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Asymptotic theory of least squares estimator of a particular nonlinear regression model

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Abstract: The consistency and asymptotic normality of the least squares estimator are derived for a particular non-linear regression model, which does not satisfy the standard sufficient conditions of Jennrich (1969) or Wu (1981), under the assumption of normal errors.

Keywords: Consistency; least squares estimator; non-linear regression.

1. Introduction

The least squares method plays an important role in drawing inferences about the parameters in the non-linear regression model. Jennrich (1969) first rigorously proved the existence of the least squares estimator and showed its consistency of the following non-linear model:

$$y_t = f_t(\theta_0) + \varepsilon_t, \quad t = 1, 2, \ldots. \tag{1.1}$$

Jennrich proved the strong consistency of the least squares estimator $\hat{\theta}_n$ under the following assumption: $F_n(\theta_1, \theta_2)$ converges uniformly to a continuous function $F(\theta_1, \theta_2)$ for all θ_1 and θ_2 and $F(\theta_1, \theta_2) = 0$ if and only if $\theta_1 = \theta_2$, where

$$F_n(\theta_1, \theta_2) = \frac{1}{n} \sum_{t=1}^{n} \left(f_t(\theta_1) - f_t(\theta_2)\right)^2. \tag{1.2}$$

Under some stronger assumptions, asymptotic normality was proved in the same paper. Wu (1981) gave some sufficient conditions under which the least squares estimator converges to θ_0 almost surely, when the growth rate requirement of F_n is replaced by a Lipschitz type condition on the sequence f_i.

We consider the non-linear regression model

$$y_t = \cos(2\pi t \theta_0) + \varepsilon_t, \quad t = 1, 2, \ldots, \tag{1.3}$$

where $\{\varepsilon_t\}$ are i.i.d. normal random variables with mean zero and finite positive variance σ^2. θ_0 is an

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interior point of [0, 0.5]. In this situation $F_n(\theta_1, \theta_2)$ doesn't converge uniformly to a continuous function nor does it satisfy Wu's Lipschitz type condition. This is an important, well studied model in time series analysis. See for example the recent work of Rice and Rosenblatt (1988) or Hannan (1973). However, the direct proof of the consistency of the least squares estimator has been established. The aim of this paper is to give a direct proof of the consistency of the least squares estimator of the model (1.3) and to show its asymptotic normality under the assumption of the Gaussian error. Some relaxation of the normality assumption of the error is possible.

2. Consistency of the least squares estimator (LSE)

Let $\hat{\theta}_n$ be a least square estimator of θ obtained by minimizing

$$Q_n(\theta) = \sum_{t=1}^{n} (y_t - \cos(2\pi t\theta))^2. \tag{2.1}$$

To prove $\hat{\theta}_n$ is a strongly consistent estimator of θ, it is enough to prove that

$$\liminf \inf_{|\theta-\theta_0|>\delta} \frac{1}{n}\{Q_n(\theta) - Q_n(\theta_0)\} > 0 \quad \text{a.s.} \tag{2.2}$$

for all $\delta > 0$ (Wu, 1981). Now

$$\begin{aligned}\frac{1}{n}(Q_n(\theta) - Q_n(\theta_0)) &= \frac{1}{n}\sum_{t=1}^{n}\left\{(y_t - \cos(2\pi t\theta))^2 - \varepsilon_t^2\right\} \\ &= \frac{1}{n}\sum_{t=1}^{n}(\cos(2\pi t\theta_0) - \cos(2\pi t\theta))^2 \\ &\quad + \frac{2}{n}\sum_{t=1}^{n}\varepsilon_t(\cos(2\pi t\theta_0) - \cos(2\pi t\theta)).\end{aligned} \tag{2.3}$$

