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DYNAMICS OF A FAMILY OF NON-CRITICALLY FINITE EVEN TRANSCENDENTAL MEROMORPHIC FUNCTIONS

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The dynamics of one parameter family of non-critically finite even transcendental meromorphic function $\xi_\lambda(z) = \lambda \dfrac{\sinh^2 z}{z^4}$, $\lambda > 0$ is investigated in the present paper. It is shown that bifurcations in the dynamics of the function $\xi_\lambda(x)$ for $x \in \mathbb{R} \setminus \{0\}$ occur at two critical parameter values $\lambda = \dfrac{x_1^5}{\sinh^2 x_1}$ (≈ 1.26333) and $\lambda = \dfrac{\tilde{x}^5}{\sinh^2 \tilde{x}}$ ($\approx 2.7.715$), where x_1 and $\tilde{x}$ are the unique positive real roots of the equations $\tanh x = \dfrac{2x}{3}$ and $\tanh x = \dfrac{2x}{5}$ respectively. For certain ranges of parameter values of λ, it is proved that the Julia set of the function $\xi_\lambda(z)$ contains both real and imaginary axes. The images of the Julia sets of $\xi_\lambda(z)$ are computer generated by using the characterization of the Julia set of $\xi_\lambda(z)$ as the closure of the set of points whose orbits escape to infinity under iterations. Finally, our results are compared with the recent results on dynamics of (i) critically finite transcendental meromorphic functions $\lambda \tan z$ having polynomial Schwarzian Derivative [10, 15, 19] and (ii) non-critically finite transcendental entire functions $\lambda \dfrac{e^z - 1}{z}$ [14].

1. Introduction

In early twentieth century, the iteration theory of complex functions originated in works of Julia and Fatou. There had been a long period of inactivity after that. During the end of 20th century, a renewed interest in the study of iteration theory started due to beautiful computer graphics and wide ranging applications in engineering and science [5, 8, 16, 17, 18, 20] associated with it. In iteration theory, complex dynamics has so far been extensively studied for rational and entire functions. However, in comparison to the investigations on dynamics of rational and entire functions not much work has been done in this direction for transcendental meromorphic functions. The initial work on the study of the iteration of transcendental meromorphic functions may be found in [3, 6, 10, 11].

One of the major difficulties arising in the study of dynamics of transcendental meromorphic functions is the fact that iterations of meromorphic maps do not lead to a dynamical system. The point at infinity is an essential singularity for such a map, so the map can not be extended continuously to infinity. Hence the forward orbit of a pole terminates. All other points have well defined forward orbits. Despite the fact that certain orbits of a meromorphic function are finite, the study of iterations of such functions are important. For example, the iterative processes associated with Newton's method applied to an entire function often yields a meromorphic function as the root finder.

The singular values of a function play an important role in determining the dynamics of the function. Let $\mathbb{C}$ and $\hat{\mathbb{C}}$ denote the complex plane and the extended complex plane respectively. A point $w \in \mathbb{C}$ is said to be a critical point of f if $f'(w) = 0$. The value $f(w)$ corresponding to a critical point w is called a critical value of f. A point $w \in \hat{\mathbb{C}}$ is said to be an asymptotic value of $f(z)$, if there is

Mathematics Subject Classification 30D05, 37F70, 37F50 a continuous curve $\gamma(t)$ satisfying $\lim_{t\to\infty}\gamma(t)=\infty$ and $\lim_{t\to\infty} f(\gamma(t))=w$. A function is said to be critically finite if it has only finitely many asymptotic and critical values. If a function $f(z)$ is not critically finite, then it is said to be non-critically finite. A singular value of f is defined to be either a critical value or an asymptotic value of f.

The dynamics of transcendental entire and meromorphic functions are somewhat different from the dynamics of polynomials or rational functions, mainly because of the essential singularity at ∞ for transcendental functions. By Picard's theorem, for meromorphic (entire) transcendental function, any neighbourhood of ∞ is mapped over the entire plane infinitely often missing at most two points (one point) which, in the language of dynamical systems, means that meromorphic (entire) map exhibits a tremendous amount of hyperbolicity near ∞. However, the dynamical behaviour of critically finite meromorphic transcendental functions share some of the properties of entire and rational functions, for instance, these functions do not have wandering domains [4] and Baker domains [6]. In contrast, non-critically finite transcendental meromorphic functions may have wandering domains and Baker domains [22].

Though, the dynamics of critically finite meromorphic transcendental functions has been studied for several interesting classes during last two decades [4, 7, 9, 10, 15], the dynamics of non-critically finite transcendental meromorphic functions has not been explored so for, probably because of non-applicability of Sullivan's Theorem [6] to these functions. Also, the presence of infinitely many critical values and the behaviour of the orbits of critical values for non-critically finite transcendental meromorphic functions make it difficult to study the dynamics of such functions. In the present work, an effort is made to fill this gap by studying the dynamics of a one parameter family of non-critically finite even transcendental meromorphic functions. For this purpose, a one paramter family $\mathcal{H}$ is considered. It is found that functions in the family $\mathcal{H}$ have bounded singular values. Bifurcations in the dynamics on real axis for the functions in our family occur at two parameter values. It is obsereved that taming effect occurs in the Julia set of function in family $\mathcal{H}$ after crossing the first paramter value while explosion occurs in the Julia set after crossing the second paramter value. It is observed that the characterization of the Julia set of a function in $\mathcal{H}$ as the closure of the set of all its escaping points continues to hold for functions in $\mathcal{H}$. The Julia set of a function in $\mathcal{H}$ is found to contain both real and imaginary axes for certain parameter values. The comparison of salient features of dynamics of functions in the family $\mathcal{H}$ with recent results on dynamics of (i) critically finite transcendental meromorphic function $\lambda\tan z$, $\lambda>0$, having polynomial Schwarzian Derivative [10, 15, 19] and (ii) non-critically finite transcendental entire function $\lambda\frac{e^z-1}{z}$, $\lambda>0$ [14], demonstrate a qualitative resembelance in their dynamics even though their Julia sets have seemingly different nature. Further, it is noted that while the functions considered in [10] have polynomial Schwarzian Derivatives, the Schwarzian Derivatives of functions in our family $\mathcal{H}$ are in general transcendental meromorphic functions.

The following definitions and results are needed in the sequel. The Fatou set (or stable set) of a function f, denoted by $F(f)$, is defined to be the set of all complex numbers where the family of iterates $\{f^n\}$ of f forms a normal family in the sense of Montel. The Julia set (or chaotic set), denoted by $J(f)$, is the complement of the Fatou set of f. The *escaping points set* of meromorphic function $f(z)$, denoted by $I(f)$, is defined as

$$I(f)=\{z\in\mathbb{C}: f^n(z)\to\infty \text{ as } n\to\infty \text{ and } f^n(z)\neq\infty\}.$$

Let B be a class of meromorphic functions having bounded singular values. Zheng [23] proved that if $f\in B$ is a meromorphic function, then for $z\in F(f)$, the orbit $\{f^n(z)\}_{n=0}^{\infty}$ does not tend to ∞. Dominguez [11] gave a characterization of the Julia set of meromorphic transcendental functions as the boundary of the set of all escaping points, i.e., if $f(z)$ is a meromorphic transcendental function, then $J(f)=\partial I(f)$. The following characterization, given by Zheng [21] for meromorphic transcendental function, is quite useful for computer generation of the Julia sets:

Theorem 1.1. *Let $f_\lambda \in B$ be transcendental, then $J(f) = \overline{I(f)}$.*

For the class of meromorphic functions with polynomial Schwarzian Derivatives, the characterization of the Julia set in Theorem 1.1 is given by Hoggard [13].

The following results exhibit the importance of singular values in the dynamics of a transcendental meromorphic function:

Theorem 1.2 ([7]). *Let $f(z)$ be a transcendental meromorphic function. Suppose z_0 lies on an attracting cycle or a parabolic cycle $f(z)$. Then, the orbit of at least one critical value or asymptotic value is attracted to a point in the orbit of z_0.*

An analogue of Denjoy-Carleman-Ahlfors Theorem [1], guaranteeing finite number of asymptotic values of a meromorphic function, is given by the following:

Theorem 1.3 ([2]). *Let f be a meromorphic function of a finite Nevanlinna order ρ and $\underline{\lim}_{r\to\infty} \dfrac{n(r,\infty,f)}{\log r} < +\infty$. Then, the number of finite asymptotic values of the function f counted according to their multiplicity is not greater than 2ρ.*

2. One parameter family $\mathcal{H}$ of non-critically finite functions

Let

$$\mathcal{H} = \left\{ \xi_\lambda(z) = \lambda \frac{\sinh^2 z}{z^4} : \lambda > 0, z \in \hat{\mathbb{C}} \right\}$$

be one parameter family of even transcendental meromorphic functions. The following proposition shows that the functions in the family $\mathcal{H}$ are indeed non-critically finite and have bounded singular values:

Proposition 2.1. *Let $\xi_\lambda \in \mathcal{H}$. Then, the function $\xi_\lambda(z)$ is non-critically finite and all of its singular values are bounded.*

Proof.

The derivative of the function $\xi_\lambda(z)$ for $z \neq 0$ is given by

$$\xi_\lambda'(z) = \lambda \frac{2\sinh z(z\cosh z - 2\sinh z)}{z^5}$$

The critical points of the function $\xi_\lambda(z)$ are solutions of the equation $\xi_\lambda'(z) = 0$. This implies that $z = m\pi i$, where m is a non-zero integer and solutions of the equation

$$z\cosh z - 2\sinh z = 0 \tag{2.1}$$

are critical points of $\xi_\lambda(z)$. The solutions of Equation (2.1) are the same as the solutions of the equation $\tanh z = \frac{z}{2}$. This equation has a solution z_0 if only if the equation $\tan w = \frac{w}{2}$ has a solution iz_0. Now equating real and imaginary parts of the equation $\tan w = \frac{w}{2}$ we get that, for a non-zero $z = x + iy$, $\dfrac{\sin 2x}{\cos 2x + \cosh 2y} = \dfrac{1}{2}x$ and $\dfrac{\sinh 2y}{\cos 2x + \cosh 2y} = \dfrac{1}{2}y$. This implies that

$$\frac{\sin 2x}{x} = \frac{\sinh 2y}{y}. \tag{2.2}$$

It is easily seen that, for $x, y \neq 0$, $|\frac{\sin 2x}{x}| < 2$ and $\dfrac{\sinh 2y}{y} > 2$ so that (2.2) is not possible in this case. Therefore, at least one of x, y must vanish for (2.2) to hold. Equivalently, (2.2) has only real or purely imaginary roots. If $x = 0$, then $\tan w = \frac{w}{2}$ implies that $\tanh y = \frac{1}{2}y$ and this equation has two zeros. Therefore, $\tan w = \frac{w}{2}$ has two purely imaginary solutions. If $y = 0$, then $\tan w = \frac{w}{2}$ implies that $\tan x = \frac{1}{2}x$ and the latter equation has infinitely many real solutions. Consequently, it follows that Equation (2.1) has two real solutions and infinitely many purely imaginary solutions. Thus, the function $\xi_\lambda(z)$ has two real and infinitely many imaginary critical points.

