SYNOPSIS

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The origin of complex dynamics started from the Cayley's problem on Newton's Method of root finder. In 1879, Arther Cayley suggested the extension of what he called the Newton-Fourier Method (used for locating real zeros of functions) to find the complex roots of a polynomial $p(z)$ using the iterative process $z_{k+1} = z_k - \frac{p(z_k)}{p'(z_k)}$. Further, he proposed that one should study the problem globally, i.e., regions of the plane should be determined, such that orbits of an initial point, taken at pleasure anywhere within one of these regions, ultimately arrive at a root of the polynomial $p(z)$. The *Complex dynamics* is an intricate and fascinating area of dynamical systems in which *deterministic fractals* appear often as chaotic sets. During the early twentieth century, the study of chaotic dynamics started implicitly in the works of P. Fatou and G. Julia. There had been a long period of inactivity but during the end of 20th century there has been a renewed interest in the study of dynamics of complex functions due to the beautiful computer graphics associated with it. It has become quite popular area of research in recent years due to its wide ranging applications in engineering problems.

In complex dynamics, most of the work has centered around the dynamics of rational functions that originated in the pioneering work of P. Fatou and G. Julia. The initiative to study the iterations of meromorphic functions has been taken up mainly by I.N. Baker, W. Bergweiler, R.L. Devaney and L. Keen. There are two basic approaches in the study of dynamics of a function. The first one is to investigate the iterative behaviour of an individual function, while the second one is to study the iterative behavioural changes due to slight perturbations in the function. In the latter approach, which has received considerable attention during recent years, the simplest case is that of a family of functions that depends on one parameter. We consider, in the present work, both of these approaches to study the dynamics of certain families of transcendental meromorphic functions that depend on one parameter.

One of the principal differences arising in the iterations of transcendental meromorphic functions in comparison to the iterations of entire functions is the fact that the iterations of meromorphic maps do not lead to a dynamical system. The point at infinity is an essential singularity for such a map, so the map can not be extended continuously to infinity. Hence the forward orbit of a pole terminates. All other points have well defined forward orbits. Despite the above fact, the investigation of iterations of transcendental meromorphic functions is important in several situations. For example, the iterative processes associated with Newton's method applied to entire functions often yields a meromorphic function as the root finder. The central objects studied in complex dynamics of a function are its Fatou set and Julia set. The *Fatou set* (or *stable set*) of a function f, denoted by $F(f)$, is defined to be the set of all complex numbers where the family of iterates $\{f^n\}$ of f forms a normal family in the sense of Montel. The *Julia set* (or *chaotic set*), denoted by $J(f)$, is the complement of the Julia set of f. The Julia sets for even simple quadratic polynomials are often *fractals*.

In the present thesis, we consider the iterations and study the resulting chaotic dynamics of certain classes of critically and non-critically finite transcendental meromorphic functions. It is well known that there are rational and transcendental entire functions for which the Julia set is the whole complex plane. It is interesting to examine whether this is also true for certain families of transcendental meromorphic functions. This and related properties are discussed for our classes of certain one parameter families of transcendental meromorphic functions. In each case, we also provide algorithms to computer generate the images of the Julia sets for functions in the families considered in various chapters.

The thesis consists of six chapters. Chapter 1, being the introduction, consists of a brief review of the basic theory and results concerning the dynamics, growth and Schwarzian Derivative of meromorphic transcendental functions that are required in the subsequent chapters in the present study.

