Synopsis

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The problem of classical interpolation is to find a continuous or a differentiable function such that the graph of the function contains a given set of data points. In most of the cases, classical interpolations are about constructing a very smooth function passing through the given data. However, in several physical experiments like those in the study of cost lines, mountains, signal processing, image processing, bio-engineering, etc., the data arises from highly irregular curves and surfaces found in nature and may not be generated from a smooth function. For efficiently modeling of such a data, Barnsley [Fractal functions and interpolation, Constr. Approx., 2:303-329, 1986] used the notion of *Fractal Interpolation Functions* (**FIF**), based on the work of Hutchinson [Fractals and self-similarity, Indiana Univ. Math. J., 30:713-747, 1981] and Iterated Function System (IFS) theory. Mandelbrot introduced the term ***'Fractal'*** to describe objects that are too irregular to fit in to the theory of Euclidean geometry. The objects found in nature such as ferns, coastlines, clouds, etc. have fractal structure and many mathematical sets like Cantor set, Sirpinski gasket, Peano curve, Koch snowflake, etc., are now recognized as fractals. These are better represented by the *'fractal geometry'*, a notion that was first formalized by Mandelbrot [The Fractal Geometry of Nature, W.H.Fremman and Co., New York, 1982]. The advent of Fractals represents a rebirth of experimental mathematics, nurtured by computers and enhanced by powerful evidence of its applications. Fractal Interpolation Functions, in general, represent objects found in nature that show some kind of self-similarity on magnification and the Fractal dimensions of their graph are non-integers. To approximate non-self-similar objects found in nature, the hidden variable FIF are constructed by Barnsley, et. al. [Hidden variable fractal interpolation functions, SIAM. J. Math. Anal., 20(5):1218-1242, 1989], employing projection of vector valued FIF for generalized interpolation data. In some of the practical applications of FIF, the interpolation data might be generated simultaneously from self-similar and non-self-similar functions. To study such data, the notion of Coalescence Fractal Interpolation Function (CFIF) is introduced [Stability of affine coalescence hidden variable fractal interpolation functions, Nonlinear Analysis, 68:3757-3770, 2008].

Fractal Interpolation Functions are continuous but are not generally differentiable. However, Barnsley and Harrington [The calculus of fractal interpolation functions, J. Approx. Theory, 57:14-34, 1989] proved the existence of differentiable FIF and developed a method for construction of C^p-FIF. Navascúes and Sebastían [Generalization of Hermite functions by fractal interpolation, J. Approx. Theory, 131:19-29, 2004] constructed a C^p-FIF f_α, called Hermite FIF, for data $\{(x_n, y_{n,k}) \in \mathbb{R}^2;\ n = 0, 1, \ldots, N$ and $k = 0, 1, \ldots, p\}$ such that $f_\alpha^{(k)}(x_n) = y_{n,k}$. The Hermite FIF interpolates only self-similar data. However, in several practical applications, the data is available in terms of functional values at all the nodal points and the values of certain order derivatives only at the end points of the interval. For the interpolation of such a data, the concept of Spline FIF and Spline CFIF are introduced [Generalized cubic Spline fractal interpolation functions, SIAM J. Numer. Anal., 44(2):655-676, 2006].

In several practical applications of FIF, the Hermite FIF is not suitable for efficient simulation of a data which is in part self-similar and in part non-self-similar. Thus, there is a need to construct a C^p-FIF which efficiently simulates such a data. This is achieved in the present work by the construction of Hermite CFIF. For Hermite FIF and Hermite CFIF, the data needs to consist of the values of its generating function and the values of its derivatives up to order p. In certain cases, the given data, at nodal points, consists of values of its generating function and the values of its even order derivatives up to $2p$. To achieve interpolation of such a data, Lidstone FIF and Lidstone CFIF are introduced in the present work. Further, in several practical applications, the data is available in terms of functional values at all the nodal points and the values of even order derivatives only at the end points of the interval. For the interpolation of such a data, the notions of Lidstone Spline FIF and Lidstone Spline CFIF are introduced in our work and their existence are proved. Fractal surfaces frequently arise in nature and several areas of science and engineering problems. Several interpolation methods for the construction of fractal surfaces are recently developed. However, the Lidstone Spline interpolation with certain LIdstone boundary conditions of such fractal surfaces is not considered so far. In the present work, this need is fulfilled by constructing a Cubic Lidstone Spline Fractal Surface through the introduction of the space of Natural Cubic Lidstone Spline FIF.

The organization of the thesis is as follows:

Chapter 1, being the introduction, consists of a brief review of history, concepts, basic definitions that are needed in the context of subsequent chapters.