To find the critical values of the function $\xi_\lambda(z)$, we note that $\xi_\lambda(m\pi i) = 0$, where m is a non-zero integer. Let $\{iy_k\}_{k=-\infty}^{\infty}$, y_k real, are critical points of $\xi_\lambda(z)$ other than the critical points $m\pi i$, $m = \pm 1, \pm 2, \ldots$. Since,

$$\xi_\lambda(iy_k) = \lambda \frac{\sin^2 y_k}{y_k^4}$$

and the values $\frac{\sin^2 y_k}{y_k^4}$ are real and distinct for distinct k, it follows that the values in the set $\{\xi_\lambda(iy_k)\}_{k=-\infty}^{\infty}$ are real and distinct. Therefore, the function $\xi_\lambda(z)$ possesses infinitely many real critical values. Since $\underline{\lim}_{r\to\infty} \frac{n(r,\infty,\xi)}{\log r} < +\infty$, and Nevanlinna order [12] of the function $\xi_\lambda(z)$ is 1, by an analogue of Denjoy-Carleman-Ahlfors theorem (*c.f. Theorem 1.3*), the function $\xi_\lambda(z)$ has at most two finite asymptotic value. This proves that the function $\xi_\lambda(z)$ is non-critically finite.

Since,

$$|\xi_\lambda(iy_k)| = \lambda \frac{|\sin^2 y_k|}{|y_k|^4} \leqslant \frac{\lambda}{y_k^4} \leqslant \lambda M$$

where, $M = \max\limits_{1 \leqslant k < \infty} \{\frac{1}{y_k^4}\}$, all critical values of the function $\xi_\lambda(z)$ are bounded. Further, $\xi_\lambda(z)$ takes finite values on both the real solutions of Equation (2.1). Thus, it follows that all of singular values of the function $\xi_\lambda(z)$ are bounded. ■

The dynamical behaviour of functions in the family $\mathcal{H}$ is now described in the sequel. In Section 3, the fixed points of the function $\xi_\lambda(x)$, $x \in \mathbb{R}\backslash\{0\}$ are obtained and their nature is investigated. The dynamics of the function $\xi_\lambda(x)$ is described in Section 4. It is shown that there exist critical parameter values $\lambda_1, \lambda_2 > 0$ such that bifurcations in the dynamics of the function $\xi_\lambda(x)$ occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where $\lambda_1 = \frac{x_1^5}{\sinh^2 x_1}$ (≈ 1.26333), $\lambda_2 = \frac{\tilde{x}^5}{\sinh^2 \tilde{x}}$ ($\approx 2.7.715$); x_1, $\tilde{x}$ being the unique positive real roots of the equations $\tanh x = \frac{2x}{3}$ and $\tanh x = \frac{2x}{5}$ respectively. If the parameter value crosses the value λ_1 or λ_2, then a change in the dynamics of $\xi_\lambda(x)$ is found to occur (*Fig. 4*). It is obsereved that taming effect occurs in the Julia set of function in family $\mathcal{H}$ after crossing the paramter value λ_1 while explosion occurs in the Julia set after crossing the paramter value λ_2 (*Fig. 6*). The dynamics of the function $\xi_\lambda(z) \in \mathcal{H}$ for $z \in \hat{\mathbb{C}}$ and $0 < \lambda \leqslant \lambda_1$ is investigated in Section 5.1. The characterization of the Julia set of the function $\xi_\lambda(z)$ as the closure of the set of all escaping points of the function $\xi_\lambda(z)$ is established for $0 < \lambda < \lambda_1$ and $\lambda = \lambda_1$ in this section. Further, for $0 < \lambda < \lambda_1$, it is proved in this section that the Julia set of the function $\xi_\lambda(z)$ contains both real and imaginary axes. For $\lambda = \lambda_1$, it is found in the same section that the Fatou set of the function $\xi_\lambda(z)$ contains a parabolic domain. In Section 5.2, the characterization of the Julia set of the function $\xi_\lambda(z)$ as the closure of the set of all escaping points of the function $\xi_\lambda(z)$ is found for $\lambda_1 < \lambda < \lambda_2$, $\lambda = \lambda_2$ and $\lambda > \lambda_2$. Further, for the case $\lambda_1 < \lambda < \lambda_2$, it is proved in this section that the Fatou set of the function $\xi_\lambda(z)$ does not contain any basin of attraction or parabolic domain except the basin of attraction $A(a_\lambda)$ of the real attracting fixed point a_λ of $\xi_\lambda(z)$. For $\lambda > \lambda_2$, it is found in this section that the Julia set of the function $\xi_\lambda(z)$ contains both real and imaginary axes. In Section 6, the results of the present paper are applied to generate computer images of Julia sets of the function $\xi_\lambda(z)$ using the algorithm developed in this section. Finally, our results concerning the dynamics of functions in $\mathcal{H}$ are compared with recent results on dynamics of (i) functions $\lambda \tan z$, $\lambda \in \mathbb{C} \setminus \{0\}$ having polynomial Schwarzian Derivative due to Devaney and Keen [10], Keen and Kotus [15] and Stallard [19] (ii) non-critically finite transcendental entire functions $\lambda \frac{e^z - 1}{z}$, $\lambda > 0$ [14].

3. Fixed points and their nature for functions in $\mathcal{H}$

In this section, we find the fixed points of the function $\xi_\lambda(x) = \lambda \frac{\sinh^2 x}{x^4}$ and describe their nature. Let

$$\phi(x) = \begin{cases} \dfrac{x^5}{\sinh^2 x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{3.1}$$

Properties of Function $\phi(x)$:

It follows easily from (3.1) that

i. **$\phi(x)$ is continuous in $\mathbb{R}$.**

ii. **$\phi(x)$ is positive in $(0, \infty)$, is negative in $(-\infty, 0)$.**

iii. **$\phi(x) \to 0$ as $x \to -\infty$ and $\phi(x) \to 0$ as $x \to \infty$.**

Further,

iv. **$\phi'(x)$ is continuous in $\mathbb{R}$:**
Since $\phi'(x) = \dfrac{5x^4 \sinh x - 2x^5 \cosh x}{\sinh^3 x}$, it follows easily that $\phi'(0) = \lim_{x \to 0} \phi'(x)$ so that $\phi'(x)$ is continuous in $\mathbb{R}$.

v. **$\phi'(x)$ has a unique positive real zero at $x = \tilde{x} (\approx 2.46406)$, where $\tilde{x}$ is a real positive solution of $\tanh x = \frac{2x}{5}$:**
Since $\phi'(x) = 0$ gives $\tanh x = \frac{2x}{5}$ and by Newton-Rapson's Method, $\tilde{x} \approx 2.46406$ is a real positive solution of $\Psi(x) = \tanh x - \frac{2x}{5} = 0$ (*c.f. Fig.* 1(*a*)), the Property (v) follows.

vi. **$\phi(x)$ is strictly increasing in $(0, \tilde{x})$, is strictly decreasing in $(\tilde{x}, \infty)$ and has a maximum at $x = \tilde{x}$, where $\tilde{x}$ is a real positive solution of $\tanh x = \frac{2x}{5}$:**
By Property (v), $\phi'(\tilde{x}) = 0$, where $\tilde{x}$ is the real positive solution of $\tanh x = \frac{2x}{5}$.

$$\begin{aligned} \phi''(\tilde{x}) &= \frac{20\tilde{x}^3 \sinh \tilde{x} - 5\tilde{x}^4 \cosh \tilde{x} - 2\tilde{x}^5 \sinh \tilde{x}}{\sinh^3 \tilde{x}} \\ &= \frac{\tilde{x}^3 \{(20 - 2\tilde{x}^2) \tanh \tilde{x} - 5\tilde{x}\}}{\tanh \tilde{x} \sinh^2 \tilde{x}} = \frac{\tilde{x}^3 (15 - 4\tilde{x}^2)}{2 \sinh^2 \tilde{x}}. \end{aligned}$$

Since $\tilde{x} \approx 2.46406$, $\phi''(\tilde{x}) < 0$. Therefore, the function $\phi(x)$ has exactly one maxima in $(0, \infty)$ at $x = \tilde{x}$. It therefore follows by Property (iii) that $\phi(x)$ decreases to 0 in $(\tilde{x}, \infty)$ and increases in $(0, \tilde{x})$.

The graph of $\phi(x)$ therefore is as shown in Fig. 1(b).

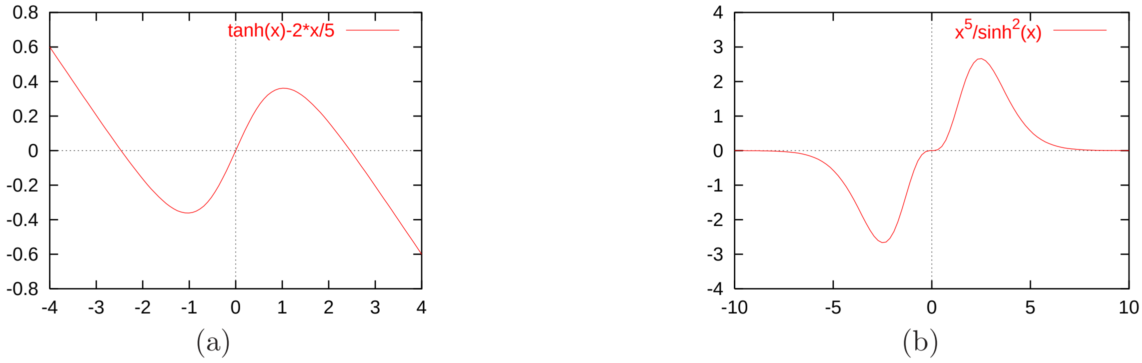


Fig. 1. (a) Graph of $\Psi(x)$ (b) Graph of $\phi(x)$

Throughout in the sequel, we denote

$$\lambda_2 = \phi(\tilde{x}) \tag{3.2}$$

where, $\tilde{x}$ is the unique positive real solution of the equation $\tanh x = \dfrac{2x}{5}$. The following proposition gives the number and locations of real fixed points of the function $\xi_\lambda(x)$ for $\lambda > 0$:

Proposition 3.1. *Let $\xi_\lambda \in \mathcal{H}$. Then, the locations of real fixed points of the function $\xi_\lambda(x) = \lambda \dfrac{\sinh^2 x}{x^4}$ are given by the following:*

i. For $0 < \lambda < \lambda_2$, $\xi_\lambda(x)$ has exactly one fixed point in each of the intervals $(0, \tilde{x})$ and $(\tilde{x}, \infty)$, where $\tilde{x}$ is solution of the equation $\tanh(x) = \dfrac{2x}{5}$.

ii. For $\lambda = \lambda_2$, the only fixed point of $\xi_\lambda(x)$ is at $x = \tilde{x}$, where $\tilde{x}$ is as in (i).

iii. For $\lambda > \lambda_2$, $\xi_\lambda(x)$ has no fixed points.

Proof: The fixed points of the function $\xi_\lambda(x)$ are the solutions of the equation

$$\lambda = \phi(x)$$

where, $\phi(x)$ is given by (3.1). We have the following cases:

i. $0 < \lambda < \lambda_2$
Since $\phi(\tilde{x}) = \lambda_2$ and $\lambda < \lambda_2$, in view of Properties (i), (iii) and (vi) of the function $\phi(x)$, the line $u = \lambda$ intersects the graph of $\phi(x)$ (*Fig. 1(b)*) at exactly two points. Using Properties (ii), (iii) and (vi), it follows in view of $\phi(\tilde{x}) = \lambda_2$ that one of the solutions of $\phi(x) = \lambda$ for $0 < \lambda < \lambda_2$ lies in the interval $(0, \tilde{x})$. Similarly, since by Property (vi) $\phi(x)$ is decreasing in the interval $(\tilde{x}, \infty)$ and $\phi(\tilde{x}) = \lambda_2$, the other solution of $\phi(x) = \lambda$ for $0 < \lambda < \lambda_2$ lies in the interval $(\tilde{x}, \infty)$. Thus, $\xi_\lambda(x)$ has two real fixed points lying in the intervals $(0, \tilde{x})$ and $(\tilde{x}, \infty)$.

ii. $\lambda = \lambda_2$
The function $\phi(x)$ has exactly one maxima at $x = \tilde{x}$ (*c.f. Property (vi)*) and the maximum value of $\phi(x)$ is $\phi(\tilde{x}) = \lambda_2$, the line $u = \lambda_2$ intersects the graph of $\phi(x)$ at exactly one point $x = \tilde{x}$. Therefore, the equation $\phi(x) = \lambda_2$ has exactly one solution at $x = \tilde{x}$. Thus, $\xi_\lambda(x)$ has only one real fixed point at $x = \tilde{x}$ for $\lambda = \lambda_2$.

iii. $\lambda > \lambda_2$
By Property (vi), the maximum value of $\phi(x)$ is $\phi(\tilde{x}) = \lambda_2$, therefore, for $\lambda > \lambda_2$, the line $u = \lambda$ does not intersect the graph of $\phi(x)$. Consequently, the equation $\phi(x) = \lambda$ has no solution for $\lambda > \lambda_2$. Thus, $\xi_\lambda(x)$ has no fixed point for $\lambda > \lambda_2$. ■

Let

$$\lambda_1 = \phi(x_1) \tag{3.3}$$

where, x_1 is a positive solution of the equation $\tanh x = \frac{2x}{3}$.

