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Dynamics of a family of transcendental meromorphic functions having rational Schwarzian derivative

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Abstract

In the present paper, a class $\mathcal{F}$ of critically finite transcendental meromorphic functions having rational Schwarzian derivative is introduced and the dynamics of functions in one parameter family $\mathcal{K} \equiv \{f_\lambda(z) = \lambda f(z)\colon\ f \in \mathcal{F},\ z \in \hat{C} \text{ and } \lambda > 0\}$ is investigated. It is found that there exist two parameter values $\lambda^* = \phi(0) > 0$ and $\lambda^{**} = \phi(\tilde{x}) > 0$, where $\phi(x) = \frac{x}{f(x)}$ and $\tilde{x}$ is the real root of $\phi'(x) = 0$, such that the Fatou sets of $f_\lambda(z)$ for $\lambda = \lambda^*$ and $\lambda = \lambda^{**}$ contain parabolic domains. A computationally useful characterization of the Julia set of the function $f_\lambda(z)$ as the complement of the basin of attraction of an attracting real fixed point of $f_\lambda(z)$ is established and applied for the generation of the images of the Julia sets of $f_\lambda(z)$. Further, it is observed that the Julia set of $f_\lambda \in \mathcal{K}$ explodes to whole complex plane for $\lambda > \lambda^{**}$. Finally, our results found in the present paper are compared with the recent results on dynamics of one parameter families $\lambda \tan z$, $\lambda \in \hat{\mathbb{C}} \setminus \{0\}$ [R.L. Devaney, L. Keen, Dynamics of meromorphic maps: Maps with polynomial Schwarzian derivative, Ann. Sci. École Norm. Sup. 22 (4) (1989) 55–79; L. Keen, J. Kotus, Dynamics of the family $\lambda \tan(z)$, Conform. Geom. Dynam. 1 (1997) 28–57; G.M. Stallard, The Hausdorff dimension of Julia sets of meromorphic functions, London Math. Soc. 49 (1994) 281–295] and $\lambda \frac{e^z-1}{z}$, $\lambda > 0$ [G.P. Kapoor, M. Guru Prem Prasad, Dynamics of $(e^z - 1)/z$: The Julia set and bifurcation, Ergodic Theory Dynam. Systems 18 (1998) 1363–1383].

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1. Introduction

Extensive researches have centered around investigations of the dynamics of rational functions and polynomials. However, not much work to study the dynamics of transcendental meromorphic functions have been done so far; although, for instance, the iterative processes associated with Newton's method applied to an entire function often yields a meromorphic function as the root finder. Devaney and Keen [9] and Stallard [18] studied the dynamics of one parameter family of transcendental meromorphic functions $T_\lambda(z) = \lambda \tan z$, $\lambda \in \hat{\mathbb{C}} \setminus \{0\}$ by exploiting that the function $T_\lambda(z)$ has no critical values, only finite number of asymptotic values and its Schwarzian derivative $SD(T_\lambda) = \big(\frac{T_\lambda''(z)}{T_\lambda'(z)}\big)' - \frac{1}{2}\big(\frac{T_\lambda''(z)}{T_\lambda'(z)}\big)^2$ is a polynomial so that the linearly independent solutions g_1 and g_2 of the differential equation $g''(z) + 2SD(T_\lambda)g(z) = 0$ satisfy $\frac{g_1}{g_2} = T_\lambda(z)$. While investigating the dynamics of $T_\lambda(z)$, it is found that the bifurcation occurs at a real parameter value $\lambda = 1$ and the whole real line is contained in its Julia set for $\lambda > 1$ [9]. Further, the Julia set of $T_\lambda(z)$ is found to explode to whole complex plane at the parameter value $\lambda = i\pi$ [18]. The purpose of the present paper is to investigate the dynamics of a one parameter family of transcendental meromorphic functions that have rational Schwarzian derivative, are critically finite, possess both critical values as well as asymptotic values even then, for a certain range of parameter values, their Julia sets are the whole complex plane. Despite the Schwarzian derivative of functions in our family being rational, these functions are shown to have their dynamical behavior somewhat similar to that of the function $T_\lambda(z)$.

We first describe the basic concepts and results concerning transcendental meromorphic functions that are needed in the sequel in the study of dynamics of our class of functions. Let C and $\hat{C}$ denote the complex plane and the extended complex plane, respectively. A point w is said to be a critical point of f if $f'(w) = 0$. The value $f(w)$ corresponding to a critical point w is called a critical value of f. A point $w \in \hat{C}$ is said to be an asymptotic value for $f(z)$, if there is a continuous curve $\gamma(t)$ satisfying $\lim_{t\to\infty} \gamma(t) = \infty$ and $\lim_{t\to\infty} f(\gamma(t)) = w$. A function is said to be critically finite if it has only finitely many asymptotic and critical values. If a function $f(z)$ is not critically finite, then it is said to be non-critically finite. A singular value of f is defined to be either a critical value or an asymptotic value of f. The Nevanlinna characteristic of a meromorphic function $f(z)$ is defined by $T(r, f) = m(r, f) + N(r, f)$, where $m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\phi})|\, d\phi$, $N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t}\, dt + n(0, f)\log r$ and $n(r, f) = n(r, \infty, f)$ are the number of poles of f in the disk $z \leqslant r$, counted according to its multiplicity. The Nevanlinna order [11] ρ of the function f is defined as $\rho = \overline{\lim}_{r\to\infty} \frac{\log T(r, f)}{\log r}$.

Bergweiler and Eremenko [6] proved the following result, guaranteeing finite number of asymptotic values of a meromorphic function having finite Nevanlinna order.

Theorem 1.1. *If f is meromorphic function of finite order having only finitely many critical values, then f has at most 2ρ asymptotic values.*

It is well known that the Julia set of a polynomial never equals the whole complex plane, since infinity is an attracting fixed point for polynomials. However, Julia sets of rational and entire transcendental functions in certain cases are the whole complex plane [3,8,14,17]. This property was seen to hold for the transcendental meromorphic function $f(z) = i\pi \tan z$ by Stallard [18], i.e. $J(i\pi \tan z) = \hat{\mathbb{C}}$. Let B be a class of meromorphic function $f(z)$ having bounded singular values. The following theorem, by Zheng [19], gives a criterion for Julia set of transcendental meromorphic functions in B to be the extended complex plane:

Theorem 1.2. *Let f be a transcendental meromorphic function in Class B. If $(P(f))'$ is finite and $(P(f))' \cap J_\infty \setminus \{\infty\} = \phi$, and for $b \in \operatorname{sing}(f^{-1})$, b is (pre)periodic or $f^n(b) \to \infty$ as $n \to \infty$, then $J(f) = \hat{\mathbb{C}}$.*

Let S be the class of critically finite transcendental meromorphic function $f(z)$. Baker et al. [2] and Bergweiler [4] proved that a function $f \in S$ has no wandering domains or Baker domains [12].