In Chapter 2, a class of critically finite transcendental meromorphic functions is introduced and it is proved that explosion occurs in the Julia sets of each one parameter family of functions in this class. For this purpose, a class $\mathcal{S}$ of functions f is considered such that (i) f is a transcendental meromorphic function of finite order, (ii) all poles of f are of odd multiplicity, (iii) all zeros of f' are of even multiplicity and (iv) Schwarzian Derivative of f is a rational function and Let, $\mathcal{F} \subset \mathcal{S}$ be the class of functions $f(z)$ defined by (i) $g(z) = \frac{(z-\alpha)f(z)}{z}$, $\alpha < -1$, is a nonvanishing transcendental entire function having atmost one finite asymptotic value (ii) $z(z-\alpha)g'(z) - \alpha g(z) = 0$ has only real roots (iii) $g(x)$ is positive and strictly increasing in $\mathbb{R}$, $g(x) \to 0$ as $x \to -\infty$ and $g(x) \to \infty$ as $\to \infty$ and (iv) $g'(x)$ is strictly increasing in $\mathbb{R}$, $g'(x) \to \infty$ as $x \to \infty$ and $(x-\alpha)\frac{g'(x)}{g(x)} > 1$ for all $x > 0$. In the present chapter, the dynamics of the functions $f_\lambda(z) = \lambda f(z)$ for $z \in \hat{\mathbb{C}}$ (extended complex plane) is studied, where $f \in \mathcal{F}$ and $\lambda > 0$. The bifurcation in the dynamics of $f_\lambda(z)$ is described in detail and the chaotic burst in the Julia sets of functions in the family $\mathcal{K} = \{f_\lambda(z) : \lambda > 0\}$ is shown to occur at parameter values $\lambda = \phi(0)$ and $\lambda = \phi(\tilde{x})$, where, $\phi(x) = \frac{x}{f(x)}$ and $\tilde{x}$ is the real root of $\phi'(x) = 0$. The characterizations of the Julia set of $f_\lambda(z)$, obtained in the present chapter, are applied to computationally generate the images of the Julia sets of sample functions $f_\lambda(z) = \lambda \frac{z}{z+4} e^z \in \mathcal{K}$ for different values of the parameter λ. Our results found here are compared with those of Devaney and Keen [*Ann. Sci. Ec. Norm. Sup.*, **22 (4)** *1989*, *55-79*], Keen and Kotus [*Conformal Geometry and Dynamics: An Elect. J. AMS*, **vol. 1**, *Aug. 1997*, *28-57*] and Stallard [*J. London Math. Soc.* **49(2)** *1994*, *281-295*], obtained for the function $\lambda \tan z$ that has polynomial Schwarzian Derivative.

In Chapter 3, the dynamics of one parameter family of functions $h_\lambda(z) = \lambda h(z)$ for $z \in \hat{\mathbb{C}}$, $\lambda > 0$, where, $h(z) = \frac{z+\mu_0}{z+\mu_0+4} e^z$, $\mu_0 > 0$, is studied. It is shown that $h(z)$ is a critically finite transcendental meromorphic function in class $\mathcal{S}$. The bifurcations in the dynamics of $h_\lambda(z)$ are found to occur at the parameter values $\lambda = \lambda^*_{\mathcal{L}}$, $\lambda = \lambda^{**}_{\mathcal{L}}$ and $\lambda = \lambda^{***}_{\mathcal{L}}$ ($\lambda^*_{\mathcal{L}} < \lambda^{**}_{\mathcal{L}} < \lambda^{***}_{\mathcal{L}}$), where, $\lambda^*_{\mathcal{L}} = \tilde{x}_3(\tilde{x}_3 + \mu_0 + 4)\, e^{-\tilde{x}_3}/(\tilde{x}_3 + \mu_0)$, $\lambda^{**}_{\mathcal{L}} = \tilde{x}_2(\tilde{x}_2 + \mu_0 + 4)\, e^{-\tilde{x}_2}/(\tilde{x}_2 + \mu_0)$, $\lambda^{***}_{\mathcal{L}} = \tilde{x}_1(\tilde{x}_1 + \mu_0 + 4)\, e^{-\tilde{x}_1}/(\tilde{x}_1 + \mu_0)$; $\tilde{x}_1$, $\tilde{x}_2$ and $\tilde{x}_3$ ($\tilde{x}_1 < \tilde{x}_2 < 0 < \tilde{x}_3$) being the real roots of the equation $x^3 + (2\mu_0 + 3)x^2 + \mu_0(\mu_0 + 2)x - \mu_0(\mu_0 + 4) = 0$. If the parameter value λ crosses any one of the value $\lambda^*_{\mathcal{L}}, \lambda^{**}_{\mathcal{L}}$ or $\lambda^{***}_{\mathcal{L}}$, a sudden dramatic change in the dynamics of $h_\lambda(z)$ is shown to occur leading to chaotic bursts in the Julia sets of functions in the family $\mathcal{L} = \{h_\lambda(z) \ : \ \lambda > 0\}$. The characterizations of the Julia set of $h_\lambda(z)$, obtained in this chapter, are applied to computationally generate the images of the Julia sets of sample functions $h_\lambda(z) = \lambda \frac{(z+1)}{z+5} e^z \in \mathcal{L}$, for different values of parameter λ. Finally, our results found in this chapter are compared with the results obtained in Chapter 2.

