DYNAMICS OF CERTAIN ENTIRE FUNCTIONS ARISING FROM SEPARATELY CONVERGENT CONTINUED FRACTIONS

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Motivation

The chaotic dynamics and fractals have become quite popular in recent years due to its wide ranging application in engineering problems. The *complex analytic dynamics* is an intricate and fascinating area of dynamical systems in which *deterministic fractals* appear often as a chaotic sets. During the last decade there has been a renewed interest in the dynamics of analytic functions due to the beautiful computer graphics related to it.

The central objects studied in complex analytic dynamics of a function are its Julia set and Fatou set. There are two basic approaches in the study of dynamics of a function.

The first one is to investigate the iterative behaviour of an individual function, while the second one is the study of the iterative behaviour changes due to slight perturbations in the function. In the latter approach, which has received considerable attention during recent years, the simplest (but sufficiently intricate) case being that of a family of functions that depends on one parameter.

The dynamics of an entire transcendental function is much more interesting than the dynamics of a polynomial, since in the case of an entire transcendental function substantial hyperbolicity occurs in dynamics. The Julia set of a polynomial is always bounded. But, it is obvious from Picard's theorem that the Julia set of an entire transcendental function is unbounded, so that its Fatou set no longer forms a neighborhood of ∞. In the dynamics of polynomials, the basin of attraction of any finite attracting periodic point is bounded. But, in the dynamics of entire transcendental functions, the basins of attraction of finite attracting periodic points may become unbounded. The Julia sets of certain entire transcendental functions are often Cantor bouquets giving beautiful examples of *fractals*. Devaney and Durkin [33] observed the burst nature in the Julia set of the exponential function. If $0 < \lambda < (1/e)$, the chaotic region for the function $\lambda \exp(z)$ is a nowhere dense set entirely contained in the right half plane, while if $\lambda > (1/e)$ the chaotic region is the entire complex plane. This phenomena is referred to as *explosion* (or *chaotic burst*) in the Julia sets of functions in one parameter family of functions $\mathcal{E} = \{\lambda \exp(z) : \lambda > 0\}$. Devaney [25, 28] observed similar explosion in the Julia sets of functions in the family $\mathcal{C} = \{i\,\lambda\, \cos z \,:\, \lambda > 0\}$. This kind of explosion in the chaotic set does not occur in the dynamics of a polynomial. Further, certain new types of stable domains like a wandering domain [8] and a domain at infinity [42] exist for transcendental entire functions but are not found for polynomials.

In the dynamics of entire functions, the dynamics of polynomials and dynamics of certain classes of transcendental entire functions are hitherto studied by taking advantage of the presence of finitely many critical values and asymptotic values of their functions. The dynamical behaviour of critically finite (*i.e.*, having only finitely many critical values and asymptotic values) entire transcendental functions share many of the properties of polynomials and rational functions; for instance, these functions do not have wandering domains. Exploiting the critical finiteness, Devaney and coworkers studied exhaustively the dynamics of some of the most interesting periodic entire transcendental functions like λe^z, $\lambda \sin z$ and $\lambda \cos z$. However, the dynamics of non-critically finite entire functions has not been explored so far, probably because of non-applicability of Sullivan's theorem (c.f. Theorem 1.1.6) to these functions. Also, the presence of infinitely many critical values and the behaviour of the orbits of critical values make it difficult to study the dynamics of non-critically finite entire functions. In the present work an effort is made in this direction.

The non-critically finite entire functions considered here for the study are obtained as numerators or denominators of certain separately convergent continued fractions and include entire functions like $(e^z - 1)/z$, $\sinh z/z$, the modified Bessel function $I_0(z)$ of order zero.

Organization

The present work is organized into six chapters; Chapter 1 being the Introduction. gives a brief review of the basic theory and results relevant to our study in the subsequent chapters.