The organization of the paper is as follows. In Section 2, a class of critically finite transcendental meromorphic functions having rational Schwarzian derivative is introduced and some basic properties of functions in this class are developed. In Section 3, the nature of real fixed points of a function $f_\lambda \in \mathcal{K}$, $\lambda > 0$, are found and the dynamics of $f_\lambda(x)$ for $x \in \mathbb{R} \setminus \{\alpha\}$ is described. Further, in this section, it is shown that there exist two critical parameter values $\lambda^* = \phi(0)$ and $\lambda^{**} = \phi(\tilde{x})$, where $\phi(x) = \frac{x}{f(x)}$ and $\tilde{x}$ is the real root of $\phi'(x) = 0$, such that bifurcations in the dynamics of $f_\lambda(x)$ occur at $\lambda = \lambda^*$ and $\lambda = \lambda^{**}$ (Fig. 2). In Section 4, the characterization of the Julia set $J(f_\lambda)$, $0 < \lambda < \lambda^*$ and $\lambda^* < \lambda < \lambda^{**}$, as the complement of the basin of attraction of an attracting real fixed point of $f_\lambda(z)$ is established. Further, it is proved that the Fatou set of the function $f_\lambda(z)$ for $\lambda = \lambda^*$ and $\lambda = \lambda^{**}$ contains a parabolic domain. It is observed in the same section that chaotic burst occurs in the Julia set $J(f_\lambda)$ such that, for $\lambda > \lambda^{**}$, $J(f_\lambda)$ is the whole of complex plane. In Section 5, the characterizations of the Julia set for the function $f_\lambda(z)$, obtained in Section 4, are applied to computationally generate their images for sample functions $f_\lambda(z) = \lambda \frac{z}{z+4} e^z \in \mathcal{K}$. Finally, the results of our investigations in this paper are compared with those of [9,15,18] obtained recently for the critically finite function $\lambda \tan z$ that has polynomial Schwarzian derivative and for the non-critically finite entire transcendental functions $\lambda \frac{e^z - 1}{z}$, $\lambda > 0$ [13].

2. Class $\mathcal{F}$ and some basic properties

We consider the following class of functions having rational Schwarzian derivative:

Definition 2.1. Let $\mathcal{T}$ be the class of transcendental meromorphic functions f of finite order satisfying:

(a) all poles of f are of odd multiplicity,
(b) all zeros of f' are of even multiplicity,
(c) Schwarzian derivative $SD(f)$ of f is a rational function.

It is easily seen that the functions $f_1(z) = \tan z$, $f_2(z) = \frac{e^z}{e^z - e^{-z}}$ and $f_3(z) = \frac{z \tan z + 1}{\tan z - z}$ investigated in [7,9] are in class $\mathcal{T}$. In addition, it is observed the function $f_4(z) = \frac{z+\mu}{z+\mu+4} e^z$, $\mu \in \mathbb{R}$ is in class $\mathcal{T}$.

Since for a function $f \in \mathcal{T}$, (i) Schwarzian derivative $SD(f)$ is rational, (ii) the pole of $f(z)$ is of odd multiplicity, and (iii) the zero of $f'(z)$ is of even multiplicity, it follows that [16] the differential equation $g''(z) + 2SD(f)g(z) = 0$ has two linearly independent solutions g_1 and g_2, each having a finite Nevanlinna order, such that $\frac{g_1}{g_2} = f(z)$.

Let $\mathcal{F}$ be the family of functions $f(z)$ defined by

$$\mathcal{F}=\left\{ f\in\mathcal{T}\colon \begin{array}{rl} \text{(i)} & g(z)=\frac{(z-\alpha)f(z)}{z},\ \alpha<1,\ \text{is a non-vanishing transcendental} \\ & \text{entire function having at most one finite asymptotic value} \\ \text{(ii)} & z(z-\alpha)g'(z)-\alpha g(z)=0 \text{ has only real roots} \\ \text{(iii)} & g(x) \text{ is positive and strictly increasing in } \mathbb{R},\ g(x)\to 0 \text{ as} \\ & x\to-\infty \text{ and } g(x)\to\infty \text{ as } x\to\infty \\ \text{(iv)} & g'(x) \text{ is strictly increasing in } \mathbb{R},\ g'(x)\to\infty \text{ as } x\to\infty \\ & \text{and } (x-\alpha)\frac{g'(x)}{g(x)}>1 \text{ for all } x>0 \end{array} \right\}. \tag{1}$$

An example of a function belonging to the family $\mathcal{F}$ is $f(z)=\frac{z}{z+4}e^z$. It turns out that though the functions in class $\mathcal{F}$ have essentially different properties than those considered by Devaney and Keen [9], Devaney and Tangerman [10] and Stallard [18] their dynamical properties are somewhat similar to the functions considered in these works.

The following proposition shows that the functions in the class $\mathcal{F}$ are critically finite having only real critical values and one finite asymptotic value:

Proposition 2.1. *The functions in the class $\mathcal{F}$ are critically finite, have only real critical values and* 0 *is their only finite asymptotic value.*

Proof. We first show that a function $f\in\mathcal{F}$ has only finite number of critical values. Let $q(z)=(f'(z))^{-\frac{1}{2}}$. Then, it is easily seen that

$$q''(z)+\frac{1}{2}SD(f)q(z)=0. \tag{2}$$

Since $\mathcal{F}\subset\mathcal{T}$, the function $f(z)$ satisfies conditions (a), (b), and (c) of Definition 2.1, so that [16, Theorem 6.5], $f(z)=\frac{g_1(z)}{g_2(z)}$, where meromorphic functions q_1 and q_2 are linearly independent solutions of (2). Therefore, the Wronskian $W(q_1,q_2)=\left|\begin{smallmatrix} q_1 & q_2 \\ q_1' & q_2' \end{smallmatrix}\right|$ and $f'(z)$ can be written as

$$f'(z)=-\frac{W(q_1,q_2)}{q_2^2(z)}. \tag{3}$$

Observe that the order of poles on both sides of the equation $q''(z)=-\frac{1}{2}SD(f)q(z)$ must agree. Suppose $q_2(z)$ has a pole at the point z_p of order n. Then, $q_2''(z)$ has a pole at the point z_p of order $(n+2)$. Consequently, $SD(f)$ has a pole at the point z_p of order 2 implying that the poles of $q_2(z)$ are also the poles of $SD(f)$ and there is no pole of $SD(f)$ other than those of $q_2(z)$. The function $SD(f)$, being a rational function, has a finite number of poles and hence $q_2(z)$ also has a finite number of poles. Therefore, by (3), $f'(z)$ can vanish only at finite number of points so that the function $f(z)$ has only finite number of critical points and, consequently, has only finite number of critical values. Further, condition (i) and Theorem 1.1 give that the function $f(z)$ has finite number of asymptotic values. Thus, the function $f\in\mathcal{F}$ is critically finite. Using (1(ii)), $f'(z)=0$ has only real zeros. It therefore follows that $f(z)$ has only real critical values. By (1(iii)), $g(x)\to 0$ as $x\to-\infty$ and hence $f(x)\to 0$ as $x\to-\infty$. Therefore, 0 is an asymptotic value for $f(z)$ and by condition (1(i)), it follows that 0 is the only finite asymptotic value of $f(z)$. □

3. Nature of fixed points in one parameter family $\mathcal{K}$ and bifurcation

The existence of the real fixed points of $f_\lambda(x) = \lambda f(x)$ in $\mathbb{R}$ and their nature is described in this section. For a function $f \in \mathcal{F}$, let

$$\mathcal{K} \equiv \left\{ f_\lambda(z) = \lambda f(z)\colon\ f \in \mathcal{F},\ z \in \hat{C} \text{ and } \lambda > 0 \right\}$$

be one parameter family of transcendental meromorphic functions.

