\lambda(\Gamma_{1,\mu})$, $\mu = a, b$, is a starlike curve with respect to origin. Thus, it follows that $f_\lambda(z)$ is one-to-one on $\partial R_{a,b,c} = \Gamma_{1,a} \cup \Gamma_{1,b} \cup \Gamma_{2,c} \cup \Gamma_{2,c+2\pi}$, for $b \ge b_0$ (see Figure 1). Consequently [**16**, vol. 2, p. 118], $f_\lambda(z)$ is one-to-one in the closed rectangle $R_{a,b,c}$ for $b \ge b_0$. Since $R_{a,b,c} \subset R_{a,b_0,c}$ for $a \le b < b_0$, $f_\lambda(z)$ is one-to-one on any closed rectangle $R_{a,b,c}$. This proves (c). □

Let $B_\delta(z_0)$ denote the open ball of radius δ, centred at z_0. If $f_\lambda(z)$ is one-to-one, and $|f'_\lambda(z)| > \mu > 1$, for all $z \in B_\delta(z_0)$, the function $f_\lambda(z)$ expands the circular neighbourhood $B_\delta(z_0)$ with scaling ratio greater than μ. The following proposition, for $\mu > 1$, exhibits the characteristic property of $f_\lambda(z)$ to expand certain neighbourhoods $U \subseteq B_\delta(z_0)$ of the point z_0 in such a way that $f_\lambda(z)$ is a homeomorphism from U to $B_{\mu\delta}(f_\lambda(z_0))$.

PROPOSITION 1.5. *If $f_\lambda(z)$ is one-to-one and $|f'_\lambda(z)| > \mu > 1$ for all $z \in B_\delta(z_0)$, then there exists an open set $U \subseteq B_\delta(z_0)$ such that $f_\lambda : U \to B_{\mu\delta}(f_\lambda(z_0))$ is a homeomorphism.*

Proof. Since $f_\lambda : B_\delta(z_0) \to f_\lambda(B_\delta(z_0))$ is a bijection, its inverse map $L_\lambda : f_\lambda(B_\delta(z_0)) \to B_\delta(z_0)$ exists. Since $f_\lambda(z)$ is analytic in $B_\delta(z_0)$, $L_\lambda(z)$ is analytic in $f_\lambda(B_\delta(z_0))$ [**16**, vol. 2, p. 86]. It therefore follows that $f_\lambda : B_\delta(z_0) \to f_\lambda(B_\delta(z_0))$ is a homeomorphism. Thus [**6**, p. 326], $f_\lambda(B_\delta(z_0))$ contains a disk of radius $\mu\delta$ centred at $f_\lambda(z_0)$. Since $f_\lambda : B_\delta(z_0) \to f_\lambda(B_\delta(z_0))$ is continuous and one-to-one, there exists an open set $U \equiv f_\lambda^{-1}(B_{\mu\delta}(f_\lambda(z_0))) \subseteq B_\delta(z_0)$ such that $f_\lambda : U \to B_{\mu\delta}(f_\lambda(z_0))$ is a homeomorphism.

COROLLARY 1.1. *Let $|f'_\lambda(z)| > \mu > 1$ for all $z \in B_\delta(z_0) \subseteq H_2$, where $\delta \le \pi$ and $H_2 = \{z \in \mathbb{C} : \Re(z) \ge 2\}$. Then, there exists an open set $U \subseteq B_\delta(z_0)$ such that $f_\lambda : U \to B_{\mu\delta}(f_\lambda(z_0))$ is a homeomorphism.*

Proof. Since $B_\delta(z_0) \subseteq H_2$ and $\delta \le \pi$, with suitable choices of a, b and c, there exists a rectangle $R_{a,b,c}$ containing the disk $B_\delta(z_0)$. Therefore, it follows from Proposition 1.4(c) that $f_\lambda(z)$ is one-to-one in $B_\delta(z_0) \subseteq H_2$. The corollary now follows immediately from Proposition 1.5. □

PROPOSITION 1.6. *Let U be an open set containing z_0. Let $z_n = f_\lambda^n(z_0)$, $n = 1, 2, 3, \ldots$. Define $D = \{z \in \mathbb{C} : |f'_\lambda(z)| > \sqrt{2}$ and $\Re(z) \ge 2\}$. Suppose $B_{\sqrt{2}\pi}(z_n) \subset D$ for $n = 0, 1, 2, \ldots$. Let $S_{2\pi}(z_n)$ be the interior of the square with centre at z_n and having sides of length 2π, parallel to the coordinate axes. Then there exists an integer $N > 0$ and open sets $U_n \subseteq U$ for $n > N$, such that $f_\lambda^n : U_n \to S_{2\pi}(z_n)$ is a homeomorphism.*

Proof. Suppose $B_\delta(z_0) \subset U$. Choose N so that $\delta(\sqrt{2})^{N-1} < \pi$ and $\delta(\sqrt{2})^N \ge \pi$. A repeated application of Corollary 1.1 gives an open set $W \subseteq B_\delta(z_0)$ that is mapped homeomorphically onto $B_{\delta(\sqrt{2})^N}(f_\lambda^N(z_0))$ by f_λ^N. Clearly, $B_{\delta(\sqrt{2})^N}(z_N) \supseteq B_\pi(z_N)$.

Let $n = 1$. By Corollary 1.1, there exists an open set $V_1 \subseteq B_\pi(z_N)$ such that $f_\lambda : V_1 \to B_{\sqrt{2}\pi}(z_{N+1})$ is a homeomorphism. Similarly, for $n = 2$, an application of Corollary 1.1 twice, gives an open set $V_2 \subseteq B_{\pi/\sqrt{2}}(z_N)$ such that $f_\lambda^2 : V_2 \to B_{\sqrt{2}\pi}(z_{N+2})$ is a homeomorphism. Continuing this process, for each integer $n > 2$, an open set $V_n \subseteq B_{\pi/(\sqrt{2})^{n-1}}(z_N) \subset B_\pi(z_N)$ is obtained such that $f_\lambda^n : V_n \to B_{\sqrt{2}\pi}(z_{N+n})$ is a homeomorphism. Consequently, since $B_{\sqrt{2}\pi}(z_{N+n}) \supseteq S_{2\pi}(z_{N+n})$, there exists a smaller set $V'_n \subseteq V_n$ such that $f_\lambda^n : V'_n \to S_{2\pi}(z_{N+n})$ is a homeomorphism, for each integer $n > 0$. Now, set for each integer $n > 0$ $U_{N+n} \equiv f_\lambda^{-N}(V'_n) \subset W \subset B_\delta(z_0)$. Since $f_\lambda^N : W \to B_{\delta(\sqrt{2})^N}(z_N)$ is a homeomorphism and $B_{\delta(\sqrt{2})^N}(z_N) \supseteq B_\pi(z_N) \supseteq V_n \supseteq V'_n$, it follows that $f_\lambda^N : U_{N+n} \to V_n'$ is a homeomorphism. Consequently, for each integer $n > 0$, $f_\lambda^{N+n} : U_{N+n} \subset U \to S_{2\pi}(z_{N+n})$ is a homeomorphism. Thus, for each integer $n > N$, there is an open set $U_n \subset U$ for which $f_\lambda^n : U_n \subset U \to S_{2\pi}(z_n)$ is a homeomorphism. □

PROPOSITION 1.7. *Let* $\mathrm{Esc}(f_\lambda) = \mathrm{clo}\{z \in \mathbb{C} : f_\lambda^n(z) \to \infty\}$ *be the closure of the set of escaping points of* $f_\lambda(z)$. *Suppose* $z_0 \in \mathrm{Esc}(f_\lambda)$ *and* U *is any open set containing* z_0. *Then, there exist an integer* $N > 0$ *and points* $z_1, z_2 \in U$ *such that* $f_\lambda^N(z_1)$ *is a real number and* $\Re(f_\lambda^N(z_2)) < 0$.

