On Categorical Relationship among various Fuzzy Topological Systems, Fuzzy Topological Spaces and related Algebraic Structures

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Seminar on Logic and Cognition, 2013, Jadavpur, India
31.10.2013

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- For each pair (A, B) of $\mathbb{A} - objects$, a set $hom(A, B)$, whose members are called $\mathbb{A} - morphisms$ from A to B.
- For each $\mathbb{A} - object$ A, a morphism $id_A : A \longrightarrow A$ called $\mathbb{A} - identity$ on A.

cont.

Definition

- A composition law associating with each $\mathbb{A} - morphism$ $f : A \longrightarrow B$ and each $\mathbb{A} - morphism$ $g : B \longrightarrow C$ an A $- morphism$ $g \circ f : A \longrightarrow C$, called the composite of f and g, subject to the following conditions

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- composition is associative.
- $\mathbb{A}-identities$ act as identities with respect to composition.

Functors

Definition

If $\mathbb{A}$ and $\mathbb{B}$ are categories, then the functor F from $\mathbb{A}$ to $\mathbb{B}$ is a function that assigns to each $\mathbb{A}-object\ A$ a $\mathbb{B}-object\ F(A)$, and to each $\mathbb{A}-morphism\ f : A \longrightarrow A'$ a $\mathbb{B}-morphism$ $F(f) : F(A) \longrightarrow F(A')$ in such a way that

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- $F-$ preserves compositions i.e. $F(f \circ g) = F(f) \circ F(g)$ whenever $f \circ g$ is defined and
- $F-$ preserve identity morphisms i.e. $F(id_A) = id_{F(A)}$ for each $\mathbb{A} - object$ A.

Right Adjoint and Left Adjoint

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A functor $G : \mathbb{A} \longrightarrow \mathbb{B}$ is said to be right adjoint provided that for every $\mathbb{B} - object\ B$ there exists a G-universal arrow with domain B.

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- Papert and Papert (1959): Construct adjunction between *Top* and Frm^{op}.

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- Abramski and Vickers (1993): Generalized the concept of quantale module.

Top Sys

Topological System (Vickers 1989)

A Topological system is a triple $(X, \models, A)$ where X is a set, A is a frame and $\models$, is a relation $\models \subseteq X \times A$, matches the logic of finite observations. Formally,

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Continuous map

Let $D = (X, \models, A)$ and $E = (Y, \models', B)$ be topological systems. A continuous map $f : D \longrightarrow E$ is a pair (f_1, f_2) where,

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- $f_1 : X \longrightarrow Y$ is a function.
- $f_2 : B \longrightarrow A$ is a frame homomorphism and
- $x \models f_2(b)$ iff $f_1(x) \models' b$, for all $x \in X$ and $b \in B$.

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Let $D = (X, \models', A)$, $E = (Y, \models'', B)$, $F = (Z, \models''', C)$.

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Top Sys

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Topological systems together with continuous maps form the category Top Sys.

Top

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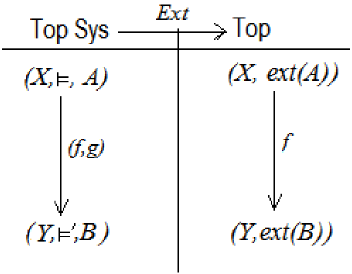
Topological spaces together with continuous maps form the category Top.

Frm

Frm

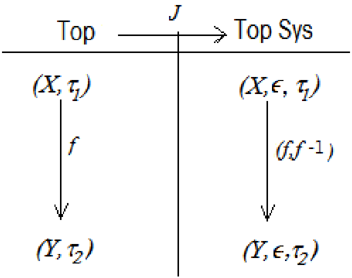
Frames together with frame homomorphisms form the category Frm.

Ext

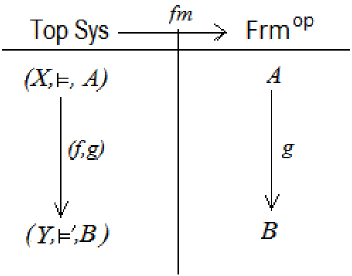


where $ext(a) = \{x \mid x \models a\}$ and $ext(A) = \{ext(a)\}_{a \in A}$.

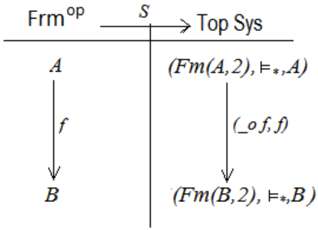
J



fm



S



where $Fm(A,2) = \{frame\ homomorphism : A \longrightarrow 2\}$ and $x \models_* a$ iff $x(a) = \top$.