Chapter 2 starts with the definition, study of existence and a method of construction of Hermite CFIF for an interpolation data in $\mathbb{R}^2$. For this purpose, in Section 2.2, an IFS is constructed by considering a set of generalized data in $\mathbb{R}^3$. The projection of the attractor of the IFS on $\mathbb{R}^2$ gives a C^p-Hermite CFIF, which is self-affine or non-self-affine depending upon the generalized data. The convergence of Hermite CFIF to data generating function and to Classical Hermite interpolation are studied in Section 2.3. In Section 2.4, for a given data set, a bound on L^∞-norm $\|\Phi^{(k)} - \hbar_{1\omega}^{(k)}\|_\infty$, for $k = 1, 2, \ldots, p$, is obtained for derivatives $\Phi^{(k)}$ of the classical Hermite interpolation function Φ and derivatives $\hbar_{1\omega}^{(k)}$ of Hermite CFIF $\hbar_{1\omega}$. Using this result, bounds on L^∞-error, in approximation of derivatives of data generating function by derivatives of Hermite CFIF, are found. It follows from our results that these errors are of the order $\|\Delta\|^{p-k}$, for a partition Δ of the interval corresponding to the data set. For a data arising from the Bessel function of order zero, the results found in the present chapter are used in modeling a Hermite CFIF in Section 2.5.

The notion of Lidstone FIF is introduced in Chapter 3. The definition, investigation of existence and a method of construction of Lidstone FIF for an interpolation data in $\mathbb{R}^2$ are given in Section 3.2,. The convergence of Lidstone FIF to the data generating function and to the classical Lidstone interpolating polynomial are investigated in Section 3.3. Further, in this section, a bound on L^∞-norm $\|\phi^{(2k)} - \ell_\alpha^{(2k)}\|_\infty$, $k = 1, 2, \ldots, p$, is obtained for derivatives $\phi^{(2k)}$ of the classical Lidstone interpolation function ϕ and derivatives $\ell_\alpha^{(2k)}$ of Lidstone FIF ℓ_α. Using these results, bounds on L^∞-error, in approximation of derivatives of data generating function by derivatives of Lidstone FIF, are found. To approximate a data generating function, whose graph is in-part self similar and in-part non-self similar, by a C^{2p}-fractal function, the notion of *Lidstone Coalescence Fractal Interpolation Function* (***Lidstone CFIF***) is introduced in Section 3.4. In Section 3.5, the convergence of Lidstone CFIF to the data generating function along with its derivatives is discussed. Finally, certain computational models of Lidstone FIF and Lidstone CFIF for a data are generated in Section 3.6.

In Chapter 4, Lidstone Spline Fractal Interpolation is introduced. The definition, existence and construction of Cubic Lidstone Spline FIF, for an interpolation data in $\mathbb{R}^2$, are given in Section 4.2. The convergence of Cubic Lidstone Spline FIF to the data generating function and to the classical Lidstone Spline interpolating polynomial in L^∞-norm are studied in Section 4.3. Section 4.4 deals with the definition, existence and construction of Quintic Lidstone Spline FIF for an interpolation data in $\mathbb{R}^2$. The convergence of Quintic Lidstone Spline FIF to the data generating function and to the classical Lidstone Spline interpolating polynomial in L^∞-norm are investigated in Section 4.5. The results found in this chapter are computationally modeled for construction of Cubic Lidstone Spline FIF and Quintic Lidstone Spline FIF for a data, in Section 4.6.

By developing the theory of Lidstone Spline Fractal Surface, the reconstruction of fractal surfaces found in nature is studied in Chapter 5 . In Section 5.2, the notion of natural cubic Lidstone Spline FIF is introduced and it is shown that the set of all natural cubic Lidstone Spline FIF is a linear space. A basis for this space is constructed in this section. Using this basis, Cubic Lidstone Spline Fractal Interpolation Surface for a given surface data is constructed in Section 5.3. An estimate of error in approximation of a data generating function by a bivariate Cubic Lidstone Spline Fractal Interpolate is found in Section 5.3. Finally, the results of this chapter are computationally modeled through an example in Section 5.5.

Chapter 6 introduces a new notion of Lidstone Spline CFIF. The definition, study of existence and a construction method of Cubic Lidstone Spline CFIF, for an interpolation data in $\mathbb{R}^2$ are given in Section 6.2. The convergence of Cubic Lidstone Spline CFIF to the data generating function and to the classical Lidstone Spline interpolating polynomial in L^∞-norm are investigated in Section 6.3. The definition, existence and construction of Quintic Lidstone Spline CFIF for an interpolation data in $\mathbb{R}^2$ are given in Section 6.4. The convergence of Quintic Lidstone Spline FIF to the data generating function and to the classical Lidstone Spline interpolating polynomial in L^∞-norm are studied in Section 6.5. Finally, the constructions of Cubic Lidstone Spline CFIF and Quintic Lidstone Spline CFIF are computationally modeled for a generalized data and various boundaries conditions, in Section 6.6.