Proof. Let $z_0 \in \mathrm{Esc}(f_\lambda)$ and U be any neighbourhood of z_0. Then, either $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$ or there exists a point $\tilde{z} \in U$ such that $f_\lambda^n(\tilde{z}) \to \infty$ as $n \to \infty$. In the latter case, rename $\tilde{z}$ as z_0, so that without loss of generality, if $z_0 \in \mathrm{Esc}(f_\lambda)$ and U is any neighbourhood of z_0, it may be assumed that $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$.

For any $z = x + iy \neq 0$, the absolute value of $f_\lambda(z)$ is given by

$$|f_\lambda(x+iy)| = \lambda \left(\frac{1 + e^{2x} - 2e^x \cos y}{x^2 + y^2} \right)^{1/2}.$$

It is easily seen that

$$\begin{aligned} |f_\lambda(x_0 + iy)| &\to 0 \quad \text{as} \quad |y| \to \infty \text{ for any fixed } x_0 \in \mathbb{R} \\ |f_\lambda(x + iy_0)| &\to 0 \quad \text{as} \quad x \to -\infty, \text{ for any fixed } y_0 \in \mathbb{R} \\ |f_\lambda(x + iy_0)| &\to \infty \quad \text{as} \quad x \to \infty, \text{ for any fixed } y_0 \in \mathbb{R}. \end{aligned}$$

Consequently, if $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$ then both $\Re(f_\lambda^n(z_0)) \to \infty$ and $\Im(f_\lambda^n(z_0)) \to \infty$ as $n \to \infty$ cannot hold simultaneously. Further, if $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$, then $\Re(f_\lambda^n(z_0)) \to \infty$ and $\Im(f_\lambda^n(z_0))$ remains bounded. Since $f_\lambda^n(z_0) \to \infty$ as $n \to \infty$, there exists a positive integer N_0 such that $f_\lambda^{N_0}(z_0) = w_0$ and $B_{\sqrt{2}\pi}(f_\lambda^n(w_0)) \subseteq D = \{z \in \mathbb{C} : |f_\lambda'(z)| > \sqrt{2} \text{ and } \Re(z) \geq 2\}$ for all $n = 0, 1, 2, \ldots$. Let V be a neighbourhood of w_0 such that $V \subseteq f_\lambda^{N_0}(U)$. Now, applying Proposition 1.6 to the point w_0 and V, it follows that there exists an integer $n_0 > 0$ such that, if $n > n_0$, there exists an open set $V_n \subset V$ for which $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(w_0))$ is a homeomorphism.

Fix $n = n_1 > n_0$. By Proposition 1.4(a), it follows that any vertical line segment of length 2π in the open square $S_{2\pi}(f_\lambda^n(w_0))$ is mapped by $f_\lambda(z)$ to a curve, which is starlike with respect to the origin. Further, it follows easily from the proof of Proposition 1.4(c) that the image curve of any vertical line segment of length 2π in $S_{2\pi}(f_\lambda^n(w_0))$ intersects both the real and the imaginary axis. Thus, $f_\lambda^{n_1+1}(V_n) \cap \mathbb{R} \neq \emptyset$. Therefore, there exists a point $\tilde{z_1} \in V_n \subseteq V$ such that $f_\lambda^{n_1+1}(\tilde{z_1})$ is a real number. By setting $z_1 = f_\lambda^{-N_0}(\tilde{z_1})$, it follows that $f_\lambda^{N_0+n_1+1}(z_1)$ is a real number for $z_1 \in U$.

The square $S_{2\pi}(f_\lambda^n(w_0))$ meets one of the horizontal lines $L_k = \{z = x + iy : x \geq 2 \text{ and } y = (2k+1)\pi\}$ for some integer k. Since $f_\lambda(L_k)$ is contained in $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$, there exists a point $\tilde{z_2} \in V_n \subseteq V$ such that $\Re(f_\lambda^{n_1+1}(\tilde{z_2})) < 0$. By setting $z_2 = f_\lambda^{-N_0}(\tilde{z_2})$, it follows that $\Re(f_\lambda^{N_0+n_1+1}(z_2)) < 0$, where $z_2 \in U$. □

2. *Dynamics of $f_\lambda(x)$ and bifurcation*

The dynamics of $f_\lambda(x) = \lambda(e^x - 1)/x$ for $x \in \mathbb{R} \setminus \{0\}$, $f_\lambda(0) = \lambda$, are studied in this section. The nature of the fixed points of $f_\lambda(x)$ is described in Proposition 2.1. Proposition 2.2 describes the dynamics of $f_\lambda(x)$. It follows from Proposition 2.2 that there exists a parameter value $\lambda^* > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs at $\lambda = \lambda^*$.

Set $\Psi(x) = (e^x - 1)/x$ for $x \in \mathbb{R} \setminus \{0\}$ and $\Psi(0) = 1$. It is easily seen that the functions $\Psi(x)$ and $\Psi'(x)$ are strictly increasing positive valued functions. Therefore, the function $\phi(x) = \Psi(x) - x\Psi'(x)$ is strictly decreasing in the interval $[0, \infty)$ and is positive in the interval $(-\infty, 0]$. Consequently, since $\Psi(0) = 1$ and $\Psi(2) < 2\Psi'(2)$, there exists a unique $x^* \in (0, 2)$ such that

$$\phi(x) \begin{cases} > 0 & \text{for } x < x^* \\ = 0 & \text{for } x = x^* \\ < 0 & \text{for } x > x^*. \end{cases} \tag{2.1}$$

Throughout the following, we denote

$$\lambda^* = \frac{1}{\Psi'(x^*)} \tag{2.2}$$

where x^* is the unique real root of the equation $\phi(x) = 0$.

The following proposition describes the nature of the fixed points of $f_\lambda(x)$ for $x \in \mathbb{R}$.

PROPOSITION 2.1. *Let $f_\lambda(x) = \lambda(e^x - 1)/x$ for $x \in \mathbb{R} \setminus \{0\}$ and $f_\lambda(0) = \lambda$.*
(a) *If $0 < \lambda < \lambda^*$ then $f_\lambda(x)$ has an attractive fixed point and a repelling fixed point.*
(b) *If $\lambda = \lambda^*$ then $f_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$.*
(c) *If $\lambda > \lambda^*$ then $f_\lambda(x)$ has no fixed points.*

Proof. Define $g_\lambda(x) = f_\lambda(x) - x = \lambda\Psi(x) - x$ for $x \in \mathbb{R}$. The zeros of $g_\lambda(x)$ are fixed points of $f_\lambda(x)$. Further,
(i) $g_\lambda(x)$ is continuously differentiable in $\mathbb{R}$ and positive for $|x|$ sufficiently large;
(ii) $g_\lambda(x)$ has a unique local minimum at $\tilde{x} = \tilde{x}(\lambda)$.

The assertion (i) is easily seen to hold. To see that (ii) holds, observe that the function $g'_\lambda(x)$ is strictly increasing, $g'_\lambda(x) \to -1$ as $x \to -\infty$ and $g'_\lambda(x) \to \infty$ as $x \to \infty$. Therefore, due to continuity of $g'_\lambda(x)$, there exist a unique real number $\tilde{x} = \tilde{x}(\lambda)$ such that $g'_\lambda(\tilde{x}) = 0$, $g'_\lambda(x) < 0$ for $x < \tilde{x}$ and $g'_\lambda(x) > 0$ for $x > \tilde{x}$. Thus, in view of $g''_\lambda(\tilde{x}) > 0$, $g_\lambda(x)$ attains a local minimum value at $x = \tilde{x}$.