Chapter 2

In Chapter 2, the growth of the entire functions arising as the numerator and the denominator of a separately convergent continued fraction is studied. Let $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{F_n z}{1 + G_n z} \right)$, $F_n \neq 0$, $n \geq 1$ be a general T-fraction satisfying the conditions $\sum_{n=1}^{\infty} |F_n| < \infty$ and $\sum_{n=1}^{\infty} |G_n| < \infty$. Let $A_n(z)$ and $B_n(z)$ denote respectively the numerator and the denominator of the nth approximant of the general T-fraction and $\lim_{n\to\infty} A_n(z) = A(z)$, $\lim_{n\to\infty} B_n(z) = B(z)$ uniformly on compact subsets of $\mathbb{C}$. In the present chapter, the growth of the entire functions $A(z)$ and $B(z)$ is studied by investigating the influence of the elements F_n and G_n of a general T-fraction on the order of $A(z)$ and $B(z)$. In Section 2.1, It is proved that if F_n and G_n tend to zero sufficiently rapidly then the entire functions $A(z)$ and $B(z)$ are of order zero. This result generalizes a result of Thron [98] obtained for a regular C-fraction $\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{F_n z}{1}\right)$. An attempt to find the influence of the elements of a general T-fraction on the generalized (α, α)- order of its numerator and denominator is made in Section 2.2. It is seen that our results do give significant information about the comparison of growth of $B(z)$ (or $A(z)$) even for regular C-fractions in the situations where the relevant result of Thron [98] does not give any nontrivial information. The results pertaining to generalized (α, α)- order of $A(z)$ and $B(z)$ are used in Chapter 3 to study the dynamics of certain entire transcendental functions of slow growth. The results concerning influence of the elements of a general T-fraction on the generalized (α, β)- order of $A(z)$ and $B(z)$ are found in Section 2.3 and one of our results in this section, giving a sufficient condition on the elements F_n, G_n that forces the numerator (*or* denominator) of a general T-fraction to have generalized (α, β)- order not less than a prespecified constant, generalizes a result of Maillet [70]. Finally, since even entire functions can not be obtained as a denominator (*or* numerator) of a separately convergent general T-fraction, a new type of continued fraction, called modified general T-fraction, is introduced in Section 2.4 and the growth of the numerator and the denominator of such a modified general T-fraction when they are slow growth, is studied.

Chapter 3

Chapter 3 is devoted to the study of the dynamics of slow growth entire functions arising as the numerator $A(z)$ and the denominator $B(z)$ of separately convergent general T-fractions having generalized (α, α)- order μ with $\alpha(x) \equiv \log x$ and $2 < \mu < 3$. For this purpose, one parameter families $\mathcal{A} \equiv \{A_\lambda(z) = \lambda A(z) : \lambda > 0\}$ and $\mathcal{B} \equiv \{B_\lambda(z) = \lambda B(z) : \lambda > 0\}$ are considered. In Section 3.2, the dynamics of $A_\lambda \in \mathcal{A}$, $\lambda > 0$, is studied. In particular, the nature of the fixed points of $A_\lambda(z)$ on the positive real line is investigated and the dynamics of $A_\lambda(x)$ for $x \geq 0$ is described. Further, in this section, the dynamics of $A_\lambda(z)$ for $z \in \mathbb{C}$ is described for the three different cases, viz, $0 < \lambda < \lambda_A^*$, $\lambda = \lambda_A^*$ and $\lambda > \lambda_A^*$ where $\lambda_A^* = \frac{1}{A'(0)}$. In all the three cases, we obtain computationally useful characterization of the Julia set of $A_\lambda(z)$ as the closure of the set of points with orbits escaping to infinity under iteration of A_λ. Such a characterization was hitherto known only for critically finite entire transcendental functions [37]. In Section 3.3, firstly, the nature of the fixed points of $B_\lambda(z)$ and the dynamics of $B_\lambda(z)$ on the positive real line are investigated. Next, a description of the basin of attraction (c.f. Theorem 1.1.7) of the real attracting fixed point a_λ of the entire function $B_\lambda(z)$ is found for $0 < \lambda < \lambda_B^* = \frac{1}{B'(x^*)}$; x^* being the unique positive real root of the equation $B(x) - xB'(x) = 0$. Similarly, a description of the parabolic domain (c.f. Theorem 1.1.7) corresponding to the rationally indifferent fixed point x^* of $B_\lambda(z)$ is found for $\lambda = \lambda_B^*$. Finally, in this section, the dynamics of $B_\lambda(z)$ for $z \in \mathbb{C}$ is described for all the three different cases, viz, $0 < \lambda < \lambda_B^*$, $\lambda = \lambda_B^*$ and $\lambda > \lambda_B^*$ and, analogous to that of $A_\lambda(z)$, a computationally useful characterization of the Julia set of $B_\lambda(z)$ is obtained. Finally, in Section 3.4, the characterizations of the Julia sets of $A_\lambda(z)$ and $B_\lambda(z)$, obtained in Sections 3.2 and 3.3, are applied to computationally generate the pictures of the Julia sets of $A_\lambda \in \mathcal{A}$ and $B_\lambda \in \mathcal{B}$ for different values of λ.