Let

$$\phi(x) = \begin{cases} \frac{x}{f(x)} & \text{for } x \neq 0, \\ \lim_{x\to 0} \frac{x}{f(x)} & \text{for } x = 0. \end{cases} \tag{4}$$

The limit in (4) exists since, by (1(i)), $f(z)$ has a simple zero at $z = 0$.

Throughout in the sequel, we denote

$$\lambda^* = \phi(0) \quad \text{and} \quad \lambda^{**} = \phi(\tilde{x}), \tag{5}$$

where $\tilde{x}$ is the solution of $\phi'(x) = 0$. Note that by Lemma 3.1(iii), it follows that λ^*, λ^{**} defined by (5) satisfy $\lambda^* < \lambda^{**}$.

The following lemmas are needed in the sequel:

Lemma 3.1. *Let $\phi(x)$ be defined by* (4). *Then,*

(i) *$\phi(x)$ is continuously differentiable in $\mathbb{R}$.*
(ii) *Solution of $\phi'(x) = 0$ exists and is unique.*
(iii) *$\phi(x)$ is strictly increasing in $(-\infty, \tilde{x})$, is strictly decreasing in $(\tilde{x}, \infty)$ and has one maximum at $x = \tilde{x}$, where $\tilde{x}$ is a negative real solution of $\phi'(x) = 0$.*

Proof. From (1(i)) and (4),

$$\phi(x) = \frac{(x-\alpha)}{g(x)}. \tag{6}$$

Since $g(z)$ is an entire function and $g(x) > 0$, (i) follows obviously.

Since, by (1(iii)), the function $g(x)$ is positive and strictly increasing in $\mathbb{R}$, it follows that $\phi'(x) > 0$ for $x < \alpha$. Further, by (1(iv)), $\phi'(x) < 0$ for $x > 0$. Thus, there exists at least one zero of $\phi'(x)$ in $\mathbb{R}$. To establish the uniqueness of the zero of $\phi'(x)$ in $\mathbb{R}$, let $h(x) = g(x) - (x-\alpha)g'(x)$. It is easily seen by using (1(iv)) that $h(x)$ is increasing for $x < \alpha$ and is decreasing for $x > \alpha$. Since $h(x)$ is continuous in $\mathbb{R}$, by (1(iii)), $h(\alpha) = g(\alpha) > 0$ and $h(x)$ has the same sign as that of $\phi'(x)$, it follows that $h(x) > 0$ for $x \leqslant \alpha$ and $h(x) < 0$ for $x > 0$. Consequently, $h(x)$ has zeros only in the interval $(\alpha, 0)$. If x_1 and x_2 are two points in $(\alpha, 0)$ such that $h(x_1) = 0 = h(x_2)$ with $x_1 < x_2$, then there exists a point $c \in (x_1, x_2)$ such that $h'(c) = 0$. This is not possible, since $h'(x)$ is negative for any $x > \alpha$. Thus, there exists a unique point $\tilde{x}$ in the interval $(\alpha, 0)$ such that $h(\tilde{x}) = 0$. Since, by (1(iii)), $g(\tilde{x}) \neq 0$, it follows that $\tilde{x}$ with $\alpha < \tilde{x} < 0$ is the unique zero of $\phi'(x)$. This proves (ii).

It is easily seen that $\phi'(x) > 0$ for $x < \tilde{x}$ and $\phi'(x) < 0$ for $x > \tilde{x}$, where $\alpha < \tilde{x} < 0$ is unique solution of $\phi'(x) = 0$. Therefore, $\phi(x)$ is strictly increasing in $(-\infty, \tilde{x})$ and is strictly decreasing in $(\tilde{x}, \infty)$. Since $\phi'(\tilde{x}) = 0$, it follows that $\phi(\tilde{x})$ is the maximum value of $\phi(x)$ in $\mathbb{R}$, completing the proof of (iii). □

The graph of $\phi(x)$ is as shown in Fig. 1.

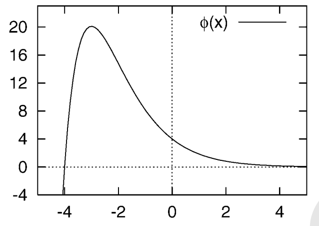

Fig. 1. Graph of $\phi(x)$.

Lemma 3.2. *Let $f \in \mathcal{F}$. Then, $f'(x)$ is positive in $(-\infty, \alpha) \cup (\alpha, \tilde{x}] \cup [0, \infty)$ and $f'(x)$ is non-negative in $(\tilde{x}, 0)$, where $\tilde{x}$ is the solution of $\phi'(x) = 0$.*

Proof. Since

$$f'(x) = \frac{\phi(x) - x\phi'(x)}{\phi^2(x)} = \frac{x(x-\alpha)g'(x) - \alpha g(x)}{(x-\alpha)^2}$$

it follows that, by Lemma 3.1(iii), $f'(x) > 0$ in $(\alpha, \tilde{x}] \cup [0, \infty)$. By (1(iii)) and(1(iv)), $f'(x) > 0$ for $x \in (-\infty, \alpha)$. Therefore, only non-negativity of $f'(x)$ in $(\tilde{x}, 0)$ remains to be established. If $f'(x) < 0$ for some point $x \in (\tilde{x}, 0)$, by continuity of $f'(x)$, $f'(x) < 0$ for $x \in (x_0 - \delta, x_0 + \delta)$, for sufficiently small δ. Then, $f(x)$ is decreasing in $(x_0 - \delta, x_0 + \delta)$. Further, $f(x)$ is continuous in $[\tilde{x}, 0]$ and is increasing at the point $x = \tilde{x}$. Since $f(0) > f(\tilde{x})$, it follows that $f(x)$ must have maxima or minima in $(\tilde{x}, 0)$. This gives a contradiction since, by condition (b) of Definition 2.1, all zeros of $f'(x)$ are of even multiplicity so that $f(x)$ can have only inflexion points, if any, in $(\tilde{x}, 0)$. Thus, $f'(x) \geqslant 0$ in $(\tilde{x}, 0)$. □

Since $f_\lambda(0) = 0$ for any $\lambda > 0$, the point $x = 0$ is a common fixed point of all the functions $f_\lambda(z)$. The non-zero fixed points of the function $f_\lambda(x)$ are solutions of the equation $\phi(x) = \lambda$, where $\phi(x)$ is given by (4). Using Lemma 3.1, it is easily seen that (Fig. 1): (i) For $0 < \lambda < \lambda^*$, $f_\lambda(x)$ has exactly one fixed point in each of the intervals, say $r_{1,\lambda} \in (\alpha, x')$ and $r_{2,\lambda} \in (0, \infty)$. (ii) For $\lambda = \lambda^*$, only fixed point of $f_\lambda(x)$ is x'. (iii) For $\lambda^* < \lambda < \lambda^{**}$, $f_\lambda(x)$ has exactly one fixed point in each of the intervals, say $r_\lambda \in (x', \tilde{x})$ and $a_\lambda \in (\tilde{x}, 0)$, where $\tilde{x}$ is the non-zero solution of $\phi'(x) = 0$. (iv) For $\lambda = \lambda^{**}$, only fixed point of $f_\lambda(x)$ is $\tilde{x}$. (v) For $\lambda > \lambda^{**}$, $f_\lambda(x)$ has no non-zero fixed points. The nature of the fixed points of $f_\lambda(x)$ for different values of parameter λ is described in the following theorem.