Consider

$$\begin{aligned}&4\sum_{t=1}^{n}(\cos(2\pi t\theta_0) - \cos(2\pi t\theta))^2 \\ &\quad = \sum_{t=1}^{n}(e^{i2\pi t\theta_0} + e^{-i2\pi t\theta_0} - e^{i2\pi t\theta} - e^{-i2\pi t\theta})^2 \\ &\quad = \sum_{t=1}^{n}(4 + e^{i4\pi t\theta_0} - 2\,e^{i2\pi t(\theta+\theta_0)} + e^{-i4\pi t\theta} + e^{i4\pi t\theta} + e^{-i4\pi t\theta_0} \\ &\qquad\qquad -2\,e^{i2\pi t(\theta-\theta_0)} - 2\,e^{-i2\pi t(\theta+\theta_0)} - 2\,e^{-i2\pi t(\theta-\theta_0)}).\end{aligned} \tag{2.4}$$

Clearly

$$\liminf \inf_{|\theta-\theta_0|>\delta} \frac{1}{n}\sum_{t=1}^{n}(\cos(2\pi t\theta_0) - \cos(2\pi t\theta))^2 > 0 \tag{2.5}$$

for any fixed $\delta > 0$. Since $(1/n)\Sigma_{t=1}^{n}\varepsilon_t \cos(2\pi t\theta_0)$ converges to zero almost surely, (2.2) follows if we can prove that $\sup_\theta(1/n)\Sigma_{t=1}^{n}\varepsilon_t \cos(2\pi t\theta)$ converges to zero almost surely.

Let A_K be a set which contains the points $\{1/K^2, 2/K^2, \ldots, [\frac{1}{2}K^2]/K^2\}$ for $K = 2, 3, \ldots$. It is known that if Z is distributed as normal random variable with mean zero and variance a^2, then for large n and for any given $\varepsilon > 0$, there exist a $c > 0$, such that

$$P\{|Z| > \varepsilon n\} \leqslant \mathrm{e}^{-cn^2/a^2}. \tag{2.6}$$

Therefore given $\varepsilon > 0$, there exist a $c > 0$, such that

$$P\left\{\left|\frac{1}{n}\sum_{t=1}^{n} \varepsilon_t \cos(2\pi t\theta)\right| > \varepsilon\right\} \leqslant \mathrm{e}^{-cn} \tag{2.7}$$

and

$$P\left\{\sup_{\theta \in A_n}\left|\frac{1}{n}\sum_{t=1}^{n} \varepsilon_t \cos(2\pi t\theta)\right| > \varepsilon\right\} \leqslant n^2\,\mathrm{e}^{-cn}. \tag{2.8}$$

Since $\sum_{n=1}^{\infty} n^2\,\mathrm{e}^{-cn} < \infty$, therefore it follows from the Borell–Cantelli lemma that

$$\sup_{\theta \in A_n} \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t \cos(2\pi t\theta) \to 0 \quad \text{a.s.} \tag{2.9}$$

Now for any α in [0, 0.5] there exists a β in A_n such that $|\alpha - \beta| \leqslant n^{-2}$. Therefore

$$\begin{aligned}\frac{1}{n}\left|\sum_{t=1}^{n} \varepsilon_t(\cos(2\pi t\alpha) - \cos(2\pi t\beta))\right| &\leqslant \frac{1}{n}\sum_{t=1}^{n} |\varepsilon_t|\,|\cos(2\pi t\alpha) - \cos(2\pi t\beta)| \\ &\leqslant \frac{1}{n}2\pi\sum_{t=1}^{n} |\varepsilon_t|\,|t(\alpha-\beta)| \\ &\leqslant 2\pi\frac{1}{n^2}\sum_{t=1}^{n} |\varepsilon_t| \to 0 \quad \text{a.s.}\end{aligned} \tag{2.10}$$

Therefore we can conclude that

$$\sup_{\theta} \frac{1}{n}\sum_{t=1}^{n} \varepsilon_t \cos(2\pi t\theta) \to 0 \quad \text{a.s.} \tag{2.11}$$

Finally we can state our result as a theorem.