(a) Clearly, $g'_\lambda(\tilde{x}) = 0$ implies that $\lambda = 1/\Psi'(\tilde{x})$. Since $\lambda^* = 1/\Psi'(x^*)$ and $\Psi'(x)$ is a strictly increasing function, it follows that $\tilde{x} > x^*$ for $0 < \lambda < \lambda^*$. Thus, by (2.1), $\phi(\tilde{x}) = \Psi(\tilde{x}) - \tilde{x}\Psi'(\tilde{x}) = \Psi'(\tilde{x})g_\lambda(\tilde{x}) < 0$. Consequently, $g_\lambda(\tilde{x}) < 0$ (see Figure 2(a)). Now, (i) and (ii) together with $g_\lambda(\tilde{x}) < 0$ imply that $g_\lambda(x)$ has only two zeros a_λ and r_λ (say) with $a_\lambda < \tilde{x} < r_\lambda$. Since $a_\lambda < \tilde{x} < r_\lambda$ implies that $g'_\lambda(a_\lambda) < 0$ and $g'_\lambda(r_\lambda) > 0$, it follows that $f'_\lambda(a_\lambda) < 1$ and $f'_\lambda(r_\lambda) > 1$. Thus, the point a_λ is an attractive fixed point and the point r_λ is a repelling fixed point of $f_\lambda(x)$. This proves (a).

(b) If $\lambda = \lambda^*$ then $\tilde{x} = x^*$, $g_\lambda(x^*) = 0$ and $g'_\lambda(x^*) = 0$ (see Figure 2(b)). Consequently, $f'_\lambda(x^*) = 1$. Thus, $f_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$. This proves (b).

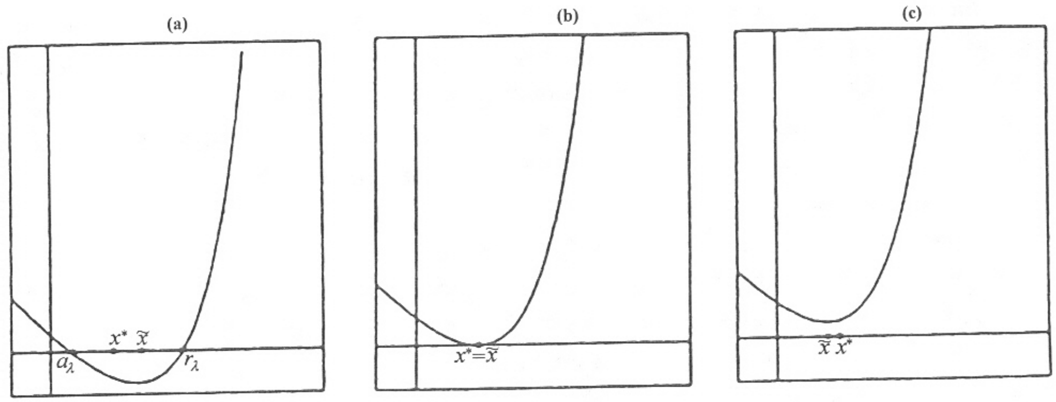


FIGURE 2. The graphs of $g_\lambda(x) = f_\lambda(x) - x$ for (a) $0 < \lambda < \lambda^*$, (b) $\lambda = \lambda^*$ and (c) $\lambda > \lambda^*$.

(c) The arguments for the proof of this part of the proposition are similar to those employed in the proof of (a). If $\lambda > \lambda^*$ then $\tilde{x} < x^*$ and $g_\lambda(\tilde{x}) > 0$ (see Figure 2(c)). Consequently, in view of (ii), $g_\lambda(x) > g_\lambda(\tilde{x}) > 0$ for all $x \in \mathbb{R}$ and hence $g_\lambda(x)$ has no zeros. Thus, $f_\lambda(x)$ has no fixed points if $\lambda > \lambda^*$, which completes the proof of (c). □

It follows from the following proposition that there exists a real parameter $\tilde{\lambda} > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R}$ occurs at $\lambda = \tilde{\lambda}$. This critical parameter $\tilde{\lambda}$ is found to be λ^*, where λ^* is given by (2.2).

PROPOSITION 2.2. *Let* $f_\lambda(x) = \lambda(e^x - 1)/x$ *for* $x \in \mathbb{R} \setminus \{0\}$ *and* $f_\lambda(0) = \lambda$.

(a) *If* $0 < \lambda < \lambda^*$ *then* $f_\lambda^n(x) \to a_\lambda$ *as* $n \to \infty$ *for* $x < r_\lambda$ *and* $f_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $x > r_\lambda$, *where* a_λ *and* r_λ *are the attractive and the repelling fixed points of* $f_\lambda(x)$, *respectively.*

(b) *If* $\lambda = \lambda^*$ *then* $f_\lambda^n(x) \to x^*$ *as* $n \to \infty$ *for* $x < x^*$ *and* $f_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $x > x^*$, *where* x^*, *given by (2.2), is the rationally indifferent fixed point of* $f_\lambda(x)$.

(c) *If* $\lambda > \lambda^*$ *then* $f_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for all* $x \in \mathbb{R}$.

Proof. The function $f'_\lambda(x)$ is strictly increasing, $f'_\lambda(x) \to 0$ as $x \to -\infty$ and $f'_\lambda(x) \to \infty$ as $x \to \infty$. Therefore, due to continuity of $f'_\lambda(x)$, there exist a unique real number $\tilde{x} = \tilde{x}(\lambda)$ such that $f'_\lambda(\tilde{x}) = 1$, $f'_\lambda(x) < 1$ for $x < \tilde{x}$ and $f'_\lambda(x) > 1$ for $x > \tilde{x}$. Thus, in view of $f''_\lambda(\tilde{x}) > 0$, $f_\lambda(x) - x$ attains a local minimum value at $x = \tilde{x}$.

(a) If $0 < \lambda < \lambda^*$, by Proposition 2.1(a), it follows that $f_\lambda(x)$ has an attractive fixed point a_λ (say) and a repelling fixed point r_λ (say) with $0 < a_\lambda < \tilde{x} < r_\lambda$. Now, $g_\lambda(x) = f_\lambda(x) - x$ is positive in the interval $(-\infty, a_\lambda) \cup (r_\lambda, \infty)$ and is negative in the interval (a_λ, r_λ). Therefore, for $a_\lambda < x < r_\lambda$,

$$f_\lambda(x) - a_\lambda < x - a_\lambda. \tag{2.3}$$

If $x < a_\lambda$ then, by the mean value theorem, $|f_\lambda(x) - a_\lambda| \le f'_\lambda(c)|x - a_\lambda|$, where $x < c < a_\lambda$. Since $f'_\lambda(\tilde{x}) = 1$, $c < \tilde{x}$ and $f'_\lambda(x)$ is strictly increasing, it follows that for $x < r_\lambda$, $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$. Further, if $x > r_\lambda$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$, since $f_\lambda(x) > x$ for $x > r_\lambda$. This completes the proof of (a).

(b) By Proposition 2.1(b), if $\lambda = \lambda^*$ then $f_\lambda(x)$ has a unique rationally indifferent fixed point at $x = x^*$. Since $f'_{\lambda^*}(x) < 1$ for $x < x^*$, $f'_{\lambda^*}(x^*) = 1$ and $f'_{\lambda^*}(x) > 1$ for $x > x^*$, it follows that if $x < x^*$, $|f_{\lambda^*}(x) - x^*| < |x - x^*|$. Therefore, $f_{\lambda^*}^n(x) \to x^*$ as $n \to \infty$, for $x < x^*$. If $x > x^*$ then $f_{\lambda^*}^n(x) \to \infty$ as $n \to \infty$, since $f_{\lambda^*}(x) > x$ for $x > x^*$. This proves (b).