Theorem 3.1. *For $f_\lambda \in \mathcal{K}$,*

(i) *the fixed points 0 for $0 < \lambda < \lambda^*$ and $a_\lambda \in (\tilde{x}, 0)$ for $\lambda^* < \lambda < \lambda^{**}$ are attracting, where $\tilde{x}$ is a solution of $\phi'(x) = 0$,*
(ii) *the fixed points 0 for $\lambda = \lambda^*$ and $\tilde{x}$ for $\lambda = \lambda^{**}$ are rationally indifferent,*
(iii) *the fixed points $r_{1,\lambda} \in (\alpha, x')$ and $r_{2,\lambda} \in (0, \infty)$ for $0 < \lambda < \lambda^*$, x' for $\lambda = \lambda^*$, $r_\lambda \in (x', \tilde{x})$ and 0 for $\lambda^* < \lambda < \lambda^{**}$ and 0 for $\lambda \geqslant \lambda^{**}$ are repelling, where x' is a non-zero solution of $\lambda^* = \phi(x)$.*

Proof. (i) For $0 < \lambda < \lambda^*$, $f'_\lambda(0) = \lambda f'(0) < \lambda^* f'(0) = \frac{\lambda^*}{\phi(0)} = 1$. Since $\alpha < 0$ and $g(0) > 0$, by condition (1(iii)) in definition of family $\mathcal{F}$, $f'_\lambda(0) = \frac{-\lambda g(0)}{\alpha} > 0$. Therefore, it follows that 0 is an attracting fixed point of $f_\lambda(x)$ for $0<\lambda<\lambda^*$. To determine the nature of the fixed point $a_\lambda \in (\tilde{x},0)$, observe that in view of Lemma 3.1(iii), for $\lambda^*<\lambda<\lambda^{**}$, $f(a_\lambda)-a_\lambda f'(a_\lambda)<0$. Since $(\tilde{x},0)\subset(\alpha,0)$ and $a_\lambda\in(\tilde{x},0)$, by (1(iii)), $f(a_\lambda)<0$. Therefore, $f'_\lambda(a_\lambda)=\frac{a_\lambda f'(a_\lambda)}{f(a_\lambda)}<1$. Further, by Lemma 3.2, $f'_\lambda(a_\lambda)\geqslant 0$. Consequently, a_λ is also an attracting fixed point of $f_\lambda(x)$ for $\lambda^*<\lambda<\lambda^{**}$.

(ii) For $\lambda=\lambda^*$, it is easily seen that $f'_\lambda(0)=\lambda^* f'(0)=\frac{\lambda^*}{\phi(0)}=1$. Thus, 0 is a rationally indifferent fixed point of $f_\lambda(x)$ for $\lambda=\lambda^*$. To determine the nature of the fixed point $\tilde{x}$, observe that, by Lemma 3.1(iii), $f'_\lambda(\tilde{x})=\frac{\tilde{x}f'(\tilde{x})}{f(\tilde{x})}=1$ for $\lambda=\lambda^{**}$. It therefore follows that $\tilde{x}$ is a rationally indifferent fixed point of $f_\lambda(x)$ for $\lambda=\lambda^{**}$.

(iii) Since, $r_{1,\lambda}\in(\alpha,x')\subset(\alpha,\tilde{x})$, by Lemma 3.1(iii), $f(r_{1,\lambda})-r_{1,\lambda}f'(r_{1,\lambda})>0$ for $0<\lambda<\lambda^*$. Further, by (1(iii)), $f(r_{1,\lambda})<0$. Therefore, using Lemma 3.2, $f'_\lambda(r_{1,\lambda})=\frac{r_{1,\lambda}f'(r_{1,\lambda})}{f(r_{1,\lambda})}>1$. Thus, $r_{1,\lambda}$ is a repelling fixed point of $f_\lambda(x)$. Similarly, since $f(r_{2,\lambda})-r_{2,\lambda}f'(r_{2,\lambda})<0$ and $f(r_{2,\lambda})>0$, it follows that $f'_\lambda(r_{2,\lambda})=\frac{r_{2,\lambda}f'(r_{2,\lambda})}{f(r_{2,\lambda})}>1$. Thus, $r_{2,\lambda}$ is a repelling fixed point of $f_\lambda(x)$.

For $\lambda=\lambda^*$, by Lemma 3.1(iii), $f(x')-x'f'(x')>0$ for $x'\in(\alpha,\tilde{x})$ and, by (1(iii)), $f(x')<0$. Therefore, by Lemma 3.2, $f'_\lambda(x')=\frac{x'f'(x')}{f(x')}>1$. Consequently, x' is a repelling fixed point of $f_\lambda(x)$.

For $\lambda^*<\lambda<\lambda^{**}$, by Lemma 3.1(iii), $f(r_\lambda)-r_\lambda f'(r_\lambda)>0$ for $x'\in(\alpha,r_\lambda)\subset(\alpha,\tilde{x})$ and, by (1(iii)), $f(r_\lambda)<0$. Therefore, by Lemma 3.2, $f'_\lambda(r_\lambda)=\frac{r_\lambda f'(r_\lambda)}{f(r_\lambda)}>1$. It now follows that r_λ is a repelling fixed point of $f_\lambda(x)$.

For $\lambda^*<\lambda<\lambda^{**}$, $f'_\lambda(0)=\lambda f'(0)>\lambda^* f'(0)=\frac{\lambda^*}{\phi(0)}=1$ so that 0 is a repelling fixed point of $f_\lambda(x)$. Similarly, for $\lambda\geqslant\lambda^{**}$, $f'_\lambda(0)=\lambda f'(0)\geqslant\lambda^{**}f'(0)>\lambda^* f'(0)=\frac{\lambda^*}{\phi(0)}=1$. This proves that 0 is a repelling fixed point of $f_\lambda(x)$. □

Using Theorem 3.1, the dynamics of the function $f_\lambda\in\mathcal{K}$ on the real line is determined by the following cases.

Case I. For $0<\lambda<\lambda^*$, (a) $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in(-\infty,x^*)\cup(r_{1,\lambda},r_{2,\lambda})\cup G$ and (b) $f_\lambda^n(x)\to\infty$ as $n\to\infty$ for $x\in(x^*,\alpha)\cup(r_{2,\lambda},\infty)\cup H$, where $x^*(<0)$ is a solution of $f_\lambda(x)=r_{2,\lambda}$, $G=(\bigcup_{n=1}^\infty f_\lambda^{-n}(-\infty,x^*))\cap(\alpha,r_{1,\lambda})$ and $H=(\bigcup_{n=1}^\infty f_\lambda^{-n}(x^*,\alpha))\cap(\alpha,r_{1,\lambda})$.