Results

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2. fm is the left adjoint to the functor S.
3. $Ext \circ S$ is the right adjoint to the functor $fm \circ J$.

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A fuzzy topological system is a triple $(X, \models, A)$, where X is a non-empty set, A is a frame and $\models$ is a $[0,1]$- fuzzy relation from X to A such that

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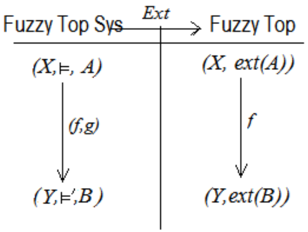
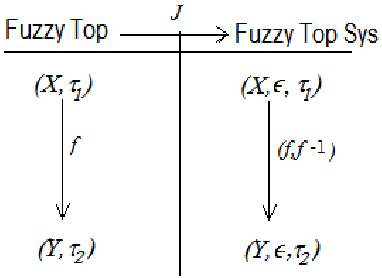


where $ext(a) : X \longrightarrow [0,1]$ s.t. $ext(a)(x) = gr(x \models a)$ and $ext(A) = \{ext(a)\}_{a \in A}$.

J


J
Fuzzy Top
Fuzzy Top Sys
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f
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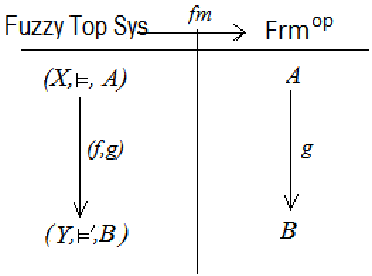

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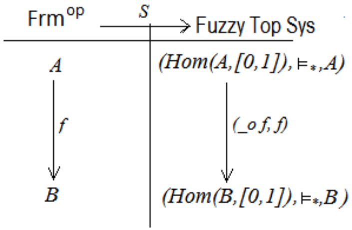

where $Hom(A, [0, 1]) = \{frame\ homomorphism\ v : A \longrightarrow [0, 1]\}$ and $gr(v \models_* a) = v(a)$.

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- If S is any subset of X, then $\alpha(u, \bigvee S) \leq \alpha(u, x)$ for some $x \in S$.
- $\alpha(u, \top) = 1$ and $\alpha(u, \bot) = 0$ for all $u \in U$.

Ł_n^c-algebra

An Ł_n^c-algebra is an MV_n algebra enriched by n constants. That is, it is an MV_n algebra $\mathcal{A} = (A, \wedge, \vee, *, \oplus, \rightarrow, ^{\perp}, 0, 1)$ in which the algebra $\bar{n}$ is embedded.

Ł$_n^c$-algebra

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Ł$_n^c$-homomorphism is a function between two Ł$_n^c$-algebras which preserves the operations.

FBSy$_n$

$\bar{n}$-fuzzy Boolean System

An $\bar{n}$-fuzzy Boolean System is a triple $(X, \models, A)$ where X is a set, A is an Ł$_n^c$-algebra and $\models$ is an $\bar{n}$ valued fuzzy relation from X to A such that

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3. $gr(x \models r) = r$ for all $r \in \bar{n}$

$FBSy_n$

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4. $x_1 \neq x_2 \Rightarrow gr(x_1 \models a) \neq gr(x_2 \models a)$ for some $a \in A$

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$\bar{n}$-fuzzy Boolean System

Composition

Let $D = (X, \models', A)$, $E = (Y, \models'', B)$, $F = (Z, \models''', C)$. Let $(f_1, f_2) : D \longrightarrow E$ and $(g_1, g_2) : E \longrightarrow F$ be continuous maps. The composition $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by

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i.e $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$.

$\bar{n}$-fuzzy Boolean System

$FBSy_n$

$\bar{n}$-fuzzy Boolean Systems together with continuous functions forms the category $\bar{n}$-fuzzy Boolean System($FBSy_n$).

FBS$_n$ (Yoshihiro Maruyama, 2010)

$\bar{n}$-fuzzy Boolean Space

For an $\bar{n}$-fuzzy topological space (X, τ) is called an $\bar{n}$-fuzzy Boolean space iff (X, τ) is zero dimensional, compact and Kolmogorov.

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FBS$_n$

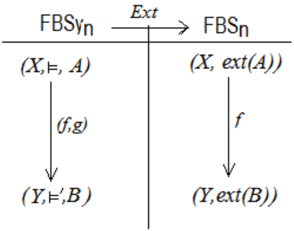
$\bar{n}$-fuzzy topological space (X, τ) together with continuous map forms the category FBS_n.