Throughout in the sequel, the fixed points of the function $\xi_\lambda(x)$ found in Proposition 3.1 are denoted by $r_{1,\lambda} \in (0, x_1)$, $r_{2,\lambda} \in (x_2, \infty)$, $a_\lambda \in (x_1, \tilde{x})$ and $r_\lambda \in (\tilde{x}, x_2)$, where $\tilde{x}$ is a positive solution of $\tanh x = \frac{2x}{5}$ and x_1, x_2 be solutions of $\lambda_1 = \phi(x)$ lying in the intervals $(0, \tilde{x})$ and $(\tilde{x}, \infty)$ respectively. The nature of these fixed points of the function $\xi_\lambda(x)$ for different values of parameterλ is described in the following theorem:

Theorem 3.1. *Let* $\xi_\lambda(x) = \lambda \frac{\sinh^2 x}{x^4}$ *for* $x \in \mathbb{R}\backslash\{0\}$ *and* $\lambda > 0$, *and* x_1, x_2 *be solutions of* $\lambda_1 = \phi(x)$ *lying in the intervals* $(0, \tilde{x})$ *and* $(\tilde{x}, \infty)$ *respectively, where* $\tilde{x}$ *is a positive solution of the equation* $\tanh x = \frac{2x}{5}$.

i. If $0 < \lambda < \lambda_1$, *then the fixed points* $r_{1,\lambda} \in (0, x_1)$ *of* $\xi_\lambda(x)$ *and* $r_{2,\lambda} \in (x_2, \infty)$ *of* $\xi_\lambda(x)$ *are repelling.*

ii. If $\lambda = \lambda_1$, *then the fixed point* x_1 *of* $\xi_\lambda(x)$ *is rationally indifferent and the fixed point* x_2 *of* $\xi_\lambda(x)$ *is repelling.*

iii. If $\lambda_1 < \lambda < \lambda_2$, *then the fixed point* $a_\lambda \in (x_1, \tilde{x})$ *of* $\xi_\lambda(x)$ *is attracting and the fixed point* $r_\lambda \in (\tilde{x}, x_2)$ *of* $\xi_\lambda(x)$ *is repelling.*

iv. If $\lambda = \lambda_2$, *then the fixed point* $\tilde{x}$ *of* $\xi_\lambda(x)$ *is rationally indifferent.*

Proof. Since the derivative of the function $\xi_\lambda(x)$ is given by

$$\xi'_\lambda(x) = \lambda \frac{2\sinh x(x\cosh x - 2\sinh x)}{x^5}$$

and the fixed points of the function $\xi_\lambda(x)$ are solutions of $\lambda = \frac{x^5}{\sinh^2 x}$, it follows that the multiplier $\xi'_\lambda(x_f)$ of the fixed point x_f is given by

$$|\xi'_\lambda(x_f)| = 2|x_f \coth x_f - 2| \tag{3.4}$$

Let

$$G(x) = \begin{cases} 2(x\coth x - 2) & \text{for } x \neq 0 \\ -2 & \text{for } x = 0. \end{cases}$$

The function $G(x)$ is differentiable and its derivative is given by

$$G'(x) = \begin{cases} 2(\coth x - x\,\mathtt{cosech}^2 x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

Since, $G'(x) \neq 0$ for $x \neq 0$, $G'(0) = 0$ and $G''(0) = \frac{4}{3} > 0$, the function $G(x)$ has exactly one minima at $x = 0$ and the minimum value is -2. Since $G'(x) > 0$ for $x \in (0, \infty)$ and $G'(x) < 0$ for $x \in (-\infty, 0)$, the function $G(x)$ is increasing from -2 to ∞ as x increases from 0 to ∞ and $G(x)$ is decreasing from $-\infty$ to -2 as x increases from $-\infty$ to 0. Thus, it follows that the function $|G(x)|$ (*Fig.* 2) satisfies

$$|G(x)| \begin{cases} < 1 & \text{for } x \in (-\tilde{x}, -x_1) \cup (x_1, \tilde{x}) \\ = 1 & \text{for } x = \pm x_1, \pm\tilde{x} \\ > 1 & \text{for } x \in (-\infty, -\tilde{x}) \cup (-x_1,\, 0) \cup (0,\, x_1) \cup (\tilde{x}, \infty). \end{cases}$$

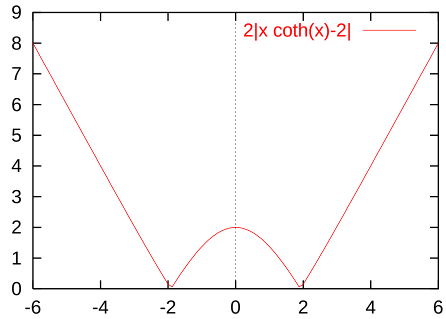


Fig. 2. Graph of $|G(x)|$

Consequently, by (3.4), we get that the multiplier $\xi'_\lambda(x_f)$ of the fixed point x_f satisfies

$$|\xi'_\lambda(x_f)| < 1 \quad \text{for } x_f \in (-\tilde{x}, -x_1) \cup (x_1, \tilde{x}) \tag{3.5}$$

$$|\xi'_\lambda(x_f)| = 1 \quad \text{for } x_f = \pm x_1, \pm\tilde{x} \tag{3.6}$$

$$|\xi'_\lambda(x_f)| > 1 \quad \text{for } x_f \in (-\infty, -\tilde{x}) \cup (-x_1,\, 0) \cup (0,\, x_1) \cup (\tilde{x}, \infty). \tag{3.7}$$

i. $0 < \lambda < \lambda_1$
Since the fixed point $r_{1,\lambda} \in (0, x_1)$, by Inequality (3.7), $|\xi'_\lambda(r_{1,\lambda})| > 1$. It therefore follows that $r_{1,\lambda}$ is a repelling fixed point of $\xi_\lambda(x)$. Similarly, since the fixed point $r_{2,\lambda} \in (x_2, \infty)$, by Inequality (3.7), $|\xi'_\lambda(r_{2,\lambda})| > 1$. Consequently, $r_{2,\lambda}$ is a repelling fixed point of $\xi_\lambda(x)$.

ii. $\lambda = \lambda_1$
By Equation (3.6), $|\xi'_\lambda(x_1)| = 1$ Therefore, $x = x_1$ is a rationally indifferent fixed point of $\xi_\lambda(x)$. Further, since $x_2 > \tilde{x} > x_1$, it follows that $x_2 \in (\tilde{x}, \infty)$. By Inequality (3.7), $|\xi'_\lambda(x_2)| > 1$. It therefore follows that x_2 is a repelling fixed point of $\xi_\lambda(x)$.

iii. $\lambda_1 < \lambda < \lambda_2$
Since the fixed point $a_\lambda \in (x_1, \tilde{x})$, by Inequality (3.5), $|\xi'_\lambda(a_\lambda)| < 1$. Thus, a_λ is an attracting fixed point of $\xi_\lambda(x)$. Further, since the fixed point $r_\lambda \in (\tilde{x}, x_2)$, by Inequality (3.7) gives that $|\xi'_\lambda(r_\lambda)| > 1$. It therefore follows that r_λ is a repelling fixed point of $\xi_\lambda(x)$.

iv. $\lambda = \lambda^{**}$
By Equation (3.6), $|\xi'_\lambda(\tilde{x})| = 1$. Consequently, $x = \tilde{x}$ is a rationally indifferent fixed point of $\xi_\lambda(x)$. ■

4. Bifurcations in Dynamics on $\mathbb{R}\backslash\{0\}$

In this section, the dynamics of functions $\xi_\lambda \in \mathcal{H}$ on the real line is described. It is proved in the following theorem that there exist parameter values $\lambda_1, \lambda_2 > 0$ such that bifurcations in the dynamics of the function $\xi_\lambda(x)$, $x \in \mathbb{R}\backslash T_0$ occur at $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where T_0 is the set of the points that are backward orbits of the pole 0 of the function $\xi_\lambda(x)$. The Phase Portrait (*Fig. 4*) describing the dynamics of the function $\xi_\lambda(x)$ for various values of parameter λ is also obtained by using the results of this theorem.

Theorem 4.1. *Let* $\xi_\lambda(x) = \lambda \dfrac{\sinh^2 x}{x^4}$ *for* $x \in \mathbb{R}\backslash\{0\}$.

a. *If* $0 < \lambda < \lambda_1$, $\xi^n_\lambda(x) \to \infty$ *as* $n \to \infty$ *for* $x \in [(-\infty,\ -r_{2,\lambda}) \cup (-\eta_1,\ 0) \cup (0,\ \eta_1) \cup (r_{2,\lambda},\ \infty))]\backslash T_0$ *and the orbits* $\{\xi^n_\lambda(x)\}$ *are chaotic for* $x \in [(-r_{2,\lambda}, -r_{1,\lambda}) \cup (-r_{1,\lambda}, -\eta_1) \cup (\eta_1, r_{1,\lambda}) \cup (r_{1,\lambda}, r_{2,\lambda})]\backslash T_0$, *where* $r_{1,\lambda}$ *and* $r_{2,\lambda}$ *are repelling fixed points of* $\xi_\lambda(x)$ *and* η_1 *is a positive solution of* $\xi_\lambda(x) = r_{2,\lambda}$.

b. *If* $\lambda = \lambda_1$, $\xi_\lambda^n(x) \to x_1$ *as* $n \to \infty$ *for* $x \in [(-x_2, -\eta_2) \cup (\eta_2, x_2)] \backslash T_0$ *and* $\xi_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $x \in [(-\infty, -x_2) \cup (-\eta_2, 0) \cup (0, \eta_2) \cup (x_2, \infty)] \backslash T_0$, *where* x_1 *is a rationally indifferent fixed point,* x_2 *is a repelling fixed point of* $\xi_\lambda(x)$ *and* η_2 *is a positive solution of* $\xi_\lambda(x) = x_2$.

c. *If* $\lambda_1 < \lambda < \lambda_2$, $\xi_\lambda^n(x) \to a_\lambda$ *as* $n \to \infty$ *for* $x \in [(-r_\lambda, -\eta_3) \cup (\eta_3, r_\lambda)] \backslash T_0$ *and* $\xi_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $x \in [(-\infty, -r_\lambda) \cup (-\eta_3, 0) \cup (0, \eta_3) \cup (r_\lambda, \infty)] \backslash T_0$, *where* a_λ *is an attracting fixed point,* r_λ *is a repelling fixed point of* $\xi_\lambda(x)$ *and* η_3 *is a positive solution of* $\xi_\lambda(x) = r_\lambda$.

d. *If* $\lambda = \lambda_2$, $\xi_\lambda^n(x) \to \tilde{x}$ *as* $n \to \infty$ *for* $x \in [(-\tilde{x}, -\eta_4) \cup (\eta_4, \tilde{x})] \backslash T_0$ *and* $\xi_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for* $x \in [(-\infty, -\tilde{x}) \cup (-\eta_4, 0) \cup (0, \eta_4) \cup (\tilde{x}, \infty)] \backslash T_0$, *where* $\tilde{x}$ *is a rationally indifferent fixed point of* $\xi_\lambda(x)$ *and* η_4 *is a positive solution of* $\xi_\lambda(x) = \tilde{x}$.

e. *If* $\lambda > \lambda_2$, $\xi_\lambda^n(x) \to \infty$ *as* $n \to \infty$ *for all* $x \in \mathbb{R} \backslash T_0$.