By Theorem 3.1, $f_\lambda(x)$ has an attracting fixed point 0 and two repelling fixed points $r_{1,\lambda}$ and $r_{2,\lambda}$ with $r_{1,\lambda}<0<r_{2,\lambda}$ for $0<\lambda<\lambda^*$. Further, it is easily seen that $f_\lambda(x)-x>0$ for $x\in(r_{1,\lambda},0)\cup(r_{2,\lambda},\infty)$ and $f_\lambda(x)-x<0$ for $x\in(\alpha,r_{1,\lambda})\cup(0,r_{2,\lambda})$. To establish the dynamics of $f_\lambda(x)$ described by Case I(a), observe that $f_\lambda(x)<x$ for $x\in(0,r_{2,\lambda})$. Further, in view of (1(iii)), $f(x)>0$ for $x>0$ and, by Lemma 3.2, $f_\lambda(x)$ is increasing. Thus, it follows that the sequence $\{f_\lambda^n(x)\}$ is decreasing and bounded below by 0. Hence, $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in(0,r_{2,\lambda})$. Similarly, since $f_\lambda(x)>x$ for $x\in(r_{1,\lambda},0)$ and, in view of (1(iii)), $f(x)<0$ for $\alpha<x<0$, by Lemma 3.2 it follows that $f_\lambda(x)$ is increasing, so that, for $x\in(r_{1,\lambda},0)$ the sequence $\{f_\lambda^n(x)\}$ is increasing and bounded above by 0. Consequently, $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in(r_{1,\lambda},0)$. Further, since $f(x^*)=r_{2,\lambda}$ and by Lemma 3.2 $f_\lambda(x)$ is increasing in $(-\infty,x^*)$, $f_\lambda(x)$ maps $(-\infty,x^*)$ into $(0,r_{2,\lambda})$. Now, using $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in(0,r_{2,\lambda})$, it follows that $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in(-\infty,x^*)$. Finally, the forward orbit of each point $x\in G$ is contained in the interval $(-\infty,x^*)$. Therefore, repeating the above arguments, we have $f_\lambda^n(x)\to 0$ as $n\to\infty$ for $x\in G$.

For establishing the dynamics of $f_\lambda(x)$ described by Case I(b), first observe that $f_\lambda(x) > x$ for $x \in (r_{2,\lambda}, \infty)$. Further, since, in view of (1(iii)), $f(x) > 0$ for $x > 0$ and by Lemma 3.2, $f_\lambda(x)$ is increasing. Consequently, the sequence $\{f_\lambda^n(x)\}$ for $x \in (r_{2,\lambda}, \infty)$ is increasing. Since there is no fixed point larger than $r_{2,\lambda}$, it now follows that $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (r_{2,\lambda}, \infty)$. Next, since $f_\lambda(x)$ is increasing and $f(x^*) = r_{2,\lambda}$, the function $f_\lambda(x)$ maps (x^*, α) into $(r_{2,\lambda}, \infty)$. Therefore, repeating the above arguments, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (x^*, \alpha)$. Further, the forward orbit of each point $x \in H$ is contained in the interval (x^*, α). Thus, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in H$.

Case II. For $\lambda = \lambda^*$, (a) $f_\lambda^n(x) \to 0$ as $n \to \infty$ for $x \in (x', 0)$ and (b) $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, \alpha) \cup (0, \infty) \cup I$, where $I = (\bigcup_{n=1}^{\infty} f_\lambda^{-n}(-\infty, \alpha)) \cap (\alpha, x')$.

By Theorem 3.1, $f_\lambda(x)$ has a rationally indifferent fixed point 0 and a repelling fixed point x' for $\lambda = \lambda^*$. Further, it is easily seen that $f_\lambda(x) - x > 0$ for $x \in (x', 0) \cup (0, \infty)$ and $f_\lambda(x) - x < 0$ for $x \in (\alpha, x')$. To describe the dynamics of $f_\lambda(x)$ for Case II(a), $f_\lambda(x) > x$ for $x \in (x', 0)$. Further, in view of (1(iii)), $f(x) < 0$ for $\alpha < x < 0$ and by Lemma 3.2, $f_\lambda(x)$ is increasing. Therefore, it follows that the sequence $\{f_\lambda^n(x)\}$ is increasing and bounded above by 0. Hence, $f_\lambda^n(x) \to 0$ as $n \to \infty$ for $x \in (x', 0)$.

To establish the dynamics of $f_\lambda(x)$ described by Case II(b), observe that $f_\lambda(x) > x$ for $x \in (0, \infty)$. Further, in view of (1(iii)), $f(x) > 0$ for $x > 0$, by Lemma 3.2, $f_\lambda(x)$ is increasing. Therefore, the sequence $\{f_\lambda^n(x)\}$ is increasing for $x \in (0, \infty)$. Since, there is no fixed point larger than 0, it follows that $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0, \infty)$. Further, since, by Lemma 3.2, $f_\lambda(x)$ is increasing in $(-\infty, \alpha)$, $f_\lambda(x)$ maps the interval $(-\infty, \alpha)$ into $(0, \infty)$, repeating the above arguments, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, \alpha)$. Finally, the forward orbit of each point $x \in I$ is contained in the interval $(-\infty, \alpha)$. Thus, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in I$.

Case III. For $\lambda^* < \lambda < \lambda^{**}$, (a) $f_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in (r_\lambda, 0)$ and (b) $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, \alpha) \cup (0, \infty) \cup L$, where $L = (\bigcup_{n=1}^{\infty} f_\lambda^{-n}(-\infty, \alpha)) \cap (\alpha, r_\lambda)$.

For $\lambda^* < \lambda < \lambda^{**}$, by Theorem 3.1, $f_\lambda(x)$ has an attracting fixed point a_λ and two repelling fixed points $0, r_\lambda$ with $r_\lambda < a_\lambda < 0$. Further, $f_\lambda(x) - x > 0$ for $x \in (r_\lambda, a_\lambda) \cup (0, \infty)$ and $f_\lambda(x) - x < 0$ for $x \in (\alpha, r_\lambda) \cup (a_\lambda, 0)$. The rest of proof now follows analogous to that of Case I.

Case IV. For $\lambda = \lambda^{**}$, (a) $f_\lambda^n(x) \to \tilde{x}$ as $n \to \infty$ for $x \in (\tilde{x}, 0)$ and (b) $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, \alpha) \cup (0, \infty) \cup M$, where $M = (\bigcup_{n=1}^{\infty} f_\lambda^{-n}(-\infty, \alpha)) \cap (\alpha, \tilde{x})$.

For $\lambda = \lambda^{**}$, by Theorem 3.1, $f_\lambda(x)$ has a rationally indifferent fixed point $\tilde{x}$ and a repelling fixed point 0. Further, $f_\lambda(x) - x > 0$ for $x \in (0, \infty)$ and $f_\lambda(x) - x < 0$ for $x \in (\alpha, \tilde{x}) \cup (\tilde{x}, 0)$. Now, the assertion follows on the lines of the proof of Case II.

Case V. For $\lambda > \lambda^{**}$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for all $x \in \mathbb{R} \setminus T_\alpha$, where T_α is the set of the points that are backward orbits of the pole a of $f_\lambda(x)$.

For $\lambda > \lambda^{**}$, by Theorem 3.1, $f_\lambda(x)$ has only one repelling fixed point at $x = 0$. Further, $f_\lambda(x) - x > 0$ for $x \in (0, \infty)$ and $f_\lambda(x) - x < 0$ for $x \in (\alpha, 0)$. Using the similar arguments as in Case II, the assertion in this case follows easily.