Theorem 2.1. *If $\hat{\theta}_n$ is a least square estimator of the non-linear regression model* (1.3) *then it is a strongly consistent estimator of θ_0.* □

3. Asymptotic normality

Since $Q_n(\theta)$ is defined as in (2.1), therefore

$$Q_n'(\hat{\theta}_n) - Q_n'(\theta_0) = (\hat{\theta}_n - \theta_0)Q''(\tilde{\theta}_n) \tag{3.1}$$

where $Q_n'(\hat{\theta}_n) = 0$, implies that

$$\hat{\theta}_n - \theta_0 = -Q_n(\theta_0)/Q_n''(\tilde{\theta}_n). \tag{3.2}$$

Since $\hat{\theta}_n$ converges to θ_0 almost surely, $\hat{\theta}_n$ also converges to θ_0. Now

$$Q_n''(\tilde{\theta}_n) = 8\pi^2 \left\{ \sum_{t=1}^{n} t^2 \left(\sin^2(2\pi t\tilde{\theta}_n) - \cos^2(2\pi t\tilde{\theta}_n) + \varepsilon_t \cos(2\pi t\tilde{\theta}_n) + \cos(2\pi t\tilde{\theta}_n) \cos(2\pi t\theta_0) \right) \right\}. \tag{3.3}$$

It can be shown similarly as before that

$$\sup_{\theta} \frac{1}{n^3} \sum_{t=1}^{n} \varepsilon_t t^2 \cos(2\pi t\theta) \to 0 \quad \text{a.s.}, \tag{3.4}$$

therefore

$$\frac{1}{n^3} Q_n''(\tilde{\theta}_n) \to 8\pi \lim_{n\to\infty} \frac{1}{n^3} \sum_{t=1}^{n} t^2 \sin^2(2\pi t\theta_0). \tag{3.5}$$

Since

$$Q_n'(\theta_0) = 4\pi \sum_{t=1}^{n} \varepsilon_t t \sin(2\pi t\theta_0), \tag{3.6}$$

we have

$$\operatorname{Var}\left(n^{-3/2} Q_n'(\theta_0)\right) = \sigma^2 16\pi^2 \frac{1}{n^3} \sum_{t=1}^{n} t^2 \sin^2(2\pi t\theta_0). \tag{3.7}$$

Since

$$\operatorname{Var}\left(n^{-3/2}(\hat{\theta}_n - \theta_0)\right) = \frac{\left[\operatorname{Var}\left(n^{-3/2} Q_n'(\theta_0)\right)\right]}{\left(Q_n''(\tilde{\theta}_n)\right)^2}, \tag{3.8}$$

from (3.2), (3.5), (3.7) and (3.8) we can conclude that

$$n^{3/2}(\hat{\theta}_n - \theta_0) \xrightarrow{d} \mathrm{N}\left(0, \frac{1}{4k\pi^2}\sigma^2\right) \tag{3.9}$$

where

$$k = \lim_{n\to\infty} \frac{1}{n^3} \sum_{t=1}^{n} t^2 \sin^2(2\pi t\theta_0).$$

4. Numerical experiments

We have performed a small simulation study to estimate the actual coverage probabilities of 95% confidence intervals. We take $n = 50$, $\sigma^2 = 1$ and $\theta_0 = 0.25$ with 1000 simulation runs. σ^2 in each case is calculated by $Q_n(\hat{\theta}_n)/n$. The confidence interval of θ_0 is constructed from (3.9). It is observed that 91.5% of time it contains the true parameter θ and the average confidence length comes out to be 0.0043.

5. Conclusions

In this note we have proved directly the strong consistency of the least squares estimator of the model (1.3) under the assumption of the normal error. But we do not require the normal error assumption except (2.6) or (2.7). In fact any other distribution which satisfies (2.6) and (2.7) can be used as the error random variable.

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