Proof.

Let x_{min} denotes the real positive solution of the equation $\tanh x = \frac{x}{2}$. Then, $\xi'(x_{min}) = 0$. By Newton-Rapson Method, $x_{min} \approx 1.91501$. Since,

$$\xi''(x) = \frac{\lambda \cosh^2 x (x^2 - 2)}{2x^4}$$

it follows that $f_\lambda''(x_{min}) > 0$. This proves that x_{min} is the minima of $\xi_\lambda(x)$.

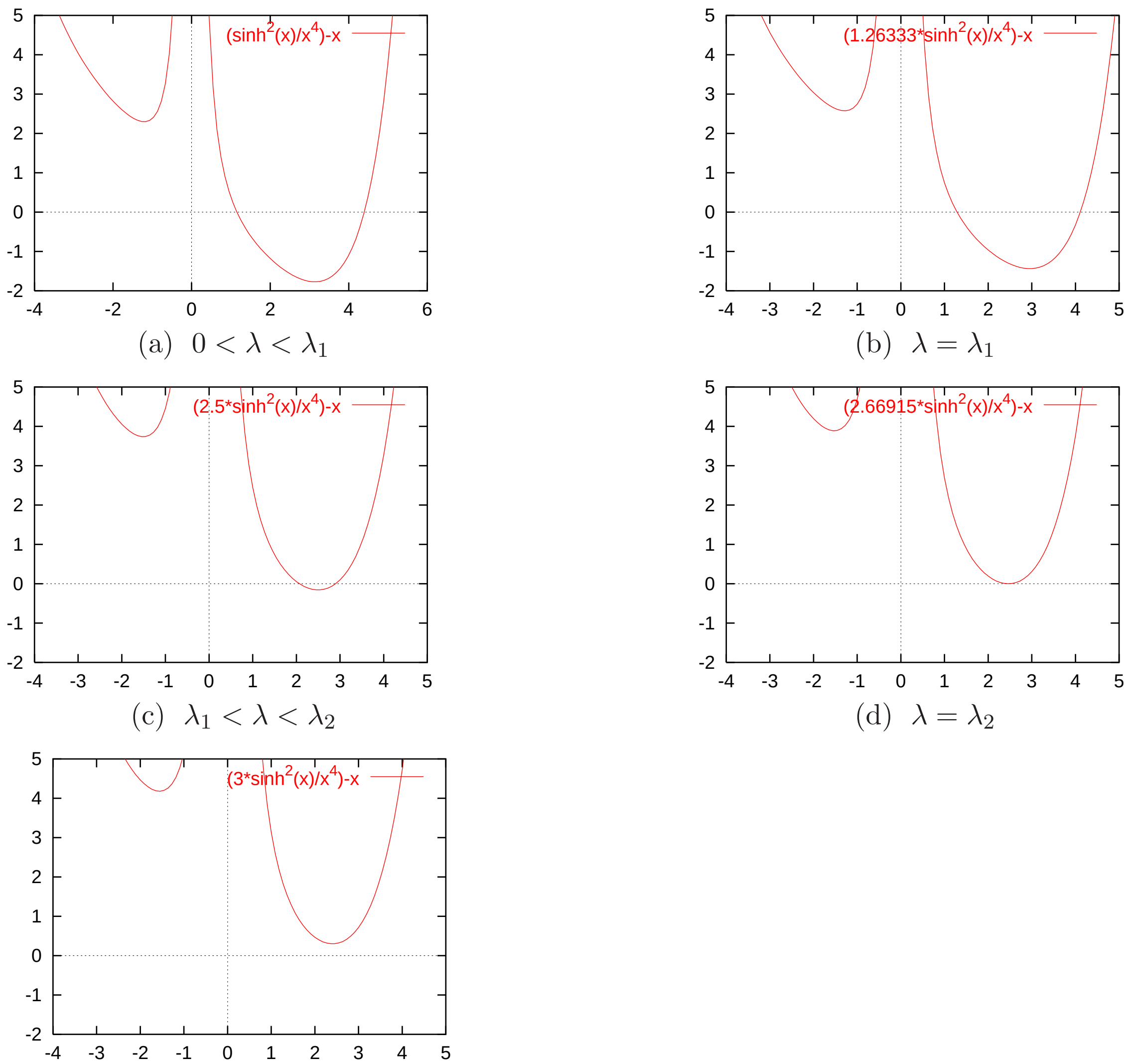
Define the function $t_\lambda(x) = \xi_\lambda(x) - x$ for $x \in \mathbb{R} \backslash \{0\}$. It is easily seen that the function $t_\lambda(x)$ is continuously differentiable for $x \in \mathbb{R} \backslash \{0\}$. Note that the fixed points of the function $\xi_\lambda(x)$ are zeros of the function $t_\lambda(x)$.

a. If $0 < \lambda < \lambda_1$, by Theorem 3.1, the function $\xi_\lambda(x)$ has only two repelling fixed points $r_{1,\lambda}$ and $r_{2,\lambda}$. Since $t_\lambda'(r_{1,\lambda}) < -2$ and in a neighbourhood of $r_{1,\lambda}$ the function $t_\lambda'(x)$ is continuous, $t_\lambda'(x) < 0$ in some neighbourhood of $r_{1,\lambda}$. Therefore, $t_\lambda(x)$ is decreasing in a neighbourhood of $r_{1,\lambda}$. By the continuity of the function $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(r_{1,\lambda} - \delta_1,\, r_{1,\lambda})$ and $t_\lambda(x) < 0$ in $(r_{1,\lambda},\, r_{1,\lambda} + \delta_1)$. Further, since $t_\lambda'(r_{2,\lambda}) > 0$ and in a neighbourhood of $r_{2,\lambda}$ the function $t_\lambda'(x)$ is continuous, $t_\lambda'(x) > 0$ in some neighbourhood of $r_{2,\lambda}$. Therefore, $t_\lambda(x)$ is increasing in a neighbourhood of $r_{2,\lambda}$. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in $(r_{2,\lambda},\, r_{2,\lambda} + \delta_2)$ and $t_\lambda(x) < 0$ in $(r_{2,\lambda} - \delta_2,\, r_{2,\lambda})$. Since $t_\lambda(x) \neq 0$ in $(0,\, r_{1,\lambda}) \cup (r_{1,\lambda},\, r_{2,\lambda}) \cup (r_{2,\lambda},\, \infty)$, it now follows that $t_\lambda(x) > 0$ in $(0,\, r_{1,\lambda}) \cup (r_{2,\lambda},\, \infty)$ and $t_\lambda(x) < 0$ in $(r_{1,\lambda},\, r_{2,\lambda})$ (*Fig. 3(a)*). Thus,

$$t_\lambda(x) = \xi_\lambda(x) - x \begin{cases} > 0 & \text{for } x \in (0,\, r_{1,\lambda}) \cup (r_{2,\lambda},\, \infty) \\ < 0 & \text{for } x \in (r_{1,\lambda},\, r_{2,\lambda}). \end{cases} \tag{4.1}$$

The dynamics of the function $\xi_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-\infty,\, -r_{2,\lambda}) \cup (-\eta_1,\, 0) \cup (0,\, \eta_1) \cup (r_{2,\lambda},\, \infty)] \backslash T_0$)**:**
By (4.1), it follows that, for $x \in (r_{2,\lambda},\, \infty)$, $\xi_\lambda(x) > x$. Since the function $\xi_\lambda(x)$ is increasing for $x \in (r_{2,\lambda},\, \infty)$, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$. Further, since $f(\eta_1) = r_{2,\lambda}$ and $\xi_\lambda(x)$ is decreasing in $(0, \eta_1)$, the function $\xi_\lambda(x)$ maps the interval $(0,\, \eta_1)$ into $(r_{2,\lambda},\, \infty)$. Now, using the above arguments, we get $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0,\, \eta_1)$. Next, since $\xi_\lambda(x)$ is an even function, using the above arguments again, we get $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty,\, -r_{2,\lambda}) \cup (-\eta_1,\, 0)] \backslash T_0$.



(a) $0 < \lambda < \lambda_1$

(b) $\lambda = \lambda_1$

(c) $\lambda_1 < \lambda < \lambda_2$

(d) $\lambda = \lambda_2$

(e) $\lambda > \lambda_2$

Fig. 3. Graphs of $t_\lambda(x) = \xi_\lambda(x) - x$ for different values of parameter λ

Case-ii $(x \in [(-r_{2,\lambda},\ -r_{1,\lambda}) \cup (-r_{1,\lambda},\ -\eta_1) \cup (\eta_1,\ r_{1,\lambda}) \cup (r_{1,\lambda},\ r_{2,\lambda})] \backslash T_0)$**:**
Since there is no attractor to attract the system dynamics, the dynamical system will keep moving indefinitely. Therefore, orbits of $\xi_\lambda(x)$ are chaotic for $x \in (\eta_1,\ r_{1,\lambda}) \cup (r_{1,\lambda},\ r_{2,\lambda})$. Further, since $\xi_\lambda(x)$ is an even function, using the above arguments again, orbits of the function $\xi_\lambda(x)$ are chaotic for $x \in [(-r_{2,\lambda},\ -r_{1,\lambda}) \cup (-r_{1,\lambda},\ -\eta_1)] \backslash T_0$.

b. If $\lambda = \lambda_1$, by Theorem 3.1, the function $\xi_\lambda(x)$ has a rationally indifferent fixed point x_1 and a repelling fixed point x_2. Since $t'_\lambda(x_1) = -2$ and in a neighbourhood of x_1 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) < 0$ in some neighbourhood of x_1. Therefore, $t_\lambda(x)$ is decreasing in a neighbourhood of x_1. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(x_1 - \delta_1,\ x_1)$ and $t_\lambda(x) < 0$ in $(x_1,\ x_1 + \delta_1)$. Further, since $t'_\lambda(x_2) > 0$ and in a neighbourhood of x_2 the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) > 0$ in some neighbourhood of x_2. Therefore, $t_\lambda(x)$ is increasing in a neighbourhood of x_2. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in $(x_2,\ x_2 + \delta_2)$ and $t_\lambda(x) < 0$ in $(x_2 - \delta_2,\ x_2)$. Since $t_\lambda(x) \neq 0$ in $(0, x_1) \cup (x_1,\ x_2) \cup (x_2,\ \infty)$, it now follows that $t_\lambda(x) > 0$ in $(0,\ x_1) \cup (x_2,\ \infty)$ and $t_\lambda(x) < 0$ in $(x_1,\ x_2)$ (*Fig.* 3(*b*)).