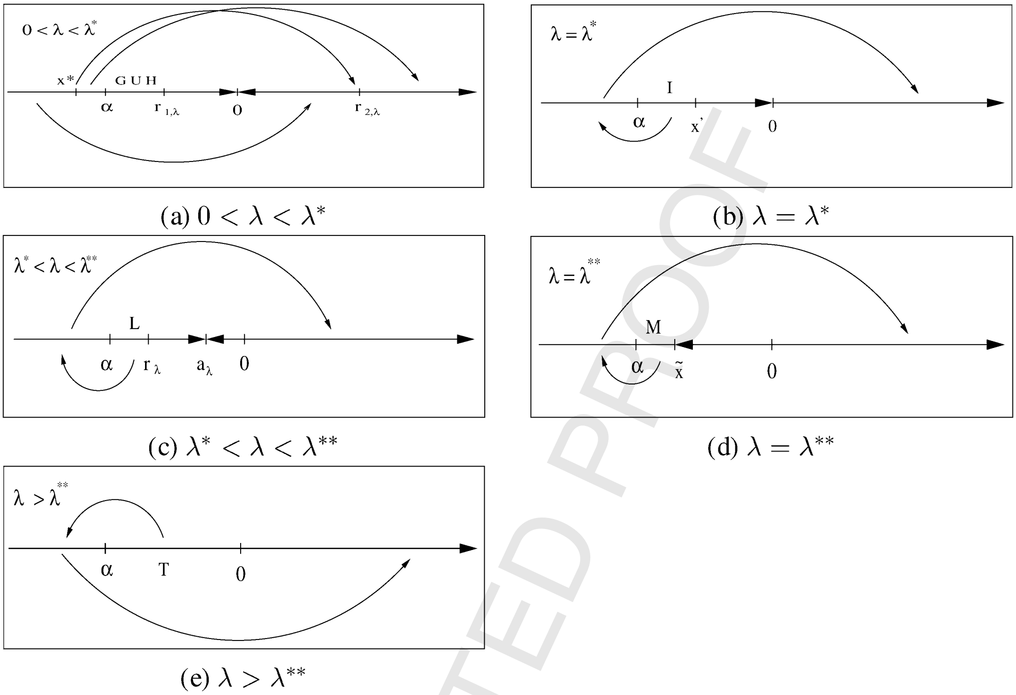

Fig. 2. Phase portraits of the function $f_\lambda(x)$ for $x \in \mathbb{R} \setminus \{\alpha\}$ and $\lambda > 0$.

Using the above cases, the phase portraits (Fig. 2) describing the dynamics of $f_\lambda(x)$ for various values of λ are obtained.

Remark 3.1. *In Case* I*, the points in the interval* $(-\infty, \alpha)$ *map either in* $(0, r_{2,\lambda})$ *or* $(r_{2,\lambda}, \infty)$*. Due to this reason, the arrow showing the behavior of points in* $G \cup H$ *for* $0 < \lambda < \lambda^*$ *are not marked in the phase portraits* (Fig. 2(a)).

It follows by Cases I–V that bifurcations in the dynamics of the function $f_\lambda(x)$ for $x \in \mathbb{R} \setminus \{\alpha\}$ occur at the critical parameter values $\lambda = \lambda^*$ and $\lambda = \lambda^{**}$. The bifurcation diagram for the function $f_\lambda(x) = \lambda f(x)$, $\lambda > 0$ is shown in Fig. 3.

4. Dynamics of $f_\lambda(z)$ for $z \in \hat{\mathbb{C}}$

In this section, the dynamics of the function $f_\lambda(z)$ for $z \in \hat{\mathbb{C}}$ and $\lambda > 0$ is investigated. Let

$$A(0) = \left\{z \in \mathbb{C}\colon\ f_\lambda^n(z) \to 0 \text{ as } n \to \infty\right\} \quad \text{and} \quad A(a_\lambda) = \left\{z \in \mathbb{C}\colon\ f_\lambda^n(z) \to x_2 \text{ as } n \to \infty\right\}$$

be the basins of attraction of the attracting fixed points 0 and a_λ of $f_\lambda(z)$. The following theorem gives the characterization of the Julia set $J(f_\lambda)$ as the complement of basin of attraction of the function $f_\lambda(z)$ for $0 < \lambda < \lambda^*$ and $\lambda^* < \lambda < \lambda^{**}$. Our characterization is more suited for computer generation of images of Julia sets than that given by Baker et al. [1].

Theorem 4.1. *Let* $f_\lambda \in \mathcal{K}$.

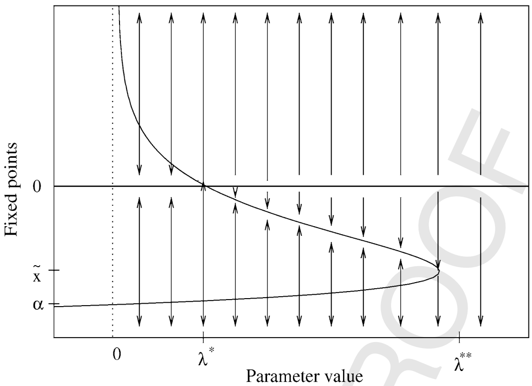

Fig. 3. Bifurcation diagram for the function $f_\lambda(x) = \lambda f(x)$, $\lambda > 0$.

(i) *For* $0 < \lambda < \lambda^*$, *the Julia set* $J(f_\lambda) = \hat{\mathbb{C}} \setminus A(0)$.
(ii) *For* $\lambda^* < \lambda < \lambda^{**}$, *the Julia set* $J(f_\lambda) = \hat{\mathbb{C}} \setminus A(a_\lambda)$.

Proof. Since $f_\lambda(z)$ is critically finite meromorphic function, it follows [2,4], that Fatou set $F(f_\lambda)$ has no wandering domains or Baker domains. Further, if U a Siegel disk or a Herman ring, the forward images of singularities of f_λ^{-1} are dense in the boundary of U [5]. Since $z = 0$ is an attracting fixed point of $f_\lambda(z)$, its only asymptotic value 0 lies in the basin of attraction $A(0)$. The number of critical values of $f_\lambda(z)$ is finite and all these critical values lie on the real line. Therefore, by Case I in Section 3, the forward orbit of critical value either tends to attracting fixed point 0 or to ∞ under iteration. Consequently, all singular values of $f_\lambda(x)$ and their orbits either lie in the same component of $A(0)$ or tend to ∞. It therefore follows that $f_\lambda(x)$ cannot have a Siegel disk or Herman ring in Fatou set $F(f_\lambda)$. If a point z_0 lies on an attracting cycle or a parabolic cycle of a meromorphic transcendental function, the orbit of at least one of its singular points is attracted to the orbit of z_0 [5]. By Theorem 3.1, $f_\lambda(z)$ has only one attracting fixed point and two repelling fixed point on real axis, consequently it follows that U is not a parabolic domain. Thus, the only possible stable component U of $F(f_\lambda)$ is the basin of attraction $A(0)$ of the real attracting fixed point 0 so that the Fatou set $F(f_\lambda) = A(0)$ or the Julia set $J(f_\lambda) = \hat{\mathbb{C}} \setminus A(0)$. This proves (i).