Ł$_n^c$-Alg

Ł$_n^c$-Alg

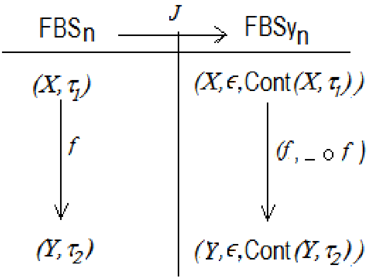
Ł$_n^c$-algebra together with Ł$_n^c$-Alg homomorphisms form the category Ł$_n^c$-Alg.

Ext

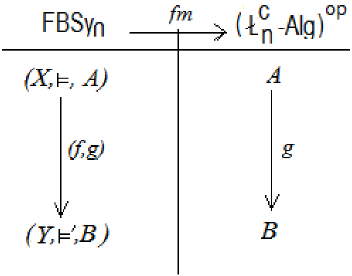


where $ext(a) : X \longrightarrow \bar{n}$ s.t. $ext(a)(x) = gr(x \models a)$ and $ext(A) = \{ext(a)\}_{a \in A}$.

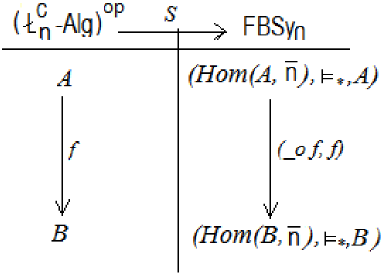
J



fm



S



where $Hom(A, \bar{n}) = \{Ł_n^c \ hom \ v : A \longrightarrow \bar{n}\}$ and $gr(v \models_* a) = v(a)$.

Results

1. *Ext* is the right adjoint to the functor J.

Results

1. *Ext* is the right adjoint to the functor J.
2. *fm* is the left adjoint to the functor S.

Results

1. Ext is the right adjoint to the functor J.
2. fm is the left adjoint to the functor S.
3. $Ext \circ S$ is the right adjoint to the functor $fm \circ J$.

Results

1. $Ł_n^c - Alg$ is dually equivalent to the category $FBSy_n$.

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1. $Ł_n^c - Alg$ is dually equivalent to the category $FBSy_n$.
2. FBS_n is equivalent to the category $FBSy_n$.

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2. FBS_n is equivalent to the category $FBSy_n$.
3. $Ł_n^c - Alg$ is dually equivalent to the category FBS_n..

$\mathscr{F}$-Top Sys

$\mathscr{F}$-Topological System

A $\mathscr{F}$-topological system is a quadruple $(X, \tilde{A}, \models, P)$, where $(X, \tilde{A})$ is a non-empty fuzzy set, P is a frame and $\models$ is a $[0, 1]$- fuzzy relation from X to P such that

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3. if S is a finite subset of P, then $gr(x \models \bigwedge S) = inf\{gr(x \models s) : s \in S\}$
4. if S is any subset of P, then $gr(x \models \bigvee S) = sup\{gr(x \models s) : s \in S\}$

$\mathscr{F}$-Top Sys

Continuous map

Let $D = (X, \tilde{A}, \models, P)$ and $E = (Y, \tilde{B}, \models', Q)$ be $\mathscr{F}$-topological systems.

$\mathscr{F}$-Top Sys

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Let $D = (X, \tilde{A}, \models, P)$ and $E = (Y, \tilde{B}, \models', Q)$ be $\mathscr{F}$-topological systems.A continuous map $f : D \longrightarrow E$ is a pair (f_1, f_2) where,

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2. $f_2 : Q \longrightarrow P$ is a frame homomorphism and
3. $gr(x \models f_2(q)) = gr(f_1(x) \models' q)$, for all $x \in X$ and $q \in Q$.

$\mathscr{F}$-Top Sys

Identity map

Let $D = (X, \tilde{A}, \models, P)$ be a $\mathscr{F}$-topological system. The identity map $I_D : D \longrightarrow D$ is a pair (I_1, I_2) defined by

$\mathscr{F}$-Top Sys

Identity map

Let $D = (X, \tilde{A}, \models, P)$ be a $\mathscr{F}$-topological system. The identity map $I_D : D \longrightarrow D$ is a pair (I_1, I_2) defined by

$$I_1 : (X, \tilde{A}) \longrightarrow (X, \tilde{A}) \; s.t. \; I_1(x_1, x_2) = \tilde{A}(x) \; iff \;\; x_1 = x_2$$
$$= 0 \qquad otherwise$$

and $I_2 : P \longrightarrow P$ *is identity morphism of* P.