(c) If $\lambda > \lambda^*$ then, for $x \in \mathbb{R}$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$, since $f_\lambda(x) > x$ for $\lambda > \lambda^*$, which completes the proof of (c). □

Remark. Let

$$\hat{x_\lambda} = \begin{cases} r_\lambda & \text{if } 0 < \lambda < \lambda^* \\ x^* & \text{if } \lambda = \lambda^*. \end{cases}$$

If $0 < \lambda \leq \lambda^*$, it follows from Proposition 2.2 that under iteration of f_λ the orbits of all the points less than $\hat{x_\lambda}$ remain bounded and the orbits of all the points greater than $\hat{x_\lambda}$ become unbounded; meanwhile, if $\lambda > \lambda^*$, there is no real point whose orbit remains bounded under iteration of f_λ. Thus, bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$, occurs at the parameter value $\lambda = \lambda^*$. The numerical computation of the root x^* of the equation $\phi(x) \equiv \Psi(x) - x\Psi'(x) = 0$ by the bisection method gives $x^* \approx 1.594$. Thus, by (2.2) the critical parameter $\lambda^* \approx 0.64761$.

3. *Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda^*$*

If $0 < \lambda < \lambda^*$, by Proposition 2.1(a) it follows that $f_\lambda(z)$ has a real attractive fixed point a_λ and a real repelling fixed point r_λ such that $\tilde{x}$ with $f'_\lambda(\tilde{x}) = 1$ satisfies $0 < a_\lambda < \tilde{x} < r_\lambda$. By Proposition 2.2(a), $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x < r_\lambda$ and $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x > r_\lambda$. The basin of attraction $W(a_\lambda)$ of the attractive fixed point a_λ is defined as

$$W(a_\lambda) = \{z \in \mathbb{C} : f_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}.$$

Clearly, $W(a_\lambda)$ contains the interval $(-\infty, r_\lambda)$.

In this section, the dynamics $f_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda^*$ are described, where λ^* is defined by (2.2). First, a general description of the basin of attraction $W(a_\lambda)$ of the real attractive fixed point a_λ of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, is found in Propositon 3.1. Theorem 3.1 gives a computationally useful characterization of the Julia set $\mathcal{J}(f_\lambda)$ as the closure of the set of escaping points of $f_\lambda(z)$. Proposition 3.2 and Corollary 3.1 describe the nature of the stable component U of the Fatou set $\mathcal{F}(f_\lambda)$. Theorem 3.2, under a certain condition, provides another characterization of the Julia set $\mathcal{J}(f_\lambda)$ as the complement of the basin of attraction $W(a_\lambda)$ of the attractive real fixed point of $f_\lambda(z)$.

PROPOSITION 3.1. *Let $0 < \lambda < \lambda^*$ and $W(a_\lambda)$ be the basin of attraction of the real attractive fixed point a_λ of $f_\lambda(z)$. Then, $W(a_\lambda) \supset D = \{z \in \mathbb{C} : |f_\lambda(z)| < \tilde{x}\}$, where $\tilde{x} > x^* \approx 1.594$ is the real number such that $f'_\lambda(\tilde{x}) = 1$ and x^* is defined by (2.2).*

Proof. Let $f_\lambda(z) = \lambda\Psi(z)$ for $z \in \mathbb{C}$, where $\Psi(z) = (e^z - 1)/z$ and $\Psi(0) = \lambda$. Since $1/\Psi'(\tilde{x}) = \lambda < \lambda^* = 1/\Psi'(x^*)$ and $\Psi'(x)$ is a strictly increasing function, it follows that

We first show that $f_\lambda(z)$ maps the open disk $B_{\tilde{x}}(0)$, centred at the origin and having radius $\tilde{x}$ into itself. Since, by (2.1), $\phi(\tilde{x}) = \Psi(\tilde{x}) - \tilde{x}\Psi'(\tilde{x}) = \Psi'(\tilde{x})((\Psi(\tilde{x})/\Psi'(\tilde{x})) - \tilde{x}) < 0$, it follows that $\Psi(\tilde{x})/\Psi'(\tilde{x}) < \tilde{x}$. Consequently, $f_\lambda(x) = \lambda\Psi(x) = (1/\Psi'(\tilde{x}))\Psi(x) < \Psi(\tilde{x})/\Psi'(\tilde{x}) < \tilde{x}$, for $x < \tilde{x}$. Therefore, $\max_{|z|=\tilde{x}} |f_\lambda(z)| \leq f_\lambda(\tilde{x}) \leq \Psi(\tilde{x})/\Psi'(\tilde{x}) < \tilde{x}$, since $|f_\lambda(z)| \leq f_\lambda(|z|)$. Now, the maximum modulus principle gives $|f_\lambda(z)| < \tilde{x}$ for $z \in B_{\tilde{x}}(0)$. Thus, $f_\lambda(z)$ maps the open disk $B_{\tilde{x}}(0)$ into itself.

The function $f_\lambda(z)$ has zeros only at $2n\pi i$, for $n = \pm 1, \pm 2, \ldots$, and $|f_\lambda(z)| \leq \lambda < 1$ for $z \in \overline{H^-} = \{z \in \mathbb{C} : \Re(z) \leq 0\}$ and $0 < \lambda < \lambda^*$. Since all the zeros of $f_\lambda(z)$ lie in $\overline{H^-}$, it follows that [**16**, vol. 1, p. 376] the curve $\gamma = \{z \in \mathbb{C} : |f_\lambda(z)| = \tilde{x}\}$ is connected and not self-intersecting. Therefore, $D = \{z \in \mathbb{C} : |f_\lambda(z)| < \tilde{x}\}$ is a simply connected domain. Since $f_\lambda(z)$ maps D into $B_{\tilde{x}}(0)$ and $f_\lambda(B_{\tilde{x}}(0)) \subseteq B_{\tilde{x}}(0)$, by the Schwarz lemma [**6**, p. 264], $f_\lambda^n(z) \to a_\lambda$ as $n \to \infty$ for all $z \in D$. Thus, $W(a_\lambda) \supset D$. □

Remark. The left half-plane $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$ is clearly contained in the basin of attraction $W(a_\lambda)$, since by Proposition 1.1, $f_\lambda(H^-) \subset B_1(0)$. Now, consider horizontal strips S of the form $S = \{z : z = x + iy, x \in \mathbb{R}$ and $(2m+1)\pi < y < ((4m+3)/2)\pi$ or $-((4m+3)/2)\pi < y < -(2m+1)\pi; m = 0, 1, 2, \ldots\}$. For $z = x + iy \in S$ and $x \geq 0$, $\Re(f_\lambda(z)) = (\lambda e^x/(x^2+y^2))[x \cos y + y \sin y - xe^{-x}] < 0$. Therefore, $W(a_\lambda)$ contains the horizontal strips S. Thus, the basin of attraction $W(a_\lambda)$ contains $D = \{z : |f_\lambda(z)| < \tilde{x}\} \cup S \supseteq H^- \cup S$. Consequently, $W(a_\lambda)$ occupies more than half of the complex plane and the complement of $W(a_\lambda)$ is very small as compared to $W(a_\lambda)$ (see the white regions in Figures 4(a) and 5(a)).