By considering the fixed point a_λ instead of 0 and using Case III instead of Case I, the proof of (ii) follows on the lines of proof similar to that of (i) above. □

Remark 4.1. (i) *The basins of attraction* $A(0)$ *and* $A(a_\lambda)$ *are completely invariant sets, since* $F(f_\lambda)$ *is completely invariant* [4] *and by Theorem* 4.1, $F(f_\lambda) = A(0)$ *or* $F(f_\lambda) = A(a_\lambda)$.

(ii) *By Case* I, *for* $0 < \lambda < \lambda^*$, $f_\lambda^n(x) \to 0$ *as* $n \to \infty$ *for* $x \in (-\infty, x^*) \cup (r_{1,\lambda}, r_{2,\lambda})$. *Therefore, it follows that the set* $(-\infty, x^*) \cup (r_{1,\lambda}, r_{2,\lambda})$ *is contained in the basin of attraction* $A(0)$ *for* $0 < \lambda < \lambda^*$. *Similarly, by Case* III, *for* $\lambda^* < \lambda < \lambda^{**}$, $f_\lambda^n(x) \to a_\lambda$ *as* $n \to \infty$ *for* $r_\lambda < x < 0$, *giving that the interval* $(r_\lambda, 0)$ *is contained in the basin of attraction* $A(a_\lambda)$ *for* $\lambda^* < \lambda < \lambda^{**}$.

The dynamics of the function $f_\lambda(z)$ for $\lambda = \lambda^*$ and $\lambda = \lambda^{**}$ found in the following theorem shows that the Fatou set of $f_\lambda(z)$ for these parameter values contains a parabolic domain.

Theorem 4.2. *Let* $f_\lambda \in \mathcal{K}$ *and* $\lambda = \lambda^*$ *or* $\lambda = \lambda^*$. *Then,* $F(f_\lambda)$ *contains a parabolic domain.*

Proof. Let $U = \{z \in \mathbb{C}\colon f_\lambda^n(z) \to 0 \text{ as } n \to \infty\}$. By Theorem 3.1(ii), $f_\lambda(z)$ has a rationally indifferent fixed point at $x = 0$. Since, by Case II, $f_\lambda^n(x) \to 0$ as $n \to \infty$ for $x \in (x', 0)$ and the points in $(0, \infty)$ tend to ∞ under iteration of $f_\lambda(z)$, the rationally indifferent fixed point 0 lies on the boundary of U. Thus, U is a parabolic domain in the Fatou set of $f_\lambda(z)$ for $\lambda = \lambda^*$. For $\lambda = \lambda^{**}$, the proof follows on the lines of the above arguments and is hence omitted. □

Remark 4.2. *By Case* II, *for $\lambda = \lambda^*$, $f_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, \alpha) \cup (0, \infty) \cup I$ and $f_\lambda^n(x) \to 0$ as $n \to \infty$ for $x \in (x', 0)$. Therefore, for $\lambda = \lambda^*$, the rationally indifferent fixed point 0 belongs to the Julia set of $f_\lambda(z)$ and the interval $(x', 0)$ is contained in the parabolic domain U. Similarly, by Case* IV, *for $\lambda = \lambda^{**}$, $f_\lambda^n(x) \to \infty$ for $x \in (-\infty, \alpha) \cup (0, \infty) \cup M$ and $f_\lambda^n(x) \to \tilde{x}$ for $x \in (\tilde{x}, 0)$. Thus, the rationally indifferent fixed point $\tilde{x}$ belongs to Julia set of $f_\lambda(z)$ and the interval $(\tilde{x}, 0)$ is contained in the parabolic domain U.*

The following theorem describes the dynamics of $f_\lambda(z)$ for $z \in \hat{\mathbb{C}}$ and $\lambda > \lambda^{**}$ showing that, in this case, the Julia set $J(f_\lambda)$ explodes to the whole complex plane.

Theorem 4.3. *Let $f_\lambda \in \mathcal{K}$ and $\lambda > \lambda^{**}$. Then, Julia set $J(f_\lambda) = \hat{\mathbb{C}}$.*

Proof. By Case V, $f_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R} \setminus \{T_\alpha \cup \{0\}\}$, it follows that the orbits of all critical values of $f_\lambda(z)$ tend to ∞. Moreover, the only asymptotic value 0 of the function $f_\lambda(z)$ is preperiodic, since it is a fixed point of the function $f_\lambda(z)$. Since all other conditions of Theorem 1.2 are satisfied, it follows that the Julia set $J(f_\lambda) = \hat{\mathbb{C}}$. □

5. Applications and comparisons

The characterizations of the Julia set $J(f_\lambda)$ in Theorems 4.1 and 4.3 can be used to develop the following useful algorithm for computationally generating the images of the Julia set $J(f_\lambda)$: (i) Select a window W in the plane and divide W into $k \times k$ grids of width d. (ii) For the midpoint of each grid (pixel), compute the orbit upto a maximum of N iterations, (iii) If, at $i < N$, the modulus of the orbit is greater than some given bound M, the original pixel is colored black and the iterations are stopped. (iv) If no pixel in the modulus of the orbit ever becomes greater than M, the original pixel is left as white.

In the output generated by the above algorithm, the black points represent the Julia set of $f_\lambda(z)$ and the white points represent the Fatou set of $f_\lambda(z)$. The Julia sets for sample functions $f_\lambda(z) = \lambda \frac{ze^z}{z+4} \in \mathcal{K}$, $\lambda = 3.9$, $\lambda = 4.1$, $\lambda = 20$ and $\lambda = 20.1$ are generated in the rectangular domain $R = \{z \in \mathbb{C}\colon -8.5 \leqslant \Re(z) \leqslant 3.5 \text{ and } -2.5 \leqslant \Im(z) \leqslant 21.5\}$. To generate these images, for each grid point in the rectangle R the maximum allowed iterations are taken as $N = 200$ for a possible escape of the bound $M = 40$.

The generated Julia set of the function $f_\lambda(z)$ for $\lambda = 3.9$ is given in Fig. 4(a). It is found that Julia sets of $f_\lambda(z)$ for all λ satisfying $0 < \lambda < \lambda^* = 4$ have the same pattern as that of Julia set of $f_\lambda(z)$ for $\lambda = 3.9$. This conforms to the result of Theorem 4.1(i). The Julia sets of the function $f_\lambda(z)$ for $\lambda = 4.1$ and $\lambda = 20$ are given in Figs. 4(b) and 4(c). It is seen that the nature of images of the Julia sets of $f_\lambda(z)$ for all λ satisfying $4 = \lambda^* < \lambda < \lambda^{**} = e^3$ also remain the same as those of the Julia sets of $f_\lambda(z)$ for $\lambda = 4.1$ and $\lambda = 20$. This verifies to the result of Theorem 4.1(ii). The nature of image of the Julia set of $f_\lambda(z)$ for $\lambda = 20.1 > e^3 \approx 20.0865$ (Fig. 4(d)) shows a sudden dramatic change from the Julia set of $f_\lambda(z)$ for $4 < \lambda < e^3$. This is due to a chaotic burst