Thus,

$$t_\lambda(x) = \xi_\lambda(x) - x \begin{cases} > 0 & \text{for } x \in (0,\, x_1) \cup (x_2,\, \infty) \\ < 0 & \text{for } x \in (x_1,\, x_2). \end{cases} \tag{4.2}$$

The dynamics of the function $\xi_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-x_2,\, -\eta_2) \cup (\eta_2,\, x_2)] \backslash T_0$)**:**
By (4.2), it follows that $\xi'_\lambda(x) < 1$ for $x \in (\eta_2,\, x_2)$, $\xi'_\lambda(x_1) = 1$ and $\xi'_\lambda(x) > 1$ for $x > x_2$, it follows that, using Mean Value Theorem, $|\xi_{\lambda_1}(x) - x_1| < |x - x_1|$ for $x \in (\eta_2, x_2)$. Therefore, $\xi^n_{\lambda_1}(x) \to x_1$ as $n \to \infty$ for $x \in (\eta_2, x_1)$. Further, since $\xi_\lambda(x)$ is an even function, using the above arguments again, $\xi^n_{\lambda_1}(x) \to x_1$ as $n \to \infty$ for $x \in (-x_2,\, -\eta_2)\backslash T_0$.

Case-ii ($x \in [(-\infty,\, -x_2) \cup (-\eta_2,\, 0) \cup (0,\, \eta_2) \cup (x_2,\, \infty)] \backslash T_0$)**:**
By (4.2), it follows that, for $x \in (x_2,\, \infty)$, $\xi_\lambda(x) > x$. Since $\xi_\lambda(x)$ is increasing for $x \in (x_2,\, \infty)$, so that $\xi^n_\lambda(x) \to \infty$ as $n \to \infty$. Further, since $\xi(\eta_2) = x_2$ and $\xi_\lambda(x)$ is decreasing in $(0, \eta_2)$, so that $\xi_\lambda(x)$ maps the interval $(0,\, \eta_2)$ into $(x_2,\, \infty)$. Now, using the above arguments, we get $\xi^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (0,\, \eta_2)$. Next, since $\xi_\lambda(x)$ is an even function, using the above arguments again, $\xi^n_\lambda(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty,\, -x_2)\backslash T_0$.

c. If $\lambda_1 < \lambda < \lambda_2$, by Theorem 3.1, the function $\xi_\lambda(x)$ has an attracting fixed point a_λ and a repelling fixed point r_λ. Since $t'_\lambda(a_\lambda) < 0$ and in a neighbourhood of a_λ the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) < 0$ in some neighbourhood of a_λ. Therefore, $t_\lambda(x)$ is decreasing in a neighbourhood of a_λ. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_1 > 0$, $t_\lambda(x) > 0$ in $(a_\lambda - \delta_1, a_\lambda)$ and $t_\lambda(x) < 0$ in $(a_\lambda, a_\lambda + \delta_1)$. Further, since $t'_\lambda(r_\lambda) > 0$ and in a neighbourhood of r_λ the function $t'_\lambda(x)$ is continuous, $t'_\lambda(x) > 0$ in some neighbourhood of r_λ. Therefore, $t_\lambda(x)$ is increasing in a neighbourhood of r_λ. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta_2 > 0$, $t_\lambda(x) > 0$ in $(r_\lambda, r_\lambda + \delta_2)$ and $t_\lambda(x) < 0$ in $(r_\lambda - \delta_2, r_\lambda)$. Since $t_\lambda(x) \neq 0$ in $(0, a_\lambda) \cup (a_\lambda, r_\lambda)$, it now follows that $t_\lambda(x) > 0$ in $(0, a_\lambda) \cup (r_\lambda, \infty)$ and $t_\lambda(x) < 0$ in (a_λ, r_λ) (*Fig. 3(c)*). Thus,

$$t_\lambda(x) = \xi_\lambda(x) - x \begin{cases} > 0 & \text{for } x \in (0, a_\lambda) \cup (r_\lambda, \infty) \\ < 0 & \text{for } x \in (a_\lambda, r_\lambda). \end{cases} \tag{4.3}$$

The dynamics of the function $\xi_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-r_\lambda, -\eta_3) \cup (\eta_3, r_\lambda)] \backslash T_0$)**:**
Since $|\xi'_\lambda(a_\lambda)| < 1$, $\xi'_\lambda(x_{min}) = 0$, $\xi'_\lambda(x)$ is increasing for $x > 0$ and $a_\lambda < x_{min}$, there exists a point $b \in [x_{min}, r_\lambda]$ such that $|\xi'_\lambda(\zeta)| < 1$ for all $\zeta \in [a_\lambda, b] \supset [a_\lambda, x_{min}]$. Using Mean Value Theorem, it follows that $|\xi_\lambda(x) - \xi_\lambda(a_\lambda)| < |x - a_\lambda|$ for $x \in [a_\lambda, b]$. Consequently, $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in [a_\lambda, b]$. For each $x \in [b, r_\lambda)$ the forward orbits contain a point from $[a_\lambda, b]$. Therefore, same as above, $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for all $x \in [b, r_\lambda)$. Hence $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in [a_\lambda, r_\lambda)$. Again, since $\xi_\lambda(\eta_3) = r_\lambda$ and $\xi_\lambda(x)$ is decreasing in the interval $(\eta_3, a_\lambda]$, $\xi_\lambda(x)$ maps the interval $(\eta_3, a_\lambda]$ into $[a_\lambda, r_\lambda)$. Therefore, using the above arguments again, $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in (\eta_3, a_\lambda]$. Thus, $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in (\eta_3, r_\lambda)$. Further, since $\xi_\lambda(x)$ is an even function, using the above arguments again, $\xi^n_\lambda(x) \to a_\lambda$ as $n \to \infty$ for $x \in (-r_\lambda, -\eta_3)\backslash T_0$.

Case-ii ($x \in [(-\infty, -r_\lambda) \cup (-\eta_3, 0) \cup (0,\, \eta_3) \cup (r_\lambda, \infty)] \backslash T_0$)**:**
By (4.3), it follows that, for $x \in (r_\lambda, \infty)$, $\xi_\lambda(x) > x$ and $\xi'_\lambda(x) > 1$ for $x > r_\lambda$. Therefore, $\xi^n_\lambda(x) \to \infty$ as $n \to \infty$. Further, since $\xi_\lambda(\eta_3) = r_\lambda$ and $\xi_\lambda(x)$ is decreasing

from ∞ to $\xi_\lambda(\eta_3)$ as x increases from 0 to η_3, $\xi_\lambda(x)$ maps the interval $(0,\ \eta_3))$ into the interval (r_λ, ∞). Therefore, using the above arguments again, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0,\ \eta_3)$. Furthermore, since $\xi_\lambda(x)$ is an even function, using the above arguments, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty, -r_\lambda) \cup (-\eta_3, 0)] \backslash T_0$.

d. If $\lambda = \lambda_2$, by Theorem 3.1, the function $\xi_\lambda(x)$ has a rationally indifferent fixed point at $\tilde{x}$. Since $t'_\lambda(\tilde{x}) = 0$ and $t''_\lambda(\tilde{x}) > 0$, so that $t_\lambda(x)$ has minima at $\tilde{x}$. Since $t_\lambda(\tilde{x}) = 0$, $t_\lambda(x) > 0$ in a neighbourhood of $\tilde{x}$. By the continuity of $t_\lambda(x)$, for sufficiently small $\delta > 0$, $t_\lambda(x) > 0$ in $(\tilde{x} - \delta,\ \tilde{x}) \cup (\tilde{x},\ \tilde{x} + \delta)$. Since $t_\lambda(x) \neq 0$ in $(0,\ \tilde{x}) \cup (\tilde{x},\ \infty)$, it now follows that $t_\lambda(x) > 0$ in $(0, \tilde{x}) \cup (\tilde{x}, \infty)$ (*Fig.* 3(*d*)). Thus,

$$t_\lambda(x) = \xi_\lambda(x) - x > 0 \text{ for } x \in (0, \tilde{x}) \cup (\tilde{x}, \infty). \tag{4.4}$$

The dynamics of the function $\xi_\lambda(x)$ is now described by the following cases:

Case-i ($x \in [(-\tilde{x}, -\eta_4) \cup (\eta_4, \tilde{x})] \backslash T_0$):
By (4.4), it follows that, $\xi'_\lambda(x) < 1$ for $x \in (\eta_4,\ \tilde{x})$, $\xi'_\lambda(\tilde{x}) = 1$ and $\xi'_\lambda(x) > 1$ for $x > \tilde{x}$, it follows that $|\xi_{\lambda_2}(x) - \tilde{x}| < |x - \tilde{x}|$ for $x \in (\eta_4, \tilde{x})$. Therefore, $\xi_{\lambda_2}^n(x) \to \tilde{x}$ as $n \to \infty$ for $x \in (\eta_4, \tilde{x})$. Again, since $\xi_\lambda(x)$ is an even function, $\xi_{\lambda_2}^n(x) \to \tilde{x}$ as $n \to \infty$ for $x \in (-\tilde{x}, -\eta_4) \backslash T_0$.

Case-ii ($x \in [(-\infty, -\tilde{x}) \cup (-\eta_4,\ 0) \cup (0,\ \eta_4) \cup (\tilde{x}, \infty)] \backslash T_0$):
By (4.4), it follows that, for $x \in (\tilde{x}, \infty)$, $\xi_{\lambda_2}(x) > x$ and $\xi'_{\lambda_2}(x) > 1$ for $x > \tilde{x}$. Therefore, $\xi_{\lambda_2}^n(x) \to \infty$ as $n \to \infty$. Next, since $\xi_{\lambda_2}(\eta_4) = \tilde{x}$ and $\xi_\lambda(x)$ is decreasing from ∞ to $\xi_\lambda(\eta_4)$ as x increases from 0 to η_4, $\xi_\lambda(x)$ maps the interval $(0,\ \eta_4)$ into the interval $(\tilde{x}, \infty)$. Therefore, using the above arguments again, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0,\ \eta_4)$. Further, since $\xi_\lambda(x)$ is an even function, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty, -\tilde{x}) \cup (-\eta_4,\ 0)] \backslash T_0$.

e. If $\lambda > \lambda_2$, by Proposition 3.1, the function $\xi_\lambda(x)$ has no fixed points. We observed that $t_\lambda(x) > 0$ for all $x \in \mathbb{R} \setminus \{0\}$ (*Fig. 3(e)*). Since $\xi_\lambda(x) > x$ for $x > 0$, so that $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (0, \infty)$. Since $\xi_\lambda(x)$ is an even function, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in (-\infty, 0) \backslash T_0$. Thus, $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$, $x \in [(-\infty, 0) \cup (0, \infty)] \backslash T_0$. ■

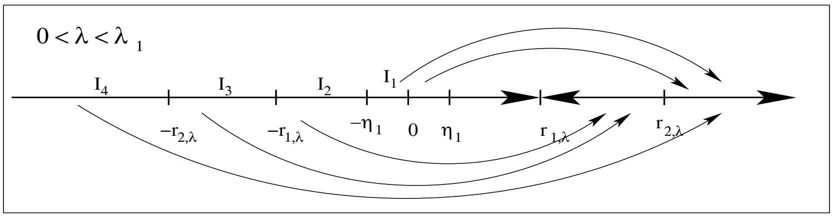
It follows by Theorem 4.1 that bifurcations in the dynamics of the function $\xi_\lambda(x)$ for $x \in \mathbb{R}\backslash\{0\}$ occur at the two critical parameter values $\lambda = \lambda_1$ and $\lambda = \lambda_2$, where $\lambda_1 = \dfrac{x_1^5}{\sinh^2 x_1}$, $\lambda_2 = \dfrac{\tilde{x}^5}{\sinh^2 \tilde{x}}$; x_1, $\tilde{x}$ being the unique positive real roots of the equations $\tanh x = \dfrac{2x}{3}$ and $\tanh x = \dfrac{2x}{5}$ respectively. The numerical computation of the root x_1 of the equation $\tanh x = \dfrac{2x}{3}$ gives $x_1 \approx 1.287.7$ and the root $\tilde{x}$ of the equation $\tanh x = \dfrac{2x}{5}$ gives $\tilde{x} \approx 2.46406$. Thus, by (3.3) and (3.2), approximation of the critical parameter values are $\lambda_1 \approx 1.26333$ and $\lambda_2 \approx 2.7.715$. Fig. 5 shows the bifurcation diagram for the function $\xi_\lambda(x) = \lambda \sinh^2 x / x^4$, $\lambda > 0$.