By Proposition 2.1(a), for $0 < \lambda < \lambda^*$, r_λ is the repelling fixed point for $f_\lambda(z)$ and therefore r_λ belongs to the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$. Proposition 2.2(a), gives that all the points $x > r_\lambda$ are escaping points of $f_\lambda(z)$. In the following theorem, we find that all points $x > r_\lambda$ belong to the Julia set $\mathcal{J}(f_\lambda)$. In fact, a characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of all escaping points of $f_\lambda(z)$ is found in Theorem 3.1. Such a characterization, hitherto known only for certain critically finite entire transcendental functions [**11**], is quite useful in computationally generating the pictures of the Julia set of $f_\lambda(z)$.

THEOREM 3.1. *Let* $\mathrm{Esc}(f_\lambda) = \mathrm{clo}\{z \in \mathbb{C} : f_\lambda^n(z) \to \infty$ *as* $n \to \infty\}$ *be the closure of the set of escaping points of* $f_\lambda(z)$. *If* $0 < \lambda < \lambda^*$ *then the Julia set* $\mathcal{J}(f_\lambda) = \mathrm{Esc}(f_\lambda)$.

Proof. Let $z_0 \in \mathcal{J}(f_\lambda)$ and U be any neighbourhood of z_0. Since $\{f_\lambda^n\}$ is not normal in any neighbourhood of z_0, by Montel's theorem [**6**, p. 274], $\bigcup_n \{f_\lambda^n(U)\}$ omits at most one point in $\mathbb{C}$. In particular, there is a point $\hat{x} > r_\lambda$ such that $\hat{x} \in \bigcup_n \{f_\lambda^n(U)\}$. Therefore, there exists a $\hat{z} \in U$ such that $f_\lambda^j(\hat{z}) = \hat{x}$ for some positive integer j. Now, by Proposition 2.2, it follows that $f_\lambda^n(\hat{x}) \to \infty$ as $n \to \infty$, since $\hat{x} > r_\lambda$. Thus, there exists a point $\hat{z} \in U$ such that $f_\lambda^n(\hat{z}) \to \infty$ as $n \to \infty$ and $z_0 \in \mathrm{Esc}(f_\lambda)$ follows. Consequently, $\mathcal{J}(f_\lambda) \subseteq \mathrm{Esc}(f_\lambda)$.

To prove that $\mathrm{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$, we note that if $z_0 \in \mathrm{Esc}(f_\lambda)$ then $z_0 \notin W(a_\lambda)$. Let U be any neighbourhood of z_0. By Proposition 1.7, there exist an integer $N > 0$ and a point $\tilde{z} \in U$ such that $\Re(f_\lambda^N(\tilde{z})) < 0$; it therefore follows from Proposition 3.1 that the point $\tilde{z} \in W(a_\lambda)$. Now the orbit $\{f_\lambda^n(\tilde{z})\}$ of $\tilde{z}$ is bounded, while the orbit of z_0 escapes to ∞ under iteration of f_λ. Thus, $\{f_\lambda^n\}$ is not normal at z_0. Consequently, $z_0 \in \mathcal{J}(f_\lambda)$ and $\text{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$ follows. □

Remark. The characterization of the Julia set of a function as the closure of the set of all escaping points of the function has so far been found only for certain critically finite entire transcendental functions [**11**]. Theorem 3.1 seems to be the first attempt to find such a characterization for the non-critically finite entire transcendental function $f_\lambda(z)$.

Any non-critically finite entire transcendental functions may have only the following types of stable component (i.e. maximal connected open set) U in its Fatou set.

(1) U is a basin of attraction of an attracting fixed point.
(2) U is a wandering domain.
(3) U is a parabolic domain.
(4) U is a Siegel disk.
(5) U is a domain at infinity (i.e. open set consisting of points whose orbits tend uniformly to ∞).

The following Proposition 3.2 shows that the stable component U of $\mathcal{F}(f_\lambda)$ can only be either a basin of attraction of an attracting fixed point or a wandering domain.

Proposition 3.2. *The only possible stable component U of $\mathcal{F}(f_\lambda)$ is either a basin of attraction of an attracting fixed point or a wandering domain of $f_\lambda(z)$.*

Proof. If a point z_0 lies on an attracting cycle or a parabolic cycle of an entire transcendental function then the orbit of at least one of the singular values (i.e. critical values or asymptotic values) is attracted to the orbit of z_0 [**8**, p. 182]. Further, if U is a Siegel disk then the orbit of at least one of the critical points is dense in the boundary of U [**8**, p. 184]. Now, since $f_\lambda(-x) \to 0$ as $x \to \infty$, it follows that 0 is the asymptotic value for $f_\lambda(z)$. All the singular values of $f_\lambda(z)$ lie in the open unit disk for $0 < \lambda < \lambda^*$, by Proposition 1.3. Since $f_\lambda(B_1(0)) \subset f_\lambda(B_{\tilde{x}}(0)) \subseteq B_{\tilde{x}}(0)$, by Proposition 2.1 the orbits of all singular values lie in $B_{\tilde{x}}(0) \subset W(a_\lambda)$, where $W(a_\lambda)$ is the basin of attraction of the real attractive fixed point a_λ of $f_\lambda(z)$. Consequently, all singular values of $f_\lambda(z)$ and their orbits lie in the same component of $W(a_\lambda)$. Therefore, it follows that $f_\lambda(z)$ has no parabolic domains and no Siegel disks in $\mathcal{F}(f_\lambda)$. Further, $\mathcal{F}(f_\lambda)$ has no attractive basins other than $W(a_\lambda)$. By Theorem 3.1, it follows that $f_\lambda(z)$ does not have a domain at infinity. Thus, the only possible stable component U of $\mathcal{F}(f_\lambda)$ is either a basin of attraction of an attracting fixed point or a wandering domain of $f_\lambda(z)$. □

Corollary 3.1. *The Fatou set $\mathcal{F}(f_\lambda)$ of $f_\lambda(z)$ does not have a wandering domain if and only if the Fatou set of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$ equals the basin of attraction $W(a_\lambda)$ of a_λ.*

The following theorem shows that if $W(a_\lambda)$ contains $D^* = \{z \in \mathbb{C} : |f_\lambda'(z)| < 1\}$ then $W(a_\lambda) = \mathcal{F}(f_\lambda)$ and so, in this case, $f_\lambda(z)$ has no wandering domains. We need the following lemma.

Lemma 3.1. *Suppose $D^* = \{z \in \mathbb{C} : |f_\lambda'(z)| < 1\} \subset W(a_\lambda)$. Then, $W(a_\lambda)$ is a dense subset of $\mathbb{C}$.*