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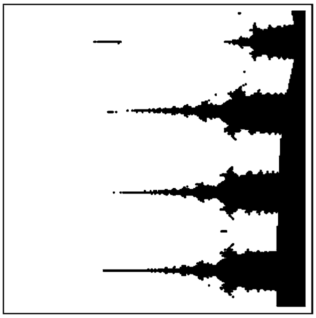
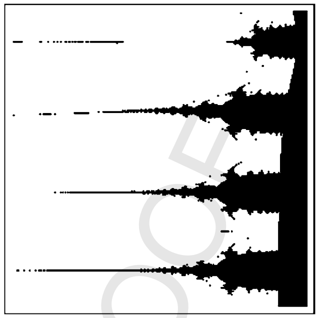
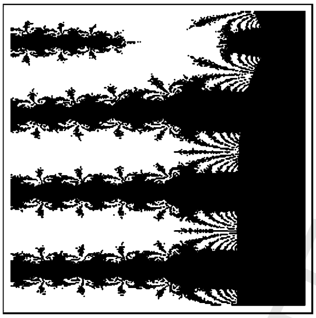

(a) $\lambda = 3.9 < \lambda^*$


(b) $\lambda = 4.1 > \lambda^*$


(c) $\lambda = 20.0 < \lambda^{**}$

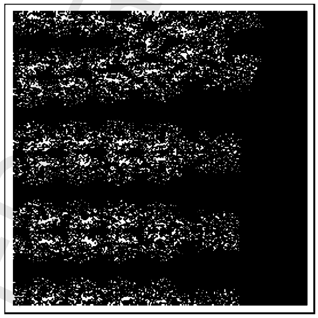

(d) $\lambda = 20.1 > \lambda^{**}$

Fig. 4. Julia sets of the function $f_\lambda(z) = \lambda \frac{ze^z}{(z+4)}$ for different values of parameter λ.

in the Julia set of $f_\lambda(z)$ as λ crosses the parameter value e^3 so that the resulting image contains significant large number of black points. This is a visualization of Theorem 4.3.

We note that explosion in the Julia set of the function $f_\lambda(z)$ does not occur at the parameter value $\lambda^* = 4$, the reason probably being just that the nature of the two fixed points are interchanged after crossing this parameter value while the nature of third fixed point remains the same.

Finally, a comparison between the dynamical properties of the functions investigated in the present paper and (i) functions considered in [9,15,18] for the dynamics of the function $T_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{C} \setminus \{0\}$ having polynomial Schwarzian derivative (ii) functions studied in [13] for the dynamics of the non-critically finite transcendental meromorphic function $E_\lambda(z) = \lambda \frac{e^z - 1}{z}$, $\lambda > 0$, is shown in Table 1.

It is observed in Table 1 that certain dynamical properties of the function $f_\lambda \in \mathcal{K}$ are different than those of the functions $T_\lambda(z) = \lambda \tan z$ and $E_\lambda(z) = \lambda \frac{e^z - 1}{z}$. For instance, the function $f_\lambda \in \mathcal{K}$ is critically finite and has critical values as well as asymptotic values while the function $T_\lambda(z)$ has only asymptotic values and the function $E_\lambda(z)$ is non-critically finite. Further, bifurcations occur at two parameter values for $f_\lambda \in \mathcal{K}$ on the real axis while bifurcation occurs at only one parameter value for $T_\lambda(z)$ and $E_\lambda(z)$ on the real axis. In spite of these differences, it is seen in Table 1 that Fatou and Julia sets of functions in our family $\mathcal{K}$ have similar characteristics as those of the functions $T_\lambda(z)$ and $E_\lambda(z)$. For all these functions, Table 1 shows that their Fatou sets equal the basins of attraction of their real attracting fixed points; Herman rings and wandering domains do not exist and their Julia sets are the closure of escaping points.

Table 1
Comparison of dynamical properties of $f_\lambda \in \mathcal{K}$, $T_\lambda(z) = \lambda \tan z$ and $E_\lambda(z) = \frac{\lambda(e^z-1)}{z}$

$f_\lambda \in \mathcal{K},\ \lambda > 0$	$T_\lambda(z) = \lambda \tan z,\ \lambda \in \hat{\mathbb{C}} \setminus \{0\}$	$E_\lambda(z) = \frac{\lambda(e^z-1)}{z},\ \lambda > 0$
$f_\lambda(z)$ has rational Schwarzian derivative	$T_\lambda(z)$ has polynomial Schwarzian derivative	$E_\lambda(z)$ has transcendental meromorphic Schwarzian derivative
$f_\lambda(z)$ is neither even nor odd function	$T_\lambda(z)$ is an odd function	$E_\lambda(z)$ is neither even nor odd function
$f_\lambda(z)$ has finitely many critical values	$T_\lambda(z)$ has no critical values	$E_\lambda(z)$ has infinitely many critical values
$f_\lambda(z)$ has one finite asymptotic value 0	$T_\lambda(z)$ has two finite asymptotic values $\pm\lambda i$	$E_\lambda(z)$ has one finite asymptotic value 0
All singular values of $f_\lambda(z)$ are bounded	All singular values of $T_\lambda(z)$ are bounded	All singular values of $E_\lambda(z)$ are bounded
The bifurcations occur in the dynamics of $f_\lambda(z)$ at two real critical parameter values λ^* and λ^{**}, defined by (5)	The bifurcation occurs in the dynamics of $T_\lambda(z)$ at only one real critical parameter value $\lambda_T = 1$	The bifurcation occurs in the dynamics of $E_\lambda(x)$ at only one real critical parameter value $\lambda_E \approx 0.6482$
For $0 < \lambda < \lambda^*$ and $\lambda^* < \lambda < \lambda^{**}$, the Fatou set $F(f_\lambda)$ equals the basin of attraction of the real attracting fixed point of $f_\lambda(z)$	For $\lvert\lambda\rvert < 1$, the Fatou set $F(T_\lambda)$ equals the basin of attraction of the real attracting fixed point 0 of $T_\lambda(z)$	For $0 < \lambda < \lambda_E$, the Fatou set $F(E_\lambda)$ equals the basin of attraction of the real attracting fixed point of $E_\lambda(z)$
Julia set $J(f_\lambda)$ contains some intervals of the real line for $0 < \lambda < \lambda^*$ and $\lambda^* < \lambda < \lambda^{**}$	Julia set $J(T_\lambda)$ contains the whole real line for $\lambda > 1$	Julia set $J(E_\lambda)$ contains the whole real line for $\lambda > \lambda_E$
Julia set $J(f_\lambda)$ is the whole complex plane for $\lambda > \lambda^{**}$	Julia set $J(T_\lambda)$ is the whole complex plane for $\lambda = i\pi$	Julia set $J(E_\lambda)$ is not the whole complex plane for any $\lambda > 0$
Herman rings, Baker domains and wandering domains do not exist	Herman rings, Baker domains and wandering domains do not exist	Herman rings and wandering domains do not exist for $0 < \lambda < \lambda_E$
Julia set $J(f_\lambda)$ is the closure of escaping points for all $\lambda > 0$	Julia set $J(T_\lambda)$ is the closure of escaping points for all $\lambda > 0$	Julia set $J(E_\lambda)$ is the closure of escaping points for all $\lambda > 0$

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