Synopsis

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Among the major recent developments in understanding the structures of objects found in nature, the notion of fractals occupies an important place. Since the introduction of the term 'Fractal' by Mandelbrot [Fractals: Form, Chance and Dimension, W. H. Freeman, 1977], a growing number of research papers have been published showing the fractal character of many systems with different physical properties. A fractal set has a highly irregular structure while it is union of many smaller copies of itself. The theory of fractal interpolation has become a powerful tool in applied sciences and engineering since Barnsley [Constr. Approx. 2, 303-329, 1986] introduced Fractal Interpolation Function (FIF) using Hutchinson's operator[Indiana Univ. Math. J. 30, 713-747, 1981] and certain Iterated Function System (IFS) whose attractor is graph of a continuous function interpolating a given set of points. FIFs are generally self-affine in nature. Barnsley et.al. [SIAM J. Math. Anal. 20(5), 1218-1242, 1989] extended the idea of FIF to produce Hidden-variable FIF which are non-self-affine curves. To simulate curves that exhibit partly self-affine and partly non-self-affine nature, Chand and Kapoor [Int. J. Nonlinear Sci. 3, 15-26, 2007] constructed a Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) depending on free variables and constrained variables. A brief review of these basics and existing contributions on fractals related to our work are given in Chapter 1.

The theory of multiresolution analysis, introduced by Mallat [Trans. Amer. Math. Soc. 315, 69–87, 1989], provides a powerful method to construct wavelets having far reaching applications in analyzing signals and images. This has led to investigations on multiresolution analysis of $L_2(\mathbb{R})$ based on various suitable functions in several contemporary researches. Hardin et.al. [J. Approx. Th. 71, 104–120, 1992] investigated multiresolution analysis of $L_2(\mathbb{R})$ based on FIFs. However, multiresolution analysis of $L_2(\mathbb{R})$ based on Coalescence Hidden-variable Fractal Interpolation Function (CHFIF) which exhibits both self-affine and non-self-affine nature has hitherto remained unexplored. In the present work, such a multiresolution analysis is accomplished in Chapter 2. The availability of a larger set of free variables and constrained variables with CHFIF in multiresolution analysis based on CHFIFs provides more control in reconstruction of functions in $L_2(\mathbb{R})$ than that provided by multiresolution analysis based only on affine FIFs. In our approach, first the vector space of CHFIFs is introduced and its dimension is determined. Using this result, Riesz bases of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, consisting of certain CHFIFs in $L_2(\mathbb{R}) \bigcap C_b(\mathbb{R})$ are constructed. The multiresolution analysis of $L_2(\mathbb{R})$ is then carried out in terms of nested sequences of vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$. As a special case, for the vector space of CHFIFs of dimension 4, an orthogonal basis consisting of dilations and translations of scaling functions, for the vector subspaces $\mathbb{V}_k, k \in \mathbb{Z}$, is explicitly constructed in the present chapter. As a natural follow-up, the orthogonality of these scaling functions is used to construct compactly supported continuous orthonormal wavelets.

Although, FIF and CHFIF are important tools in the study of highly uneven curves arising as seismic fractures, lightening, ECG, etc., they cannot be applied for the study of highly uneven surfaces such as surfaces of metals, terrains, planets, living organisms, and many other naturally occurring objects for which the generating function depends on more than one variable. Chand and Kapoor [Fractals 11(3), 227-288, 2003] constructed Coalescence Hidden-variable Fractal Interpolation Surfaces (CHFIS) which, depending on choices of free variables and constrained variable, exhibit both self-affine and non-self-affine nature in its various parts. An open problem, as observed in the above work, is to find the effect of free variables and constrained variables on smoothness and fractal dimension of a CHFIS. In the present work, the contents of Chapter 3 provide a satisfactory solution to this problem. It is shown here that the smoothness of a CHFIS depends on free variables as well as on Lipschitz exponents of certain bivariate polynomials and a CHFIS belongs to a certain class of Lipchitz functions under appropriate conditions. The results obtained in this chapter are illustrated through interpolation data generated from certain sample surfaces for which (i) CHFISs are computationally constructed for various sets of suitably chosen parameter values and (ii) Lipschitz exponents of constructed CHFISs are computed to visualize their effect on the resulting smoothness of the simulated surfaces.

It is important to know while conducting experiments in various science and engineering disciplines as to how the governing CHFIS varies with slight variations in the interpolation data. A contribution is made in this direction in Chapter 4 of the present work by establishing the stability of CHFIS. Our stability results, first found individually for perturbations in independent variables, the dependent variable and the hidden variable, comprise of estimates on errors in approximation of the data generating function by a CHFIS. These estimates together lead to the estimate on cumulative error in approximation of the data generating function by a CHFIS, when there are simultaneous perturbations in all of the above variables. Our stability results found here are illustrated through computational simulations of a data generating sample surface and the surfaces corresponding to perturbed data obtained by small variations in concerned variables.

The mathematical models and experimental studies show that the estimation of fractal dimension, having its origin in the works of Kolmogorov [Dokl. Akad. Nauk SSSR 119(5), 861–864, 1958], is a sophisticated tool in the study of natural objects and structures. The determination of estimates on fractal dimension of surfaces exhibiting both self-affine and non-self-affine nature is hitherto unexplored. An attempt is made in the present work in this direction by finding the bounds on fractal dimension of a CHFIS in first few sections of Chapter 5. These bounds give conditions on free variables so that the fractal dimension of the constructed CHFIS is close to 3. Also, these bounds give a range on free variables, confining to which forces the fractal dimension of the constructed CHFIS to be strictly greater than 2. As a test case, a Tsunami wave surface at a static moment is considered and the bounds on fractal dimension, found in this chapter, are computed for the simulated CHFIS to substantiate our results. Another important tool for the applications of Fractals in real world problems is computation of its Integral Moments. To facilitate such a computation while generating fractals based on CHFIS, a recursive formula for integral moments of a CHFIS is obtained in a later section of this chapter. This recursive formula is then employed to find an explicit representation of inner product of CHFIS in terms of integrals of certain bivariate polynomials and free variables, which is likely to be highly useful in the study of multiresolution analysis and multi-wavelets arising from CHFIS.

Although, deterministic fractals are used to model natural objects, it is often observed that the nature in fact exhibits local variations at small scales. Barnsley, Hutchinson and Stenflo [Fractals 13(2), 111–146, 2005] introduced the class of V-variable fractals, where the integer parameter V controls the number of distinct shapes and forms at each level of magnification. However, for a data set arising from nature, a solution of fractal interpolation problem based on several IFS has not been investigated so far. A solution to such an interpolation problem is found in Chapter 6 of the present work by introducing the notion of Super Fractal Interpolation Function (SFIF) based on multiple IFSs. At each level of iteration, an IFS is chosen randomly, allowing variability in structure and better geometrical modeling of nature. Further, the integral, the smoothness and sufficient conditions for differentiability of a SFIF are explored in this chapter. It is proved here that for a SFIF passing through a given interpolation data, its integral is also a SFIF, albeit for a different interpolation data. The smoothness of a SFIF, described in terms of its Lipschitz exponent, is seen to depend on free variables as well as on the extent of smoothness of certain polynomials. Certain sufficient conditions for existence of derivatives of a SFIF are also found in this chapter. Further, motivated by vast applications of classical splines in solutions of various science and engineering problems, the notion of Cubic Spline SFIF is introduced in the present chapter and its approximation properties are investigated. The convergence results for Cubic Spline SFIF found here show that any desired accuracy can be achieved in the approximation of a regular data generating function and its derivatives by a Cubic Spline SFIF and its corresponding derivatives respectively.