Chapter 4

Let $f(z) = (e^z - 1)/z$ be the non-critically finite entire function arising as the denominator of the separately convergent general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{z/(n+1)}{1 - (z/(n+1))} \right)$. In Chapter 4, the dynamics of the entire function $f_\lambda(z) = \lambda f(z)$, $\lambda > 0$ is studied. Let $\mathcal{K} \equiv \{f_\lambda(z) = \lambda f(z) : \lambda > 0\}$ be one parameter family of functions. In Section 4.2, some of the basic properties of the function $f \in \mathcal{K}$ are developed. Section 4.3 contains the study of the dynamics of $f_\lambda \in \mathcal{K}$ on the real line. In this section, it is shown that bifurcation in the dynamics of $f_\lambda(x)$ occurs at $\lambda = \lambda^* (\approx 0.64761)$ where $\lambda^* = (x^*)^2/(e^{x^*} - 1)$ and x^* is the unique positive real root of the equation $e^x(2-x) - 2 = 0$. That is, if the parameter value crosses the value λ^*, then a sudden dramatic change in the dynamics of $f_\lambda(x)$ occurs. In Section 4.4, the dynamics of $f_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < \lambda < \lambda^*$ is studied. For this case, we prove two different characterizations for the Julia set of $f_\lambda(z)$. The first characterization gives the Julia set $\mathcal{J}(f_\lambda)$ for $0 < \lambda < \lambda^*$ as the closure of the set of escaping points; while the second characterization, describes it as the complement of the basin of attraction of an attracting real fixed point of $f_\lambda(z)$. Further. in this section, it is found that, under a certain condition, the Julia set of $f_\lambda(z)$, $0 < \lambda < \lambda^*$, is a nowhere dense subset of the right half plane. In Section 4.5, the dynamical behaviour of $f_\lambda(z)$ for $\lambda > \lambda^*$ is described. We prove that the Julia set of $f_\lambda(z)$ for $\lambda > \lambda^*$ contains the entire real line. The characterization of the Julia set of $f_\lambda(z)$ as the closure of the set of escaping points, analogous to the first characterization in Section 4.4 is obtained in this case also. In Section 4.6, the characterizations of the Julia set of $f_\lambda(z)$, obtained in Sections 4.4 and 4.5, are applied to computationally generate the pictures of the Julia set of $f_\lambda(z)$ for different values of λ. Finally, the results of our investigations on the dynamics of the non-critically finite entire function $f_\lambda \in \mathcal{K}$ are compared with those of Devaney [26, 31], Devaney and Durkin [33], Devaney and Krych [36], Devaney and Tangerman [37] and Misiurewicz [75] obtained for the dynamics of the critically finite entire functions $E_\lambda(z) = \lambda\, e^z$.

Chapter 5

In Chapter 5, the dynamics of the entire function $h_\lambda(z) = \lambda h(z)$ where λ is a non-zero real parameter and $h(z) = \sinh z/z$ is an even non-critically finite entire function arising as a limit function of the sequence of denominators of the approximants of the modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathrm{K}}} \left(\frac{-z^2/((2n)(2n+1))}{1 + (z^2/(2n)(2n+1))} \right)$ is studied. Let $\mathcal{H} \equiv \{h_\lambda(z) = \lambda h(z) :$

