Generalizations of the Łoś-Tarski Preservation Theorem

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Joint work with
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Introduction

- Preservation theorems have been one of the earliest areas of study in classical model theory.
- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures, extensions, unions of chains, homomorphisms, etc.
- Most preservation theorems fail in the finite.
- Some preservation results recovered over special classes of finite structures, like those with bounded degree, bounded tree-width etc. (Dawar et al.)
- Homomorphism preservation theorem is true in the finite (Rossman).

Some assumptions and notations for the talk

- Assumptions:
  - First Order (FO) logic.
  - Arbitrary vocabularies (constants, predicates and functions)
  - Arbitrary structures typically, unless stated otherwise explicitly.
- Notations:
  - $\Sigma_1 = \exists^*(\ldots), \Pi_1 = \forall^*(\ldots)$
    $\Sigma_2 = \exists^*\forall^*(\ldots), \Pi_2 = \forall^*\exists^*(\ldots)$
  - $M_1 \subseteq M_2$ means M_1 is a substructure of M_2. For graphs, $\subseteq$ means *induced subgraph*.
  - U_M = universe of M.

A Brief Recap of the Related Talk in CLC 2012

Preservation under Substructures

Definition 1 (Pres. under subst.)

A sentence ϕ is said to be preserved under substructures, denoted $\phi \in PS$, if $\big((M \models \phi) \wedge (N \subseteq M)\big) \rightarrow N \models \phi$.

- E.g.: Consider $\phi = \forall x \forall y E(x, y)$ which describes the class of all cliques.
- Any induced subgraph of a clique is also a clique. Then $\phi \in PS$.
- In general, every Π_1 sentence (i.e. $\forall^*$ sentence) is in PS.

Theorem 1 (Łoś-Tarski, 1960s)

A FO sentence in PS is equivalent to a Π_1 sentence.

Preservation under substructures modulo finite cores

Definition 2

A sentence ϕ is said to be preserved under substructures modulo a finite core, denoted $\phi \in PSC_f$, if for each model M of ϕ, there is a finite subset C of U_M s.t. $\big((N \subseteq M) \wedge (C \subseteq U_N)\big) \rightarrow N \models \phi$.

- The set C is called a core of M w.r.t. ϕ. If ϕ is clear from context, we will call C as a core of M.
- For every $\phi \in PS$, for each model M of ϕ, the empty subset is a core of M. Then $PS \subseteq PSC_f$.

Example

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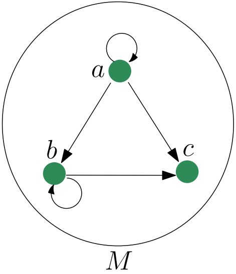

$M \models \phi$

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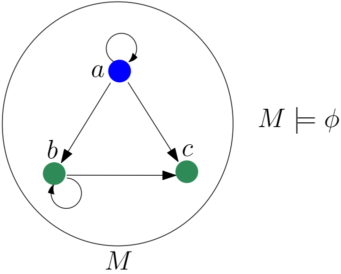

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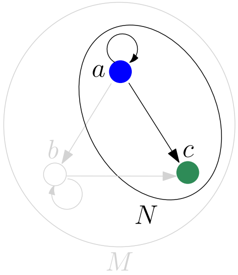

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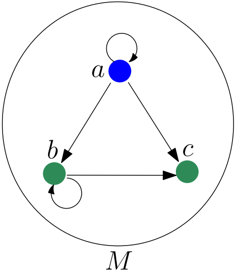

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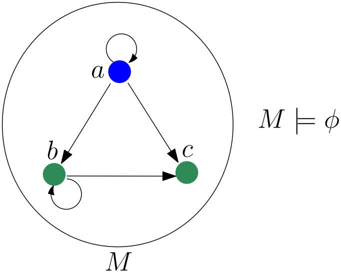

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- There can be cores that are not witnesses for x.