5. Dynamics on $\hat{\mathbb{C}}$

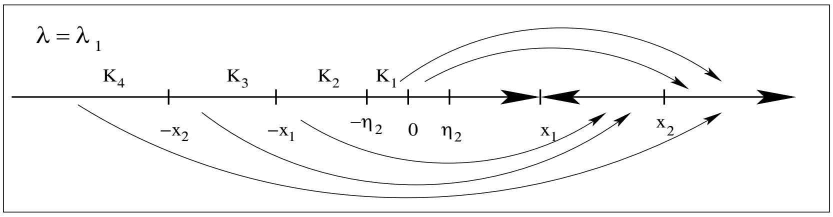
The dynamics of the functions $\xi_\lambda(z)$ in one parameter family $\mathcal{H}$ is described in the following:

5.1. Dynamics of $\xi_\lambda \in \mathcal{H}$ for $0 < \lambda \leqslant \lambda_1$

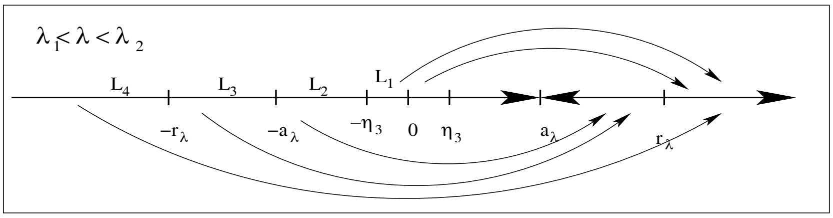
The dynamics of the function $\xi_\lambda(z)$ for $z \in \hat{\mathbb{C}}$ and $0 < \lambda \leqslant \lambda_1$ is investigated here, where λ_1 is defined by (3.3). The characterization of the Julia set of $\xi_\lambda(z)$ in this case as the closure of the set of all escaping points of $\xi_\lambda(z)$ is found in the following:



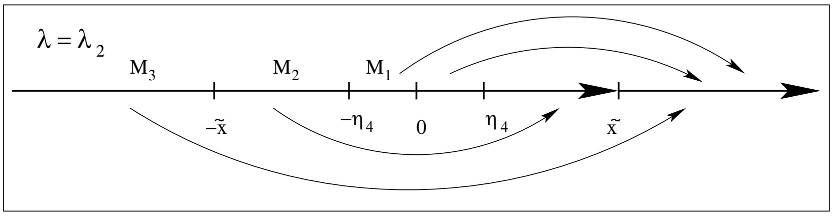
(a) $I_1 = (-\eta_1, 0)\backslash T_0$, $I_2 = (-r_{1,\lambda}, -\eta_1)\backslash T_0$, $I_3 = (-r_{2,\lambda}, -r_{1,\lambda})\backslash T_0$ and $I_4 = (-\infty, -r_{2,\lambda})\backslash T_0$



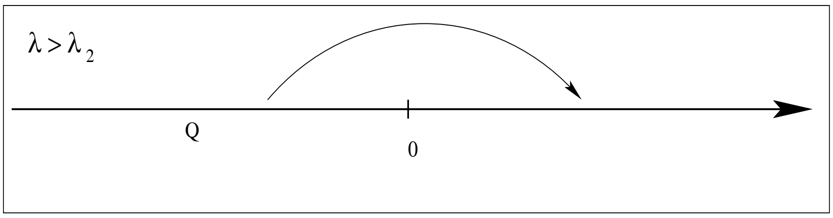
(b) $K_1 = (-\eta_2, 0)\backslash T_0$, $K_2 = (-x_1, -\eta_2)\backslash T_0$, $K_3 = (-x_2, -x_1)\backslash T_0$ and $K_4 = (-\infty, -x_2)\backslash T_0$



(c) $L_1 = (-\eta_3, 0)\backslash T_0$, $L_2 = (-a_\lambda, -\eta_3)\backslash T_0$, $L_3 = (-r_\lambda, -a_\lambda)\backslash T_0$ and $L_3 = (-\infty, -r_\lambda)\backslash T_0$



(d) $M_1 = (-\eta_4, 0)\backslash T_0$, $M_2 = (-\tilde{x}, -\eta_4)\backslash T_0$ and $M_3 = (-\infty, -\tilde{x})\backslash T_0$



(e) $Q = (-\infty, 0)\backslash T_0$

Fig. 4. Phase Portraits of the function $\xi_\lambda(x) = \lambda \dfrac{\sinh^2 x}{x^4}$ for $x \in R\backslash\{0\}$ and $\lambda > 0$

Theorem 5.1. *Let $\xi_\lambda \in \mathcal{T}$ and the set of escaping points of the function $\xi_\lambda(z)$ be defined by $I(\xi_\lambda) = \{z \in \mathbb{C} : \xi_\lambda^n(z) \to \infty$ as $n \to \infty$ and $\xi_\lambda^n(z) \neq \infty\}$. If $0 < \lambda \leqslant \lambda_1$, then the Julia set $J(\xi_\lambda) = \overline{I(\xi_\lambda)}$.*

Proof.

By Proposition 2.1, all singular values of $\xi_\lambda(z)$ are bounded. It now follows by Theorem 1.1 that the Julia set $J(\xi_\lambda) = \overline{I(\xi_\lambda)}$. ■

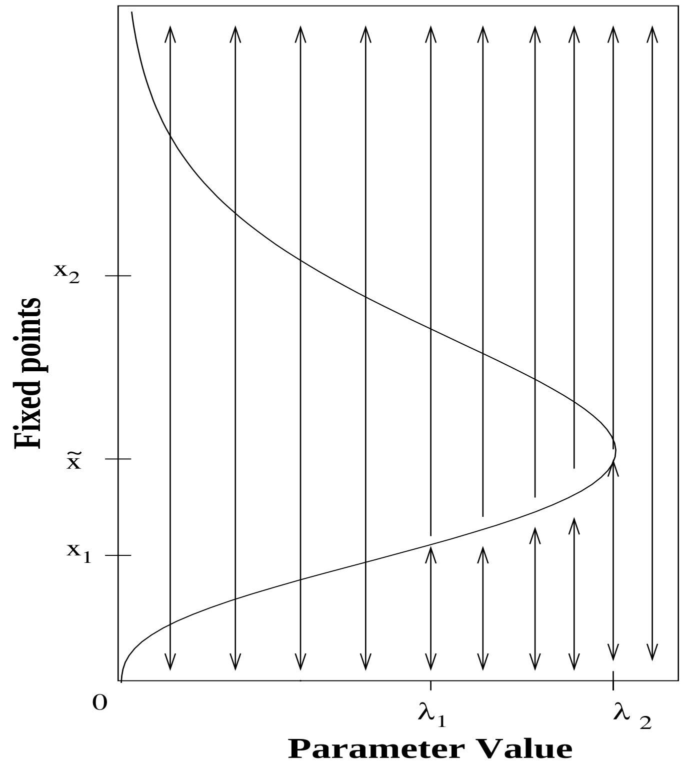


Fig. 5. Bifurcation diagram for the function $\xi_\lambda(x) = \lambda \dfrac{\sinh^2 x}{x^4}$, $\lambda > 0$

The following proposition shows that for $0 < \lambda < \lambda_1$ the Julia set $J(\xi_\lambda)$ contains both the real and imaginary axes.

Proposition 5.1. *Let $\xi_\lambda \in \mathcal{H}$ and $0 < \lambda < \lambda_1$. Then, the Julia set $J(\xi_\lambda)$ contains both real and imaginary axes.*

Proof.
By Theorem 4.1(a), $\xi_\lambda^n(x) \to \infty$ for $x \in [(-\infty, -r_{2,\lambda}) \cup (-\eta_1, 0) \cup (0, \eta_1) \cup (r_{2,\lambda}, \infty)] \backslash T_0$ and the orbits $\{\xi_\lambda^n(x)\}$ are chaotic for $x \in [(-r_{2,\lambda}, -r_{1,\lambda}) \cup (-r_{1,\lambda}, -\eta_1) \cup (\eta_1, r_{1,\lambda}) \cup (r_{1,\lambda}, r_{2,\lambda})] \backslash T_0$, it follows that $\mathbb{R} \backslash T_0 \subset J(\xi)$. Since $\xi_\lambda(x)$ maps imaginary axis on real axis and $\xi_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R} \backslash T_0$, it gives that $i\mathbb{R} \backslash iT_0 \subset J(\xi)$. Also, since 0 is an asymptotic value which is also a pole, $0 \in J(\xi)$ and since preimages of pole are contained in Julia set, the set $T_0 \subset J(\xi)$. Therefore, $J(\xi)$ contains both real and imaginary axes. ∎

Next, the dynamics of the function $\xi_\lambda(z)$ for $z \in \hat{\mathbb{C}}$ and $\lambda = \lambda_1$ is described. The following proposition shows that in this case the Fatou set of $\xi_\lambda(z)$ contains a unique parabolic domain:

Proposition 5.2. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda = \lambda_1$. Then, the Fatou set $F(\xi_\lambda)$ contains a unique parabolic domain.*

Proof.
Let $U_1 = \{z \in \mathbb{C} : \xi_\lambda^n(z) \to x_1 \text{ as } n \to \infty\}$. By Theorem 3.1(ii), it follows that $\xi_\lambda(z)$ has a rationally indifferent fixed point at $x = x_1$. Since, by Theorem 4.1(b), $\xi_\lambda^n(x) \to x_1$ as $n \to \infty$ for $x \in (x_1, x_2) \backslash T_0$ and $\xi_\lambda^n(x) \to \infty$ for $x \in (0, x_1)$, the rationally indifferent fixed point x_1 lies on the boundary of U_1. Thus, U_1 is a parabolic domain in the Fatou set of $\xi_\lambda(z)$.

Again, by Theorem 4.1(b), it follows that the forward orbits of all singular values either tend to x_1 or tend to ∞. Therefore, by Theorem 1.2, $F(\xi_\lambda)$ does not contain any parabolic domain other than U_1. ∎

In the following proposition, it is found that Julia and Fatou sets of $\xi_\lambda \in \mathcal{H}$ contain certain intervals of real line for $\lambda = \lambda_1$:

Proposition 5.3. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda = \lambda_1$. Then, the Julia set contains the intervals $(-\infty, -x_2)$, $(-x_1, 0)$, $(0, x_1)$ and (x_2, ∞) and the Fatou set contains the intervals $(-x_2, -x_1)\backslash T_0$ and (x_1, x_2), where x_1 is a rationally indifferent fixed point, x_2 is a repelling fixed point of $\xi_\lambda(x)$ and η_2 is a positive solution of $\xi_\lambda(x) = x_2$.*

Proof.