Proof. Let $z_0 \in (W(a_\lambda))^c$ and U be any open set containing z_0. We claim that $U \cap W(a_\lambda) \neq \emptyset$. If $U \cap W(a_\lambda) = \emptyset$ then $f_\lambda^n(U) \cap W(a_\lambda) = \emptyset$ for all n. Since $U \subset (W(a_\lambda))^c$, the inequality $\left|f_\lambda'(z)\right| > 1$ holds for all $z \in U$. Choose $\delta > 0$ so that $B_\delta(z_0) \subset U$ and $f_\lambda(z)$ is one-to-one in $B_\delta(z_0)$. Let μ_1 be such that $|f_\lambda'(z)| > \mu_1 > 1$ for all $z \in B_\delta(z_0)$. By Proposition 1.5 applied to $B_\delta(z_0)$, it follows that $f_\lambda(B_\delta(z_0)) \supset B_{\mu_1\delta}(f_\lambda(z_0))$. Now $B_{\mu_1\delta}(f_\lambda(z_0))$ does not meet $W(a_\lambda)$. Let μ_2 be such that $|f_\lambda'(z)| > \mu_2 > 1$ for all $z \in B_{\mu_1\delta}(f_\lambda(z_0))$. Again, by Proposition 1.5, we get $f_\lambda(B_{\mu_1\delta}(f_\lambda(z_0))) \supset B_{\mu_1\mu_2\delta}(f_\lambda^2(z_0))$. Continuing this process, and using the fact that $f_\lambda^n(U) \cap W(a_\lambda) = \emptyset$ for all n, it follows that there is an open disk $B_\rho(f_\lambda^n(z_0))$ of radius $\rho = \mu_1\mu_2\cdots\mu_n\delta$, centre $f_\lambda^n(z_0)$ and contained in $f_\lambda^n(U)$, that does not meet $W(a_\lambda)$. If n is chosen large enough so that $\rho = \mu_1\mu_2\cdots\mu_n\delta \geq \pi$, then $B_\rho(f_\lambda^n(z_0))$ must meet one of the horizontal lines $L_k = \{z = x + iy : y = (2k+1)\pi, -\infty < x < \infty\}$ $k = 0, \pm 1, \pm 2, \ldots$, say L_{k_0}. However, the line L_{k_0} is mapped by $f_\lambda(z)$ to a curve in the left half-plane $H^- = \{z \in \mathbb{C} : \Re(z) < 0\}$. This leads to a contradiction, since $W(a_\lambda) \supset H^-$. Therefore, $U \cap W(a_\lambda) \neq \emptyset$ and so $W(a_\lambda)$ is a dense subset of $\mathbb{C}$. □

Remark. Since $W(a_\lambda)$ is a dense subset of $\mathbb{C}$ and $W(a_\lambda) \supset H^-$, the complement of $W(a_\lambda)$ is a nowhere dense subset of the right half-plane. Thus, for $0 < \lambda < \lambda^*$, the Julia set of $f_\lambda(z)$ is also a nowhere dense subset of the right half-plane, if $D^* = \{z \in \mathbb{C} : |f_\lambda'(z)| < 1\} \subset W(a_\lambda)$.

THEOREM 3.2. *Suppose $D^* = \{z \in \mathbb{C} : |f_\lambda'(z)| < 1\} \subset W(a_\lambda)$, where $W(a_\lambda)$ is the basin of attraction of the real fixed point a_λ of $f_\lambda(z)$. Then, for $0 < \lambda < \lambda^*$, $\mathcal{F}(f_\lambda) = W(a_\lambda)$.*

Proof. Let $z_0 \in (W(a_\lambda))^c$ and U be any neighbourhood containing z_0. Then, by lemma 3.1, there exists a point $z_1 \in U$ such that $z_1 \in W(a_\lambda)$. Now, for sufficiently large values of n, $|f_\lambda^n(z_1) - a_\lambda|$ is very small, while $f_\lambda^n(z_0)$ must remain in the half-plane $\{z \in \mathbb{C} : \Re(z) > \tilde{x}\}$, for all $n \geq N$, where $f_\lambda'(\tilde{x}) = 1$. Therefore, $\left\{f_\lambda^n\right\}$ is not a normal family at z_0. Thus, $(W(a_\lambda))^c \subseteq \mathcal{J}(f_\lambda)$ or, equivalently, $\mathcal{F}(f_\lambda) \subseteq W(a_\lambda)$. In view of $W(a_\lambda) \subseteq \mathcal{F}(f_\lambda)$, we get $\mathcal{F}(f_\lambda) = W(a_\lambda)$. This completes the proof of Theorem 3.2. □

Remark. It follows from Theorem 3.2 that if $D^* = \{z \in \mathbb{C} : |f_\lambda'(z)| < 1\} \subset W(a_\lambda)$ then $\mathcal{J}(f_\lambda) = (W(a_\lambda))^c$, giving another characterization of the Julia set as the complement of basin of attraction of non-critically finite entire transcendental function $f_\lambda(z)$ for $0 < \lambda < \lambda^*$.

4. *Dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^*$*

In this section, we describe the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $\lambda > \lambda^*$. In this case, we mainly prove a result analogous to Theorem 3.1 that characterizes the Julia set of $f_\lambda(z)$, $\lambda > \lambda^*$, as the closure of the set of all escaping points. First we prove the following proposition giving that, for $\lambda > \lambda^*$, all the real points are contained in the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$.

PROPOSITION 4.1. *The real line $\mathbb{R}$ is contained in the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$ for $\lambda > \lambda^*$.*

Proof. Let $x_0 \in \mathbb{R}$ and U be any open set containing x_0. Set $x_n = f_\lambda^n(x_0)$, $n = 1, 2, 3, \ldots$. By Proposition 2.2(a), it follows that $x_n \to \infty$ as $n \to \infty$. Since $f_\lambda'(x) \neq 0$, for all

$x \in \mathbb{R}$, $f_\lambda(x)$ is locally one-to-one for all $x \in \mathbb{R}$. Further, $f_\lambda(x) > x$ for $x \in \mathbb{R}$ and $f'_\lambda(x) \to \infty$ as $x \to \infty$. Therefore, using the continuity of $f'_\lambda(z)$, there exist an integer $N_0 > 0$ and an open ball $B_\delta(x_0) \subseteq U$ such that:

(i) $f_\lambda^{N_0} : B_\delta(x_0) \to V$ is a homeomorphism;

(ii) $B_{\sqrt{2}\pi}(x_m) \subset \{z \in \mathbb{C} : |f'_\lambda(z)| > \sqrt{2}$ and $\Re(z) \geq 2\}$ for $m \geq N_0$.

Proposition 1.6, applied to the point x_{N_0} and the open set V containing x_{N_0}, gives that there exists an integer $N_1 > 0$ such that, if $n > N_1$, there exists an open set $V_n \subseteq V$ for which $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(x_{N_0}))$ is a homeomorphism, where $S_{2\pi}(f_\lambda^n(x_{N_0}))$ is the interior of the square with centre at $f_\lambda^n(x_{N_0})$ and having sides of length 2π, parallel to the coordinate axes. Since $f'_\lambda(x_n) \to \infty$ as $n \to \infty$, by Proposition 1.5, it follows that f_λ expands $S_{2\pi}(f_\lambda^n(x_{N_0}))$ with arbitrarily large scaling ratio as $n \to \infty$. Therefore, choose $N_2 > N_1$ large enough so that $f_\lambda(S_{2\pi}(f_\lambda^{n-1}(x_{N_0}))) \supset S_{2\pi}(f_\lambda^n(x_{N_0}))$ for $n > N_2$.

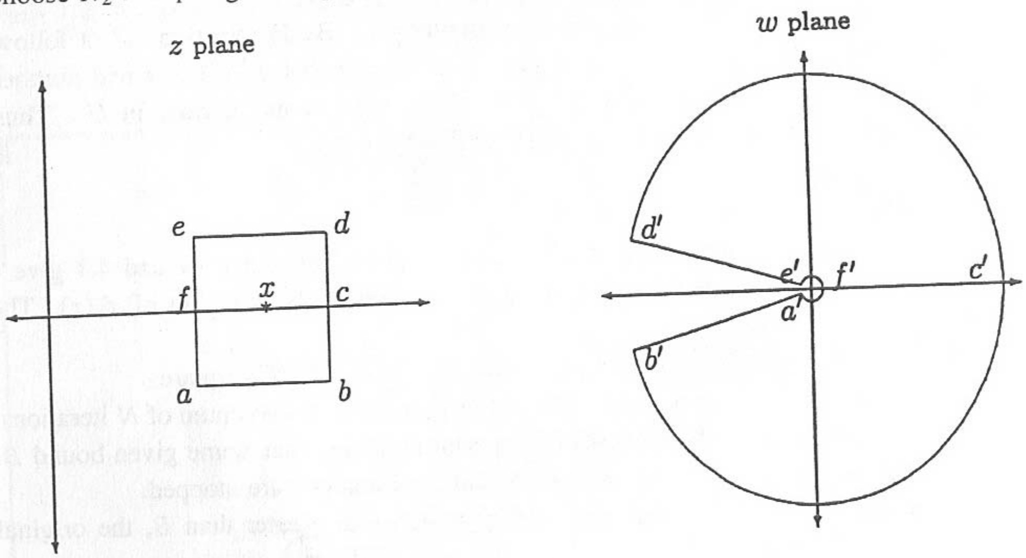


FIGURE 3. Image of the square $S_{2\pi}(x)$, under the mapping $w = f_\lambda(z)$.