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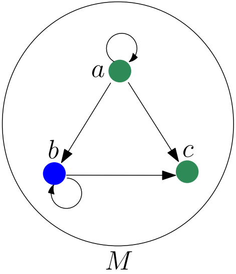

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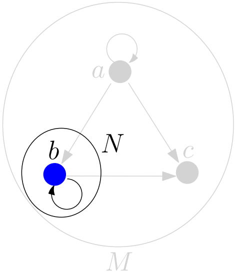

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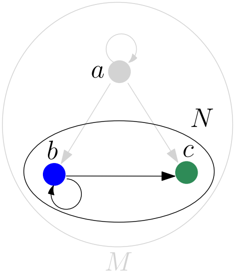

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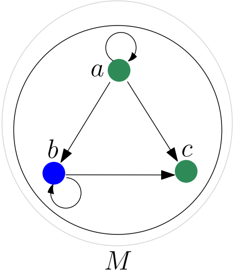

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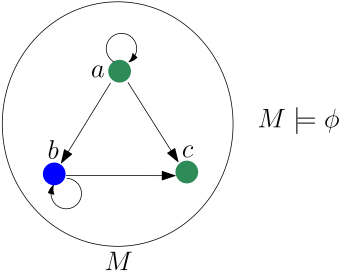
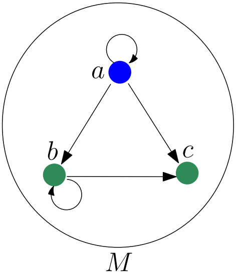


- Any witness for x is a core. Thus $\phi \in PSC_f$.
- There can be cores that are not witnesses for x.
- Every model of ϕ has a core of size ≤ 1.

Example (Contd.)



$M \models \phi$

- Observe: $\phi \notin PS$.

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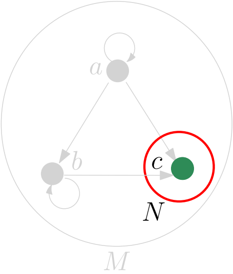

$M \models \phi$

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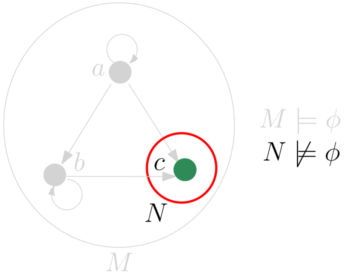

- Observe: $\phi \notin PS$.
- Easy to see: $PS \subseteq PSC_f$. Then $PS \subsetneq PSC_f$.
- In general, $\Sigma_2 \subseteq PSC_f$. In fact, for $\varphi \in \Sigma_2$, each model has a core of size $\leq$ the number of $\exists$ quantifiers of φ.

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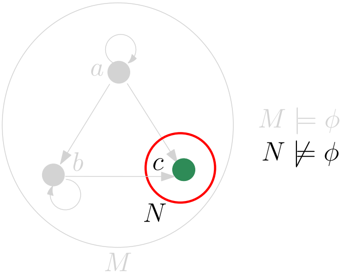

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- In general, $\Sigma_2 \subseteq PSC_f$. In fact, for $\varphi \in \Sigma_2$, each model has a core of size $\leq$ the number of $\exists$ quantifiers of φ.
- Interestingly, even for an arbitrary $\phi \in PSC_f$, there exist cores of bounded size in all models!

$PSC_f \equiv \Sigma_2$

Theorem 2

A sentence $\phi \in PSC_f$ iff ϕ is equivalent to a Σ_2 sentence.

Corollary 3 (Finite core implies bounded core)

If $\phi \in PSC_f$, there exists $k \in \mathbb{N}$ such that every model of ϕ has a core of size at most k.

Proof: Take k to be the number of $\exists$ quantifiers in the equivalent Σ_2 sentence guaranteed by Theorem 2. ■

Preservation under substructures modulo Bounded Cores

Definition 3 (Pres. under subst. modulo bounded cores)

A sentence ϕ is said to be preserved under substructures modulo a core of size k, denoted $\phi \in PSC(k)$, if $\phi \in PSC_f$ and each model M of ϕ has a core of size at most k.

- Observe that $PSC(0) = PS$.
- Easy to see that $PSC(l) \subsetneq PSC(k)$ for $l < k$. Consider ϕ which says that there are at least k distinct elements in any model. Then $\phi \in PSC(k) \setminus PSC(l)$.
- Let $PSC = \bigcup_{k \geq 0} PSC(k)$.

Towards a Syntactic Characterization of $PSC(k)$

Since finite core implies bounded core, we have

Lemma 4

$PSC = PSC_f$.

- A Σ_2 sentence ϕ with k $\exists$ quantifiers is in $PSC(k)$.
- In the converse direction, $\phi \in PSC(k)$ has an equivalent Σ_2 sentence.
- Question: For $\phi \in PSC(k)$, is there an equivalent Σ_2 sentence having k $\exists$ quantifiers?