$\lambda \in \mathbb{R} \setminus \{0\}\}$. Section 5.2 is devoted to the results on some of the basic properties of $h_\lambda \in \mathcal{H}$. In Section 5.3, the dynamics of $h_\lambda(x)$ for $x \in \mathbb{R}$ is described. In this section, it is shown that there exists a critical parameter value $\lambda^{**} > 0$ such that bifurcation in the dynamics of $f_\lambda(x)$, $x \in \mathbb{R}$ occurs at $|\lambda| = \lambda^{**}(\approx 1.104)$. The critical parameter λ^{**} is given by $\lambda^{**} = (x^{**})^2/\sinh x^{**}$ and x^{**} is the unique positive real root of the equation $\tanh x = x/2$. The dynamics of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $0 < |\lambda| < \lambda^{**}$ is studied in Section 5.4. For this case, two different characterizations for the Julia set of $h_\lambda(z)$ are obtained. The first characterization gives the Julia set $\mathcal{J}(h_\lambda)$ for $0 < |\lambda| < \lambda^{**}$ as the closure of the set of escaping points; while the second characterization, describes it as the complement of the basin of attraction of an attracting real fixed point of $h_\lambda(z)$. Further, in this section, it is found that, under a certain condition, the Julia set of $h_\lambda(z)$, $0 < |\lambda| < \lambda^{**}$ is a nowhere dense subset of the complex plane. In Section 5.5, the dynamical behaviour of $h_\lambda(z)$ for $z \in \mathbb{C}$ and $|\lambda| > \lambda^{**}$ is described. We prove that the Julia set of $h_\lambda(z)$ for $|\lambda| > \lambda^{**}$ contains all the real points and the purely imaginary points of the complex plane. The characterization of the Julia set of $h_\lambda(z)$ as the closure of the set of escaping points is obtained in this case. In Section 5.6, the characterizations of the Julia set obtained in Sections 5.4 and 5.5, are applied to computationally generate the pictures of the Julia set of $h_\lambda(z)$ for various values of λ. Further, the results obtained in this chapter for the dynamics of $h_\lambda \in \mathcal{H}$ are compared with those of Devaney and Durkin [33] obtained for the dynamics of critically finite even entire function $C_\lambda(z) = \lambda i \cos z$, $\lambda \in \mathbb{R} \setminus \{0\}$ and, finally, a comparison is made in this section between the results on the dynamics of $f_\lambda \in \mathcal{K}$ and $h_\lambda \in \mathcal{H}$, $\lambda > 0$, as found in Chapter 4 and in the present chapter.

Chapter 6

In Chapter 6, a class of non-critically finite entire functions is introduced and it is proved that explosion occurs in the Julia sets of functions in one parameter family generated from each function in this class. Let $\mathcal{F}$ be the class of functions $f(z)$ satisfying (i) $f(z)$ is an entire function having order ρ with $(1/2) \leq \rho < 1$, (ii) $f(z)$ has only negative real zeros in the complex plane, (iii) $|f(-x)| \leq f(0) = 1$, for all $x > 0$ and (iv) $\lim_{x\to\infty} f(-x) = 0$; and $\mathcal{G}$ be the class of functions defined by $\mathcal{G} = \{g(z) = f(z^2) : f \in \mathcal{F}\}$. In the present chapter, the dynamics of $g_\lambda(z) = \lambda g(z)$, $\lambda \in \mathbb{R} \setminus \{0\}$, for a function $g \in \mathcal{G}$, is studied. Let $\mathcal{S} \equiv \{g_\lambda(z) \; : \; \lambda \in \mathbb{R} \setminus \{0\}\}$. Section 6.2 describes the bifurcation in the dynamics of functions $g_\lambda \in \mathcal{S}$ for $x \in \mathbb{R}$. It is shown that there exists a critcal parameter $\lambda_g^* > 0$ such that bifurcation in the dynamics of functions in $\mathcal{S}$ for $x \in \mathbb{R}$ occurs at $|\lambda| = \lambda_g^*$. In Section 6.3, the dynamics of $g_\lambda \in \mathcal{S}$ for $z \in \mathbb{C}$ is described and the chaotic burst in the Julia sets of functions in the family $\mathcal{S}$ is exhibited. It is shown that the Fatou set of $g_\lambda(z)$ is an unbounded proper subset of the complex plane when $0 < |\lambda| \leq \lambda_g^*$ and consequently Julia set of $g_\lambda(z)$ is also unbounded proper subset of the complex plane for this case, while the Julia set of $g_\lambda(z)$ is the extended complex plane when $|\lambda| > \lambda_g^*$. Finally, certain interesting examples of the family $\mathcal{S}$, viz, (i) $\mathcal{I} \equiv \{\lambda I_0(z) \; : \; \lambda \in \mathbb{R} \setminus \{0\}\}$, where $I_0 \in \mathcal{G}$ is the well known modified Bessel function of zero order arising as the denominator of the separately convergent modified general T-fraction $\underset{n=1}{\overset{\infty}{\mathbf{K}}} \left(\frac{-z^2/(2n)^2}{1 + z^2/(2n)^2} \right)$ and (ii) $\mathcal{M}_k \equiv \{\lambda G_{2k}(z) \; : \; G_{2k}(z) = F_{2k}(iz)/F_{2k}(0), \; \lambda \in \mathbb{R} \setminus \{0\}\}$, where $G_{2k} \in \mathcal{G}$ with fixed $k = 1, 2, \ldots$ and $F_{2k}(z) = \int_0^\infty e^{-t^{2k}} \cos zt \; dt$, are given and the picture of the Julia set of functions in the family $\mathcal{I}$ is computationally generated for various values of λ.