Let $\Gamma: \widehat{abcdef}$ denote the boundary of the square $S_{2\pi}(f_\lambda^n(x_{N_0}))$ and $f_\lambda(\Gamma)$ be denoted by $a'b'\widehat{c'd'}e'f'$ (see Figure 3). Since the mappings $f_\lambda^{N_0} : B_\delta(x_0) \to V$, $f_\lambda^n : V_n \to S_{2\pi}(f_\lambda^n(x_{N_0}))$ for $n > N_1$ and $f_\lambda : S_{2\pi}(f_\lambda^n(x_{N_0})) \to f_\lambda(S_{2\pi}(f_\lambda^n(x_{N_0})))$ are homeomorphisms, for any point $w_1 \in f_\lambda(S_{2\pi}(f_\lambda^n(x_{N_0})))$ sufficiently near to the boundary point d', there exists a point in U that gets mapped to w_1 by $f_\lambda^{N_0+n+1}$ for each $n > N_1$. Similarly, for any real point $w_2 \in f_\lambda(S_{2\pi}(f_\lambda^n(x_{N_0})))$ sufficiently near the boundary point c', there exists a point in U that gets mapped to w_2 by $f_\lambda^{N_0+n+1}$, for each $n > N_1$. The points w_1 and w_2, in turn, get mapped by the function $f_\lambda(z)$ respectively to the points $f_\lambda(w_1)$ and $f_\lambda(w_2)$, one being inside the open disk $B_\lambda(0)$ and the other arbitrarily close to ∞. Therefore, for each integer $n > N_0+N_1+2$, there exist distinct points x_1 and x_2 in the open set U containing x_0 that get mapped by the function $f_\lambda^n(z)$ to the points $f_\lambda^n(x_1)$ and $f_\lambda^n(x_2)$, one being inside the open disk $B_\lambda(0)$ and the other arbitrarily close to ∞ respectively. Thus, the family of functions $\{f_\lambda^n\}$ is not normal in U and so $x_0 \in \mathcal{J}(f_\lambda)$. Since x_0 is any arbitrary real point, it follows that $\mathcal{J}(f_\lambda)$ contains the real line $\mathbb{R}$. □

Remark. For $0 < \lambda < \lambda^*$, the Julia set $\mathcal{J}(f_\lambda)$ lies entirely in the right half-plane $H^+ = \{z \in \mathbb{C} : \Re(z) > 0\}$. As soon as the real parameter λ crosses the value λ^*, there is a sudden change in the geometry of the Julia set of $f_\lambda(z)$ since, by the above proposition, the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$, spreads to the left half-plane as well. Thus, bifurcation in the dynamics of $f_\lambda(z)$ for $\lambda > 0$ occurs at $\lambda = \lambda^* (\approx 0.64761)$.

The following theorem, analogous to Theorem 3.1, provides a characterization for the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$.

THEOREM 4.1. *Let* $\mathrm{Esc}(f_\lambda) = \mathrm{clo}\{z \in \mathbb{C} : f_\lambda^n(z) \to \infty$ *as* $n \to \infty\}$ *be the closure of the set of escaping points of* $f_\lambda(z)$. *If* $\lambda > \lambda^*$ *then the Julia set* $\mathcal{J}(f_\lambda) = \mathrm{Esc}(f_\lambda)$.

Proof. The inclusion relation $\mathcal{J}(f_\lambda) \subseteq \mathrm{Esc}(f_\lambda)$ follows on the lines of a proof similar to that of Theorem 3.1. We need only to prove $\mathrm{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$.

Let $z_0 \in \mathrm{Esc}(f_\lambda)$ and U be an open set containing z_0. By Proposition 1.7, it follows that there exist an integer $N > 0$ and a point $\hat{z} \in U$ such that $f_\lambda^N(\hat{z})$ is a real number. Therefore, by Proposition 4.1, $\hat{z} \in \mathcal{J}(f_\lambda)$. Thus, $\{f_\lambda^n\}$ is not normal in U. Thus, $\mathrm{Esc}(f_\lambda) \subseteq \mathcal{J}(f_\lambda)$. This completes the proof of Theorem 4.1. □

5. *Applications*

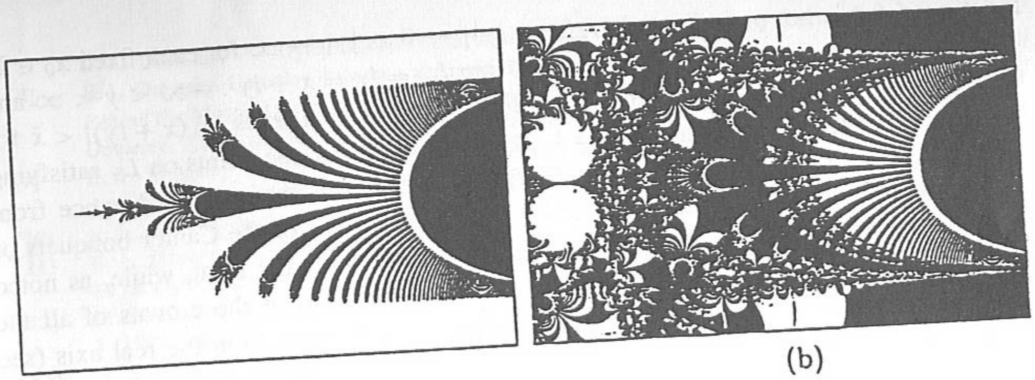
The characterizations of the Julia set $\mathcal{J}(f_\lambda)$ of $f_\lambda(z)$ in Theorems 3.1 and 4.1 give a useful algorithm to computationally generate the picture of the Julia set of $f_\lambda(z)$. The algorithm runs as follows.

(1) Select a square in the plane and construct a $k \times k$ grid in this square.
(2) For each grid point, compute the orbit of this point up to a maximum of N iterations.
(3) If, at iteration $i < N$, the real part of the orbit is greater than some given bound B, the original grid point is coloured black and the iterations are stopped.
(4) If the real part of no point in the orbit ever becomes greater than B, the original grid point is left as white.

Thus, in the output generated by this algorithm, the black points represent the Julia set of $f_\lambda(z)$ and the white points represent the Fatou set of $f_\lambda(z)$. Sometimes, either the real parts of the orbits of certain white points may take longer than N iterations to escape the bound B or the real parts of the orbits of certain black points may take longer than N iterations to become smaller than the given bound B. To overcome this difficulty, for $0 < \lambda < \lambda^*$, first a bound $B = B_0$ (say) is fixed and the pictures of the Julia set of $f_\lambda(z)$ with various values of N are computed. Then, by experimental observations, an integer $N_0 = N_0(B_0)$ is determined such that, for all $N > N_0$, the computed pictures of the Julia set of $f_\lambda(z)$ are similar for a maximum number of N iterations and $B = B_0$.