A Syntactic Characterization of $PSC(k)$

Theorem 5

A sentence is in $PSC(k)$ iff it is equivalent to a Σ_2 sentence having k existential quantifiers.

- The proof uses the notion of *saturations* from classical model theory.
- Theorem 5 works over arbitrary vocabularies and over any class of structures definable by FO theories.
- The case of $k = 0$ is exactly the Łoś-Tarski theorem for sentences.

Preservation Properties Dual to $PSC(k)$ and PSC_f

Preservation under Extensions

Definition 4

A sentence ϕ is said to be preserved under extensions, denoted $\phi \in PE$, if $\big((M \models \phi) \wedge (M \subseteq N)\big) \rightarrow N \models \phi$.

- E.g.: Let $\phi = \exists x \exists y E(x, y)$. Easy to see that $\phi \in PE$.

Following is a duality lemma.

Lemma 6

A sentence ϕ is in PS iff $\neg\phi$ is in PE.

Theorem 7 (Łoś-Tarski, 1960s)

A FO sentence in PE is equivalent to a Σ_1 sentence.

An Alternate Form of Łoś-Tarski Theorem

Definition 5

A structure M is said to be an extension of a collection R of structures, denoted $R \subseteq M$, if for each $N \in R$, we have $N \subseteq M$.

- Easy to check: Preservation under extensions of single structures $\equiv$ Preservation under extensions of collections of structures.
- Then PE can be defined to be preservation under extensions of collections of structures and the Łoś-Tarski theorem statement would still be true.

k-ary Covered Extensions

Definition 6

For $k \in \mathbb{N}$, a structure M is said to be a k-ary covered extension of a non-empty collection R of structures if (i) M is an extension of R, and (ii) for every $A \subseteq U_M$ s.t. $|A| \leq k$, there is a structure in R that contains A. We call R a k-ary cover of M.

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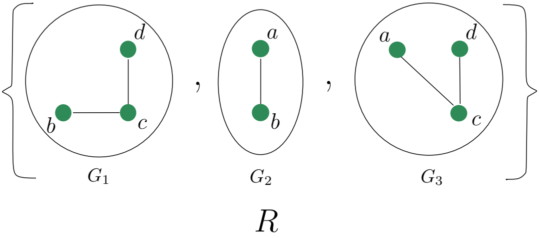

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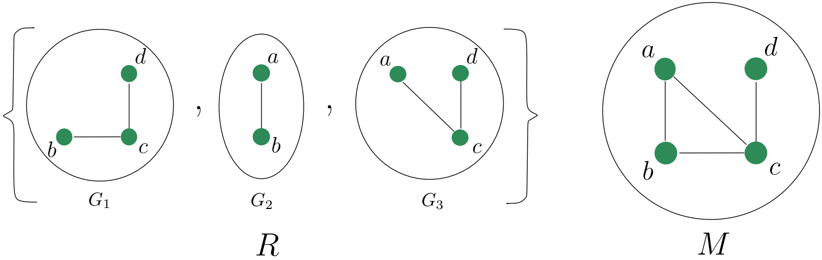

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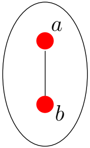
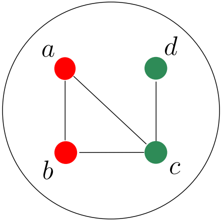


G_2



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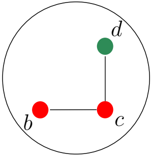
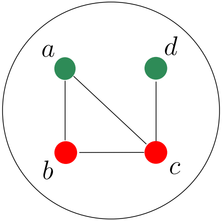


G_1



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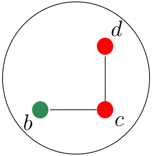

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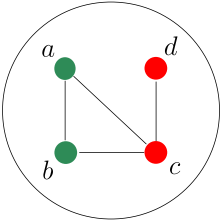

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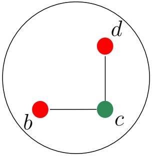

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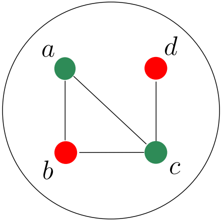

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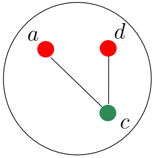
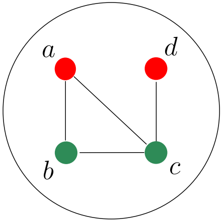


G_3



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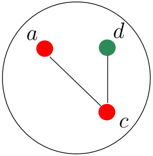