By Theorem 4.1(b), for $\lambda = \lambda_1$, $\xi_\lambda^n(x) \to \infty$ for $x \in [(-\infty, -x_2) \cup (-x_1, 0) \cup (0, x_1) \cup (x_2, \infty)]\backslash T_0$ and $\xi_\lambda^n(x) \to x_1$ for $x \in [(-x_2, -x_1) \cup (x_1, x_2)]\backslash T_0$. Therefore, the intervals $(-x_2, -x_1)\backslash T_0$ and (x_1, x_2) are contained in the parabolic domain U_1. Since Fatou set contains parabolic domains, the intervals $(-x_2, -x_1)\backslash T_0$ and (x_1, x_2) belong to the Fatou set. Further, by Theorem 5.1, the intervals $(-\infty, -x_2)\backslash T_0$, $(-x_1, 0)\backslash T_0$, $(0, x_1)$ and (x_2, ∞) are contained in the Julia set of $\xi_\lambda(z)$. Since pole and preimages of the pole also belong to the Julia set, it now follows that Julia set are contained the intervals $(-\infty, -x_2)$, $(-x_1, 0)$, $(0, x_1)$ and (x_2, ∞). ■

5.2. Dynamics of $\boldsymbol{\xi_\lambda \in \mathcal{H}}$ for $\boldsymbol{\lambda_1 < \lambda \leqslant \lambda_2}$ and $\boldsymbol{\lambda > \lambda_2}$

The present subsection is devoted to the investigation of the dynamics of the function $\xi_\lambda(z)$ for $z \in \hat{\mathbb{C}}$, $\lambda_1 < \lambda < \lambda_2$, $\lambda = \lambda_2$ or $\lambda > \lambda_2$ is described, where λ_1 and λ_2 are defined by (3.3) and (3.2). The characterization of the Julia set of $\xi_\lambda(z)$ as the closure of the set of all escaping points of $\xi_\lambda(z)$ for $\lambda_1 < \lambda \leqslant \lambda_2$ is given by the following:

Theorem 5.2. *Let $\xi_\lambda \in \mathcal{T}$ and the set of escaping points of the function $\xi_\lambda(z)$ be defined by $I(\xi_\lambda) = \{z \in \mathbb{C} : \xi_\lambda^n(z) \to \infty$ as $n \to \infty$ and $\xi_\lambda^n(z) \neq \infty\}$. If $\lambda_1 < \lambda \leqslant \lambda_2$, then the Julia set $J(\xi_\lambda) = \overline{I(\xi_\lambda)}$.*

Proof.

The proof of theorem is analogous to that of Theorem 5.2 for the case $0 < \lambda \leqslant \lambda_1$ and is hence omitted. ■

By Theorem 3.1(iii), $\xi_\lambda(z)$ has a real attracting fixed point a_λ. Let

$$A(a_\lambda) = \{z \in \mathbb{C} : \xi_\lambda^n(z) \to a_\lambda \text{ as } n \to \infty\}$$

be the basin of attraction of the attracting fixed point a_λ of $\xi_\lambda(z)$ for $\lambda_1 < \lambda < \lambda_2$. Our next theorem shows that, in this case, the Fatou set of $\xi_\lambda(z)$ does not contain any other basin of attraction of attracting fixed point a_λ except $A(a_\lambda)$:

Theorem 5.3. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda_1 < \lambda < \lambda_2$. Then, the Fatou set $F(\xi_\lambda)$ does not contain any basin of attraction or parabolic domain except the basin of attraction $A(a_\lambda)$ of the real attracting fixed point a_λ of $\xi_\lambda(z)$.*

Proof.

For any point $z \in A(a_\lambda)$, the sequence of iterates $\{\xi_\lambda^n(z)\}$ tends to a_λ as $n \to \infty$ so that the sequence of iterates $\{\xi_\lambda^n(z)\}$ forms a normal family at z. Consequently, $z \in F(\xi_\lambda)$. Thus, $A(a_\lambda) \subset F(\xi_\lambda)$.

Further, by Theorem 4.1(c), it follows that the forward orbits of all singular values either tend to a_λ or tend to ∞. Therefore, by Theorem 1.2, $F(\xi_\lambda)$ does not contain the basin of attractions other than $A(a_\lambda)$. That $F(\xi_\lambda)$ does not contain any parabolic domains follows similarly using Theorem 1.2. ■

Remark 5.1. The conditions [22] for existence of wandering domains and Baker domains are not easily verifiable for the function $\xi_\lambda(z)$, therefore the existence of wandering or Baker domains in the Fatou set of $\xi_\lambda(z)$ are not ruled out.

In the following proposition, it is shown that Julia set and Fatou set of $\xi_\lambda \in \mathcal{H}$ contain certain intervals of real line.

Proposition 5.4. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda_1 < \lambda < \lambda_2$. Then, the Julia set $J(\xi_\lambda)$ contains the intervals $(-\infty, -r_\lambda)$, $(-\eta_3, 0)$, $(0, \eta_3)$ and (r_λ, ∞) and the Fatou set $F(\xi_\lambda)$ contains the intervals $(-r_\lambda,\ -\eta_3)\backslash T_0$ and $(\eta_3,\ r_\lambda)$, where r_λ is a repelling fixed point of $\xi_\lambda(x)$ and η_3 is a positive solution of $\xi_\lambda(x) = r_\lambda$.*

Proof.
For $\lambda_1 < \lambda < \lambda_2$, by Theorem 4.1(c), $\xi_\lambda^n(x) \to \infty$ as $n \to \infty$ for $x \in [(-\infty, -r_\lambda) \cup (-\eta_3, 0) \cup (0, \eta_3) \cup (r_\lambda, \infty)]\backslash T_0$. Therefore, by Theorem 5.2, the intervals $(-\infty, -r_\lambda)\backslash T_0$, $(-\eta_3, 0)]\backslash T_0$, $(0, \eta_3)$ and (r_λ, ∞) belong to the Julia set of $\xi_\lambda(z)$. Since pole and preimages of the pole lie in the Julia set, it now follows that the Julia set contains the intervals $(-\infty, -r_\lambda)$, $(-\eta_3, 0)$, $(0, \eta_3)$ and (r_λ, ∞). Again, by Theorem 4.1(c), $\xi_\lambda^n(x) \to a_\lambda$ as $n \to \infty$ for $x \in [(-r_\lambda, -\eta_3) \cup (\eta_3, r_\lambda)]\backslash T_0$, where r_λ is a repelling fixed point of $\xi_\lambda(x)$. Therefore, it follows that the intervals $(-r_\lambda, -\eta_3)\backslash T_0$ and (η_3, r_λ) are contained in the basin of attraction $A(a_\lambda)$ of the attracting fixed point $x = a_\lambda$ of $\xi_\lambda(z)$ for $\lambda_1 < \lambda < \lambda_2$. Since the Fatou set contains basin of attractions, the intervals $(-r_\lambda, -\eta_3)\backslash T_0$ and (η_3, r_λ) belong to the Fatou set of $\xi_\lambda(z)$. ■

The following proposition describes the dynamics of function $\xi_\lambda(z)$ for $\lambda = \lambda_2$, $z \in \hat{\mathbb{C}}$ and shows that the Fatou set of $\xi_\lambda(z)$ contains a parabolic domain in this case:

Proposition 5.5. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda = \lambda_2$. Then, the Fatou set $F(\xi_\lambda)$ contains a unique parabolic domain.*

Proof.
Let $U = \{z \in \mathbb{C} : \xi_\lambda^n(z) \to \tilde{x} \text{ as } n \to \infty\}$. By Theorem 3.1(iv), it follows that $\xi_\lambda(z)$ has a rationally indifferent fixed point at $x = \tilde{x}$. Since, by Theorem 4.1(d), $\xi_\lambda^n(x) \to \tilde{x}$ as $n \to \infty$ for $x \in (\eta_4, \tilde{x})\backslash T_0$ and $\xi_\lambda^n(x) \to \infty$ for $x \in (\tilde{x}, \infty)$, the rationally indifferent fixed point $\tilde{x}$ lies on the boundary of U. Thus, U is a parabolic domain in the Fatou set of $\xi_\lambda(z)$.

By Theorem 4.1(d), it follows that the forward orbits of all singular values either tend to $\tilde{x}$ or tend to ∞. Therefore, by Theorem 1.2, the Fatou set $F(\xi_\lambda)$ does not contain any parabolic domain other than U. ■

For $\lambda = \lambda_2$ also, the Julia set and Fatou set of $\xi_\lambda \in \mathcal{H}$ contain certain intervals of real line as seen by the following:

Proposition 5.6. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda = \lambda_2$. Then, the Julia set $J(\xi_\lambda)$ contains the intervals $(-\infty, -\tilde{x})$, $(-\eta_4, 0)$, $(0, \eta_4)$ and $(\tilde{x}, \infty)$ and the Fatou set $F(\xi_\lambda)$ contains the intervals $(-\tilde{x}, -\eta_4)\backslash T_0$ and $(\eta_4, \tilde{x})$, where $\tilde{x}$ is a rationally indifferent fixed point of $\xi_\lambda(x)$ and η_4 is a positive solution of $\xi_\lambda(x) = \tilde{x}$.*

Proof.
By Theorem 4.1, for $\lambda = \lambda_2$, $\xi_\lambda^n(x) \to \infty$ for $x \in [(-\infty, -\tilde{x}) \cup (-\eta_4, 0) \cup (0, \eta_4) \cup (\tilde{x}, \infty)]\backslash T_0$ and $\xi_\lambda^n(x) \to \tilde{x}$ for $x \in [(-\tilde{x}, -\eta_4) \cup (\eta_4, \tilde{x})]\backslash T_0$. Therefore, by Theorem 5.2, the intervals $(-\infty, -\tilde{x})\backslash T_0$, $(-\eta_4, 0)\backslash T_0$, $(0, \eta_4)$ and $(\tilde{x}, \infty)$ belong to the Julia set of $\xi_\lambda(z)$. Since pole and preimages of the pole are also contained in the Julia set, it now follows that the intervals $(-\infty, -\tilde{x})$, $(-\eta_4, 0)$, $(0, \eta_4)$ and $(\tilde{x}, \infty)$ belong to Julia set of $\xi_\lambda(z)$. Again, by Theorem 4.1, the intervals $(-\tilde{x}, -\eta_4)\backslash T_0$ and $(\eta_4, \tilde{x})$ are contained in the parabolic domain U. Since the Fatou set contains the parabolic domains, the intervals $(-\tilde{x}, -\eta_4)\backslash T_0$ and $(\eta_4, \tilde{x})$ are contained in the Fatou set $\xi_\lambda(z)$. ■

The following theorem describes the dynamics of the function $\xi_\lambda(z)$ for $z \in \hat{\mathbb{C}}$ and $\lambda > \lambda_2$ showing that the Julia set $J(\xi_\lambda)$ contains the real and imaginary axes.

Theorem 5.4. *Let $\xi_\lambda \in \mathcal{H}$ and $\lambda > \lambda_2$. Then, the Julia set $J(\xi_\lambda)$ contains both real and imaginary axes.*

Proof.
By Theorem 4.1(e), $\xi_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R}\backslash T_0$, it follows that $\mathbb{R}\backslash T_0 \subset J(\xi_\lambda)$. Since $\xi_\lambda(x)$ maps imaginary axis on real axis and $\xi_\lambda^n(x) \to \infty$ for all $x \in \mathbb{R}\backslash T_0$, it gives that $i\mathbb{R}\backslash iT_0 \subset J(\xi_\lambda)$. Since the

asymptotic value 0 is also a pole of $\xi_\lambda(z)$, $0 \in J(\xi_\lambda)$ and since preimages of pole are contained in Julia set, $T_0 \subset J(\xi)$. Therefore, $J(\xi_\lambda)$ contains both real and imaginary axes. ■

Remark 5.2.

(i) Since $\xi_\lambda^n(x) \to \infty$ for all $x \in R \backslash T_0$, the forward orbit of critical values on real axis tend to ∞. Further, the asymptotic value 0 is also a pole of $\xi_\lambda(z)$ so that orbit of 0 terminates. Therefore, Fatou set can not have any basin of attraction, parabolic domain, Siegel disks or Herman rings [7] for $\lambda > \lambda_2$.

(ii) The Julia set $J(\xi_\lambda)$ for the case considered in Sections 5.1 and 5.2 contain Cantor bouquets, provided $J(\xi_\lambda)$ is shown not to be the whole complex plane [10].