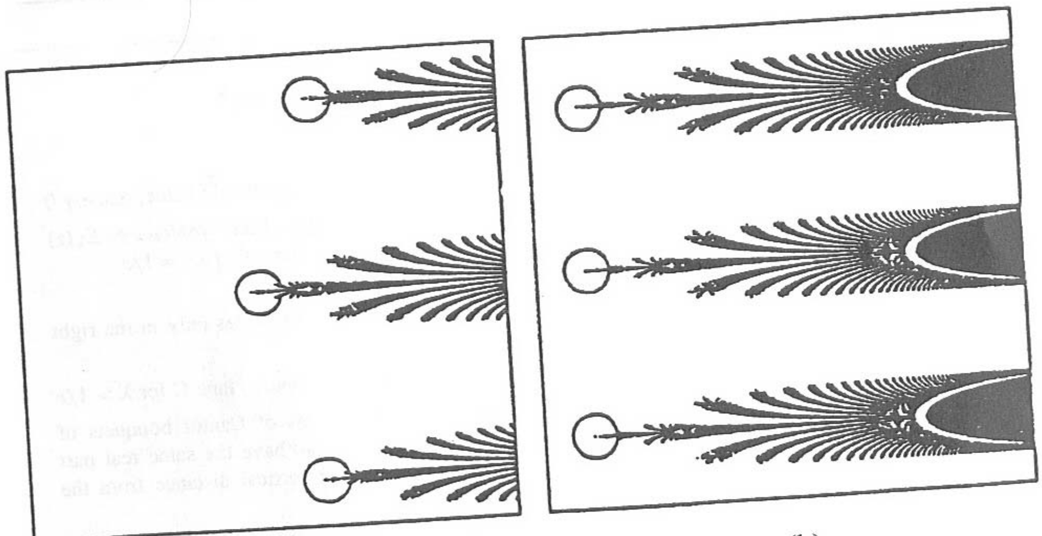
Using the above algorithm in the rectangular domain $R = \{z \in \mathbb{C} : 1.5 \le \Re(z) \le 8.5$ and $-2.5 \le \Im(z) \le 2.5\}$, the Julia set of $f_\lambda(z)$ for $\lambda = 0.64$ and $\lambda = 0.65$ are generated. To generate these pictures, a 700×500 grid is selected in the rectangle R and the maximum number of iterations $N = 240$ is allowed for a possible escape of the bound $B = 100$. The generated pictures of the Julia sets for $\lambda = 0.64$ and $\lambda = 0.65$ are shown in Figure 4.

The Julia set of $f_\lambda(z)$ for $\lambda = 0.64 < \lambda^*$ has the same pattern as those of the Julia ... it is observed that the nature



(a) (b)

FIGURE 4. Explosion in the Julia set of $f_\lambda(z)$. (a) $\lambda = 0.64 < \lambda^*$; (b) $\lambda = 0.65 > \lambda^*$. (Maximum number of iterations $N = 240$.)



(a) (b)

FIGURE 5. Comparison between the Julia sets of (a) $f_{0.33}(z)$ and (b) $E_{0.33}(z)$. ○ tip of the crown of the Cantor bouquet. (Maximum number of iterations $N = 240$.)

of the picture of the Julia set $\mathcal{J}(f_{0.64})$ remains unaltered by increasing the maximum number of iterations $N \geq 200$ for a fixed bound $B = 100$. Meanwhile, the nature of the picture of the Julia set of $f_\lambda(z)$ for $\lambda = 0.65 > \lambda^*$ shows a distinct change on increasing the number of iterations and, for a fixed bound $B = 100$, it becomes increasingly more black as the maximum number of iterations N is increased. Figure 4 suggests that the Julia set of $f_\lambda(z)$ admits Cantor bouquets for $0 < \lambda < \lambda^*$ and there is an explosion in the Julia set of $f_\lambda(z)$ as λ crosses the threshold value λ^*.

The following difference in the nature of the Julia set of $f_\lambda(z)$ for $0 < \lambda < \lambda^*$ and the nature of the Julia set of $E_\lambda(z) = \lambda e^z$ for $0 < \lambda < 1/e$, as found in [**9**], is observed. Since e^z is periodic, the Julia set of $E_\lambda(z)$ is the same in any horizontal strip of length 2π. The function $f_\lambda(z)$ is not periodic and $|f_\lambda(x_0+iy)| \to 0$ as $|y| \to \infty$ for each fixed $x_0 \in \mathbb{R}$. Therefore, it follows that, for each vertical line $L_x = \{z = x+iy : -\infty < y < \infty$ and fixed $x > 0\}$, there exists $y_x = y(x)$ such that $x+iy \in L_x$ satisfies $|f_\lambda(x+iy)| < \tilde{x}$ for $|y| < y_x$, where $\tilde{x}$ is such that $f'_\lambda(\tilde{x}) = 1$. By Proposition 3.1, the points on L_x satisfying $|f_\lambda(x+iy)| < \tilde{x}$ lie in the Fatou set $\mathcal{F}(f_\lambda)$. Therefore, as the vertical distance from the real axis increases, the real parts of the tip of the crowns of the Cantor bouquets of the Julia set of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, are pushed more towards the right, while, as noted in the case of $E_\lambda(z)$, $0 < \lambda < 1/e$, the real parts of the tip of the crowns of all the Cantor bouquets remain the same irrespective of their distances from the real axis (see Figure 5).

TABLE 1.

$f_\lambda(z) = \lambda(e^z-1)/z$, $\lambda > 0$	$E_\lambda(z) = \lambda e^z$, $\lambda > 0$
$f_\lambda(z)$ is not periodic	$E_\lambda(z)$ is periodic
$f_\lambda(z)$ has infinitely many critical values in the open disk $B_\lambda(0)$ centred at the origin and having radius λ	$E_\lambda(z)$ has no critical values
$f_\lambda(z)$ has only one asymptotic value, namely 0	$E_\lambda(z)$ has only one asymptotic value, namely 0
The bifurcation occurs in the dynamics of $f_\lambda(z)$ at the critical parameter value $\lambda^* \approx 0.64761$, defined by (2.2)	The bifurcation occurs in the dynamics of $E_\lambda(z)$ at the critical parameter value $\lambda^* = 1/e$
For $0 < \lambda < \lambda^*$, $\mathcal{J}(f_\lambda)$ lies only in the right half-plane	For $0 < \lambda < 1/e$, $\mathcal{J}(E_\lambda)$ lies only in the right half-plane
$\mathcal{J}(f_\lambda)$ contains the real line $\mathbb{R}$ for $\lambda > \lambda^*$	$\mathcal{J}(E_\lambda)$ equals the complex plane $\mathbb{C}$ for $\lambda > 1/e$
The real parts of the tip of the crowns of Cantor bouquets of $\mathcal{J}(f_\lambda)$, $0 < \lambda < \lambda^*$, are pushed towards the right as their vertical distance from the real axis increases	The tip of the crowns of Cantor bouquets of $\mathcal{J}(E_\lambda)$, $0 < \lambda < 1/e$, have the same real part irrespective of their vertical distance from the real axis
$\mathcal{J}(f_\lambda) = \mathrm{Esc}(f_\lambda)$ for $\lambda > 0$	$\mathcal{J}(E_\lambda) = \mathrm{Esc}(E_\lambda)$ for $\lambda > 0$

Finally, a general comparison between the dynamical properties of the non-critically finite function $f_\lambda(z)$ and the critically finite function $E_\lambda(z) = \lambda e^z$ is given in Table 1.

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