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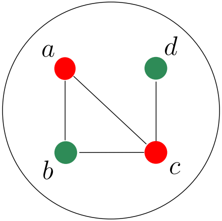

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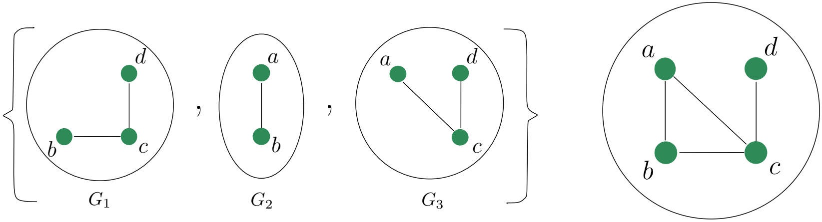

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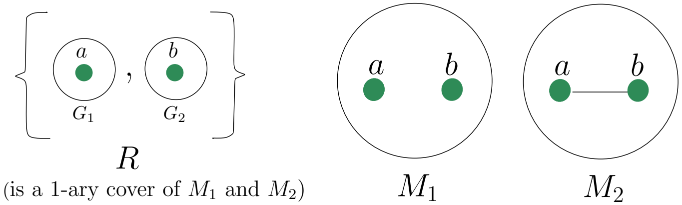

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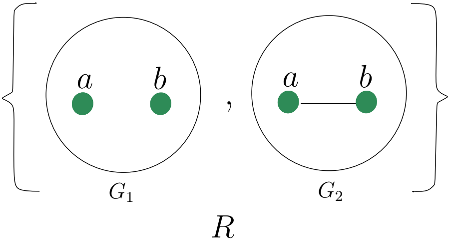

R has no extension!

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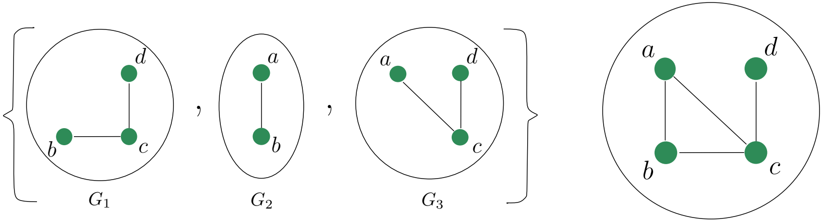

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Preservation under k-ary Covered Extensions ($PCE(k)$)

Definition 7

Given $k \in \mathbb{N}$, a sentence ϕ is said to be preserved under k-ary covered extensions, denoted $\phi \in PCE(k)$, if for each collection R of models of ϕ, (M is a k-ary covered extension of R) $\rightarrow M \models \phi$.

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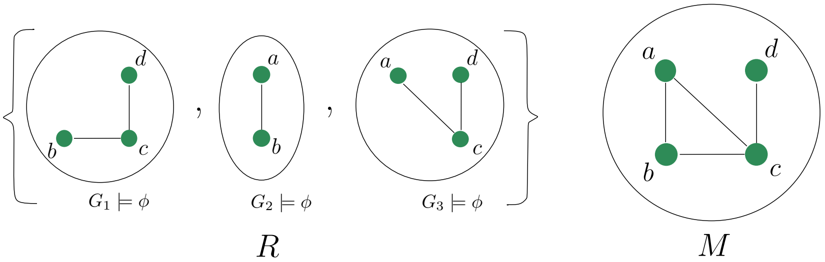

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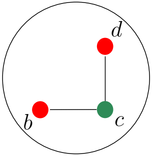
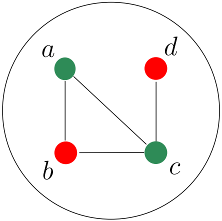


$G_1 \models \phi$



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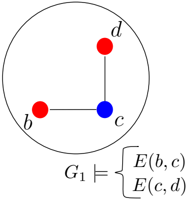

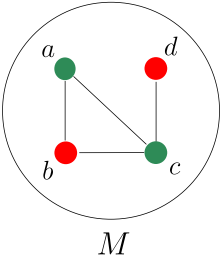

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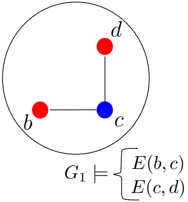