6. Applications and comparisons

The computer images of the Julia sets of the function $\xi_\lambda \in \mathcal{H}$ are generated by the following algorithm based on Theorems 5.1 and 5.2:

i. Select a window W in the plane and divide the window W into $k \times k$ grids of width d.

ii. For each grid point (i.e. pixel), compute the orbit upto a maximum of N iterations.

iii. If, at $i < N$, the modulus of the orbit is greater than some given bound M, the original pixel is colored black and the iterations are stopped.

iv. If no pixel in the modulus of the orbit ever becomes greater than M, the original pixel is left as white.

Thus, in the output generated by this algorithm, the black points represent the Julia set of $\xi_\lambda(z)$ and the white points represent the Fatou set of $\xi_\lambda(z)$.

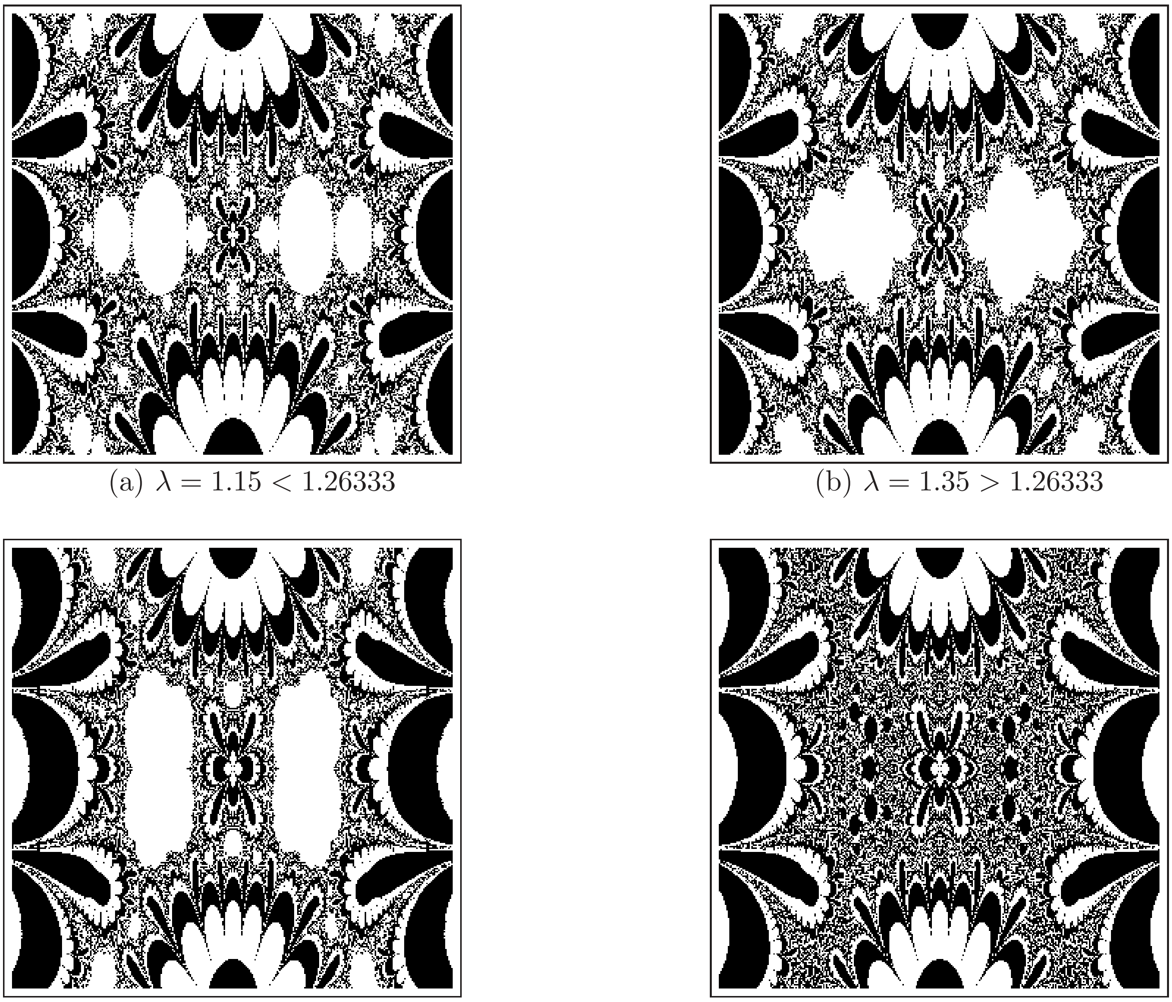
The Julia sets of a function $\xi_\lambda(z) = \lambda \dfrac{\sinh^2 z}{z^4}$, $\lambda = 1.15$, $\lambda = 1.35$, $\lambda = 2.6$ and $\lambda = 2.7$ are generated in the rectangular domain $R = \{z \in \mathbb{C} : -6 \leqslant \mathrm{Re}(z) \leqslant 6 \text{ and } -3 \leqslant \mathrm{Im}(z) \leqslant 3\}$ and resulting images of the Julia sets are shown in Fig. 6. To generate these images, for each grid point in the rectangle R the maximum number of iterations $N = 200$ is allowed for a possible escape of the bound $M = 100$.

The image of Julia set of the function $\xi_\lambda(z)$ for $\lambda = 1.15$ is shown in Fig. 6(a). It is found that the Julia sets of the function $\xi_\lambda(z)$ for all λ satisfying $0 < \lambda < \lambda_1$ have the same pattern as that of the Julia set of $\xi_\lambda(z)$ for $\lambda = 1.15$. This gives a visualization of Theorems 5.1 for $0 < \lambda < \lambda_1$. The nature of image of the Julia set of $\xi_\lambda(z)$ for $\lambda = 1.35 (> \lambda_1 = 1.26333)$ and $\lambda = 2.6 (< \lambda_2 = 2.7.715)$ given by Figs. 6(b) and (c) have the same pattern as those of the Julia sets of the function $\xi_\lambda(z)$ for any other λ satisfying $\lambda_1 < \lambda < \lambda_2$ for a fixed bound $M = 100$. It is also observed in Figs. 6(b) that taming effect occurs in the Julia set at $\lambda = \lambda_1$. This conforms to the result of Theorems 5.2 for $\lambda_1 < \lambda < \lambda_2$. The nature of image of the Julia set of the function $\xi_\lambda(z)$ for $\lambda = 2.7 (> 2.7.715)$ given in Figs. 6(d) shows a distinct change giving a significantly large number of black points in Julia set in comparison to the Julia set of $\xi_\lambda(z)$ for $\lambda_1 < \lambda < \lambda_2$. This demonstrates the chaotic behaviour in Julia set $J(\xi_\lambda)$ when the parameter value crosses λ_2. The Julia sets of $\xi_\lambda(z)$ for all λ satisfying $\lambda > \lambda_2$ is found to have the same pattern as that of Julia set of $\xi_\lambda(z)$ for $\lambda = 2.7 (> \lambda_2 = 2.7.715)$. This conforms to the result of Theorems 5.2.

Finally, Table 1 gives a comparison between the dynamical properties of non-critically finite even transcendental meromorphic function $\xi_\lambda(z) = \lambda \dfrac{\sinh^2 z}{z^4}$, $\lambda > 0$ with the recent results of Devaney and Keen [10], Keen and Kotus [15] and Stallard [19] obtained for the dynamics of critically finite odd transcendental meromorphic function $T_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{C} \backslash \{0\}$ and the recent results on the dynamics of non-critically finite transcendental entire function $E_\lambda(z) = \lambda \dfrac{e^z - 1}{z}$, $\lambda > 0$ [14].

Table 1. Comparison of dynamical properties of functions $\xi_\lambda(z) = \dfrac{\lambda \sinh^2 z}{z^4}$, $\lambda > 0$, $T_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{C}\backslash\{0\}$ and $E_\lambda(z) = \dfrac{\lambda(e^z - 1)}{z}$, $\lambda > 0$

$\xi_\lambda(z) = \lambda \dfrac{\sinh^2 z}{z^4}$, $\lambda > 0$	$T_\lambda(z) = \lambda \tan z$, $\lambda \in \mathbb{C}\backslash\{0\}$	$E_\lambda(z) = \lambda \dfrac{e^z - 1}{z}$, $\lambda > 0$
$\xi_\lambda(z)$ is an even function.	$T_\lambda(z)$ is an odd function.	$E_\lambda(z)$ is neither even nor odd function.
$\xi_\lambda(z)$ has infinitely many critical values.	$T_\lambda(z)$ has no critical values.	$E_\lambda(z)$ has infinitely many critical values.
$\xi_\lambda(z)$ has only one finite asymptotic value 0.	$T_\lambda(z)$ has two finite asymptotic values $\pm\lambda i$.	$E_\lambda(z)$ has only one finite asymptotic value 0.
All singular values of $\xi_\lambda(z)$ are bounded.	All singular values of $T_\lambda(z)$ are bounded.	All singular values of $E_\lambda(z)$ are bounded.
The Schwarzian Derivative of $\xi_\lambda(z)$ is transcendental meromorphic function.	The Schwarzian Derivative of $T_\lambda(z)$ is polynomial.	The Schwarzian Derivative of $E_\lambda(z)$ is transcendental meromorphic function.
The bifurcations occur in the dynamics of $\xi_\lambda(z)$ at two critical parameter values $\lambda_1 \approx 1.26333$ and $\lambda_2 \approx 2.7.715$.	The bifurcation occurs in the dynamics of $T_\lambda(z)$ at only one critical parameter value $\lambda^* = = 1$.	The bifurcation occurs in the dynamics of $E_\lambda(z)$ at only one critical parameter value $\lambda_E \approx 0.647.7$.
For any $\lambda > 0$, $F(\xi_\lambda)$ is not equal to the basin of attraction of the real attracting fixed point.	For $\|\lambda\| < 1$, $F(T_\lambda)$ equals the basin of attraction of the real attracting fixed point 0.	For $0 < \lambda < \lambda_E$, $F(E_\lambda)$ equals the basin of attraction of the real attracting fixed point.
Fatou set $F(\xi_\lambda)$ contains parabolic domains for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.	Fatou set $F(T_\lambda)$ does not contain essentially parabolic domains but it contains eventually parabolic domains.	Fatou set $F(E_\lambda)$ contains parabolic domains for $\lambda = \lambda_E$.
Julia set $J(\xi_\lambda)$ contains both real and imaginary axes for $0 < \lambda < \lambda_1$ and $\lambda > \lambda_2$.	$J(T_\lambda)$ contains the real axis for $\lambda > 1$.	$J(E_\lambda)$ contains the real axis for $\lambda > \lambda_E$.
Julia set $J(\xi_\lambda)$ is not the whole complex plane for any $\lambda > 0$.	Julia set $J(T_\lambda)$ is the whole complex plane for $\lambda = i\pi$.	Julia set $J(E_\lambda)$ is not the whole complex plane for any $\lambda > 0$.
Herman rings and wandering domains may exist.	Herman rings and wandering domains do not exist.	Herman rings and wandering domains do not exist for $0 < \lambda < \lambda_E$.
Julia set $J(\xi_\lambda)$ is the closure of escaping points for all $\lambda > 0$.	Julia set $J(T_\lambda)$ is the closure of escaping points for all $\lambda > 0$.	Julia set $J(E_\lambda)$ is the closure of escaping points for all $\lambda > 0$.



(a) $\lambda = 1.15 < 1.26333$

(b) $\lambda = 1.35 > 1.26333$

(c) $\lambda = 2.6 < 2.7.715$

(d) $\lambda = 2.7 > 2.7.715$

Fig. 6. Julia sets of the function $\xi_\lambda(z) = \lambda \dfrac{\sinh^2 z}{z^4}$ for different values of parameter λ

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