In Chapter 4, the dynamics of one parameter family of functions $\zeta_\lambda(z) = \lambda \frac{z}{z+1} e^{-z}$ for

$z \in \hat{\mathbb{C}}$, $\lambda > 0$, is studied. All the functions in this family are shown to be critically finite transcendental meromorphic functions. It is found that bifurcations in the dynamics of the functions $\zeta_\lambda(z)$ occur at $\lambda = 1$ and $\lambda = (\sqrt{2}+1)e^{\sqrt{2}}$. The characterizations of the Julia set of $\zeta_\lambda(z)$, for different ranges of parameter value λ, are applied to computationally generate the images of the Julia sets of functions $\zeta_\lambda(z) = \lambda \frac{z}{z+1} e^{-z} \in \mathcal{M}$. The results found in the present chapter are compared with the results obtained in Chapter 3.

In Chapter 5, the dynamics of non-critically finite odd transcendental meromorphic function $\psi_\lambda(z) = \lambda \frac{\sinh(z)}{z^2}$ for $z \in \hat{\mathbb{C}}, \lambda > 0$, is studied. It is shown that there exits a critical parameter value $\lambda^* > 0$ such that bifurcation in the dynamics of $\psi_\lambda(x)$, $x \in \mathbb{R}\backslash\{0\}$, occurs at $\lambda = \lambda^*$, where $\lambda^* = \frac{\tilde{x}^3}{\sinh(\tilde{x})}$ (≈ 2.69528) and $\tilde{x}$ is the unique positive real root of the equation $\tanh(x) = \frac{x}{3}$. The dynamics of the function $\psi_\lambda(z) = \lambda \frac{\sinh(z)}{z^2}$ for $0 < \lambda < \lambda^*$, $\lambda = \lambda^*$ and $\lambda > \lambda^*$, are separately studied. These results are then used to generate computer images of Julia sets of $\xi_\lambda(z)$ applying an algorithm developed by using the results obtained in this chapter. Finally, our results found here are compared with the recent results on the dynamics of non-critically finite entire function $E_\lambda(z) = \lambda(e^z - 1)/z$, $\lambda > 0$ due to Kapoor and Prasad [*Ergodic Theory and Dynamical Systems*, **18 (6)** *1998, 1363–1383*].

In Chapter 6, the dynamics of non-critically finite even transcendental meromorphic functions $\xi_\lambda(z) = \lambda \frac{\sinh^2(z)}{z^4}$ for $z \in \hat{\mathbb{C}}, \lambda > 0$, is studied. It is shown that there exit critical parameter values $\lambda_1, \lambda_2 > 0$ such that bifurcations in the dynamics of $\xi_\lambda(x)$, $x \in \mathbb{R}\backslash\{0\}$ occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where $\lambda_1 = \frac{\bar{x}^5}{\sinh^2(\bar{x})}$ (≈ 1.26333), $\lambda_2 = \frac{\tilde{x}^5}{\sinh^2(\tilde{x})}$ (≈ 2.66915); $\bar{x}$, $\tilde{x}$ being the unique positive real roots of the equations $\tanh(x) = \frac{2x}{3}$ and $\tanh(x) = \frac{2x}{5}$ respectively. If the parameter value crosses the value λ_1 or λ_2, then a dramatic change in the dynamics of $\xi_\lambda(x)$ is found to occur. The dynamics of the function $\xi_\lambda(z) = \lambda \frac{\sinh^2(z)}{z^4}$ for $0 < \lambda < \lambda_1$, $\lambda = \lambda_1$, $\lambda_1 < \lambda < \lambda_2$, $\lambda = \lambda_2$ and $\lambda > \lambda_2$, are studied separately. The results found in this chapter are used to generate computer images of Julia sets of $\xi_\lambda(z)$ using the same algorithm as in Chapter 5. Our results found in the present chapter are compared with (i) the results on dynamics of the function $\lambda \tan z$ having polynomial Schwarzian Derivative due to Devaney and Keen [*Ann. Sci. Ec. Norm. Sup.,* **22 (4)** *1989, 55-79*], Keen and Kotus [*Conformal Geometry and Dynamics: An Elect. J. AMS,* **vol. 1**, *Aug. 1997, 28-57*] and Stallard [*J. London Math. Soc.* **49(2)** *1994, 281-295*] and (ii) the results obtained in Chapter 5.