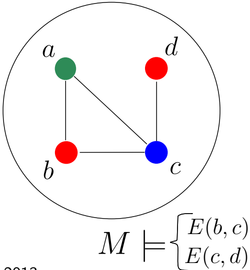

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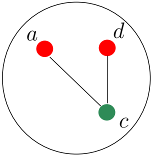
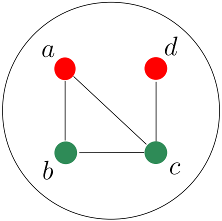


$G_3 \models \phi$



M

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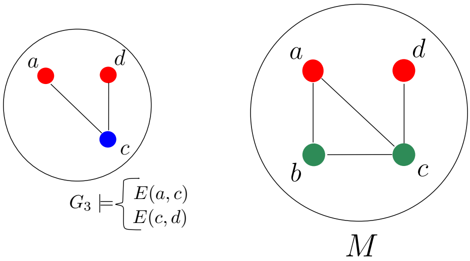

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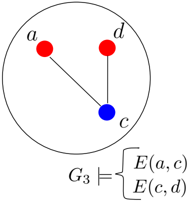

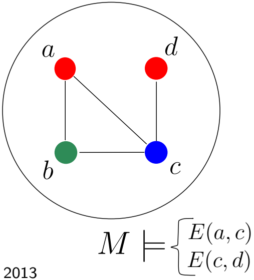

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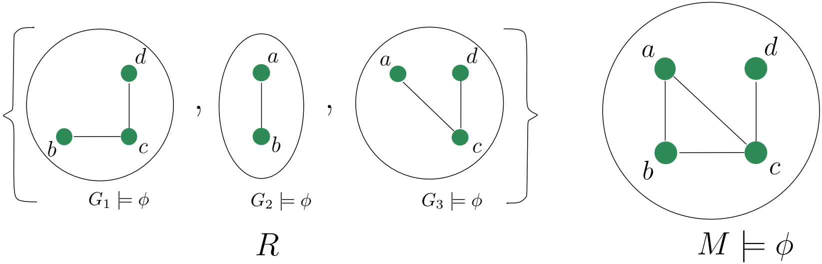

The Duality of $PSC(k)$ and $PCE(k)$

Lemma 8

A sentence ϕ is in $PSC(k)$ iff $\neg\phi$ is in $PCE(k)$.

Proof Sketch:
(We prove the 'If' direction; the 'Only If' is by a dual argument. Below, $A \subseteq_k B$ means $A \subseteq B$ and $|A| \leq k$.)

- Suppose $M \models \phi$ and there is no k-core in M.
- Then for each $A \subseteq_k U_M$, there exists $N_A \subseteq M$ containing A s.t. $N_A \models \neg\phi$.
- Then $R = \{N_A \mid A \subseteq_k U_M\}$ forms a k-ary cover of M. Since $\neg\phi \in PCE(k)$, we get $M \models \neg\phi$ – a contradiction. ∎

A Syntactic Characterization of $PCE(k)$

Theorem 9

A sentence ϕ is in $PCE(k)$ iff ϕ is equivalent to a Π_2 sentence having k universal quantifiers.

Proof Sketch:

- Let $\Gamma = \{\psi \mid \psi = \forall^k \exists^* (\ldots),\ \phi \rightarrow \psi\}$. Clearly, $\phi \rightarrow \Gamma$.
- Show that $\Gamma \rightarrow \phi$ holds over the class $\mathcal{C}$ of α-*saturated structures*, where $\alpha \geq \omega$.
- Use the fact that every structure has an elementarily equivalent structure in $\mathcal{C}$ to show that $\Gamma \rightarrow \phi$ holds over all structures.
- Finally, by Compactness theorem, the result follows. ■

A Generalization of the Łoś-Tarski Theorem

Theorem 9 and the $PSC(k)$-$PCE(k)$ duality imply the following.

Theorem 5

A sentence ϕ is in $PSC(k)$ iff ϕ is equivalent to a Σ_2 sentence with k existential quantifiers.

- Theorem 5 gives us exactly the substructural version of Łoś-Tarski theorem for $k = 0$.
- Theorem 9 gives us exactly the extensional form of the Łoś-Tarski theorem for $k = 0$.

Preservation under Finitary Covered Extensions (PCE_f)

- Finitary covered extension – replace 'k-ary' in the definition of k-ary covered extension with 'finitary'.
- Preservation under finitary covered extensions, denoted PCE_f, – replace 'k-ary' with 'finitary' in the $PCE(k)$ defn.

Lemma 10

A sentence ϕ is in PSC_f iff $\neg\phi$ is in PCE_f.

Theorem 11

A sentence ϕ is in PCE_f iff ϕ is equivalent to a Π_2 sentence.

Corollary 12

$PCE_f = \bigcup_{k \geq 0} PCE(k)$.

Comparison with Semantic Characterizations of Σ_2 and Π_2 in the Literature

- Define $PSC = \bigcup_{k \geq 0} PSC(k)$ and $PCE = \bigcup_{k \geq 0} PCE(k)$. Theorems 5 and 9 give new semantic characterizations of Σ_2 and Π_2 via PSC and PCE respectively.
- Existing characterizations in the literature for Σ_2 and Π_2 are via unions of ascending chains, intersections of descending chains, Keisler's 1-sandwiches, etc. *None* of these relate the *count* of the quantifiers to any model-theoretic properties, and hence do not generalize the Łoś-Tarski theorem.
- The PSC and PCE conditions are combinatorial in nature unlike any of the above literature notions.
- All of the above literature notions become trivial in the finite. However, there are sentences inside and outside of PSC and PCE in the finite.

Our Preservation Theorems over Finite Structures

- The failure of Łoś-Tarski theorem in the finite implies the failure of Theorems 5 and 9. In fact, the failure is stronger.

Theorem 13

$PSC(k)$, resp. $PCE(k)$, is strict semantic superset of the class of $\exists^k\forall^*$ sentences, resp. $\forall^k\exists^*$ sentences, for each $k \in \mathbb{N}$.

- However, for each k, the example witnessing the strict subsumption of $\exists^k\forall^*$ sentences by $PSC(k)$, is a $\exists^{k+1}\forall^*$ sentence – which is therefore in $PSC(k+1)$.
- This raises the possibility that $PSC(k)$ is semantically subsumed by the class of $\exists^l\forall^*$ sentences for some $l > k$.
- If so, then $PSC \equiv \Sigma_2$ and $PCE \equiv \Pi_2$ over the class of finite structures as well!

A Quick Note on Further Generalizations

- For $n \geq 1$, let $\Sigma_n(k_1, k_2, *, k_4, *, \ldots)$ be the subset of Σ_n in which each sentence has k_1 quantifiers in the first block and $k_2, k_4, \ldots$ quantifiers in the even indexed blocks. Likewise define $\Pi_n(k_1, k_2, *, k_4, *, \ldots)$.
- We have semantic characterizations for $\Sigma_n(k_1, k_2, *, k_4, *, \ldots)$ and $\Pi_n(k_1, k_2, *, k_4, *, \ldots)$ for each $n \geq 1$ and each $k_1, k_2, k_4, \ldots \in \mathbb{N}$ via variants of the $PSC(k)$ and $PCE(k)$ notions.
- These give us new and much finer characterizations of Σ_n and Π_n compared to those in the literature via unions of ascending Σ_n-chains and Keisler's n-sandwiches.

Directions for Future Work

Future Work

Over arbitrary structures:

- Semantic characterizations of Σ_n and Π_n sentences in which the number of quantifiers in each block is given.
- A syntactic characterization of *theories* in $PSC(k)$ and $PCE(k)$.

Over finite structures:

- Investigating if $PSC = \Sigma_2$ and $PCE = \Pi_2$ over the class of all finite structures.
- Characterizing $PSC(k)$ and $PCE(k)$ over interesting classes of finite structures like equivalence relations, partial orders, acyclic graphs, graphs of bounded degree, bounded tree-width, bounded split-width, etc.

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Thank you!

Appendix

An Intuitive but Incorrect Attempt at Characterizing $PSC(k)$

- Let $\phi \in PSC(k)$, $S =$ Models(ϕ), Vocab(ϕ) $= \tau$, $\tau_k = \tau \cup \{c_1, \ldots, c_k\}$.
- Let Z be the class of models of ϕ expanded with their core elements. Formally, $Z = \{(M, a_1, \ldots, a_k) \mid M \in S$ and $a_1, \ldots, a_k$ forms a core in $M\}$.
- Clearly Z is pres. under substr. Then by Łoś-Tarski theorem, Z is captured by a Π_1 sentence. Replace $c_1, \ldots, c_k$ with fresh variables $x_1, \ldots, x_k$ and existentially quantify out the latter.
- Error: Z is assumed FO definable.
- The above proof attempt fails for as simple a sentence as $\phi = \exists x \forall y E(x, y)$. (In fact, Z in this case is not definable by any FO theory too!)