$\mathscr{F}$-Top Sys

Composition

Let $D = (X, \tilde{A}, \models', P)$, $E = (Y, \tilde{B}, \models'', Q)$, $F = (Z, \tilde{C}, \models''', R)$.
Let $(f_1, f_2) : D \longrightarrow E$ and $(g_1, g_2) : E \longrightarrow F$ be continuous maps.
The composition $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by

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The composition $(g_1, g_2) \circ (f_1, f_2) : D \longrightarrow F$ is defined by

$$g_1 \circ f_1 : (X, \tilde{A}) \longrightarrow (Z, \tilde{C})$$
$$f_2 \circ g_2 : R \longrightarrow P$$

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$$f_2 \circ g_2 : R \longrightarrow P$$

i.e. $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$.

$\mathscr{F}$-Top Sys

$\mathscr{F}$-Top Sys

$\mathscr{F}$-topological systems together with continuous maps form the category $\mathscr{F}$-Top Sys.

$\mathscr{F}$-Top (M.K. Chakrabory and M. Banerjee, 1992)

$\mathscr{F}$-Top

$\mathscr{F}$-topological spaces together with continuous maps form the category $\mathscr{F}$-Top.

Frm

Frm

Frames together with frame homomorphisms form the category Frm.

Ext

Let $(X, \tilde{A}, \models, P)$ be a $\mathscr{F}$-topological system and $p \in P$. For each p, its extent in $(X, \tilde{A}, \models, P)$ is given by $ext(p) = (X, ext^*(p))$ where $ext^*(p)$ is a mapping from X to $[0, 1]$ given by $ext^*(p)(x) = gr(x \models p)$ for all $x \in X$.

Ext

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i.e. $ext^*(p) : X \longrightarrow [0, 1]$ such that $ext^*(p)(x) = gr(x \models p)$ for all $x \in X$.
Also $ext(P) = \{(X, ext^*(p))\}_{p \in P} = (X, ext^* P)$ where $ext^* P = \{ext^* p\}_{p \in P}$.

Ext

Ext is a (forgetful) functor from $\mathscr{F}$-Top Sys to $\mathscr{F}$-Top defined thus.
Ext acts on the object $(X, \tilde{A}, \models', P)$ as $Ext(X, \tilde{A}, \models', P) = (X, \tilde{A}, ext(P))$ and on the morphism (f_1, f_2) as $Ext(f_1, f_2) = f_1$.

J

J is a functor from $\mathscr{F}$-Top to $\mathscr{F}$-Top Sys defined thus. J acts on the object $(X, \tilde{A}, \tau)$ as $J(X, \tilde{A}, \tau) = (X, \tilde{A}, \in, \tau)$ where $gr(x \in \tilde{T}) = \tilde{T}(x)$ for $\tilde{T} \in \tau$ and on the morphism f as $J(f) = (f, f^{-1})$.

Loc

Loc is a functor from $\mathscr{F}$-Top Sys to Frm^{op} defined thus. *Loc* acts on the object $(X, \tilde{A}, \models, P)$ as $Loc(X, \tilde{A}, \models, P) = P$ and on the morphism (f_1, f_2) as $Loc(f_1, f_2) = f_2$.

S

S is a functor from Frm^{op} to $\mathscr{F}$-Top Sys defined thus. S acts on the object P as $S(P) = (Hom(P, [0,1]), \tilde{P}, \models_*, P)$, where $Hom(P, [0,1]) = \{frame\ hom\ v : P \longrightarrow [0,1]\}$, $gr(v \models_* p) = v(p)$ and $\tilde{P}(v) = \bigvee_{p \in P} v(p)$, and on the morphism f as $S(f) = (\_ \circ f, f)$.

Results

1. *Ext* is the right adjoint to the functor J.

Results

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2. *Loc* is the left adjoint to the functor S.

Results

1. *Ext* is the right adjoint to the functor J.
2. *Loc* is the left adjoint to the functor S.
3. $Ext \circ S$ is the right adjoint to the functor $Loc \circ J$.

For Interrelation among Top Sys, Top and Frm

Logic and Algebra

Logic of finite observations or geometric logic corresponds to the algebra frame, that is frame provides an algebraic semantics for logic of finite observations or geometric logic.

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Logic of finite observations or geometric logic corresponds to the algebra frame, that is frame provides an algebraic semantics for logic of finite observations or geometric logic.

Logic and Space

Categorical relationship between Topological Space and Frame provides a relationship between logic of finite observations or geometric logic and Topology.

For Interrelation among FBSy$_n$, FBS$_n$ and Ł$_n^c$-Alg

Logic and Algebra

Łukasiewicz n-valued logic Ł$_n^c$ with truth constants corresponds to Ł$_n^c$-algebra, that is Ł$_n^c$-algebra provides an algebraic semantics for Łukasiewicz n-valued logic Ł$_n^c$ with truth constants.

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Logic and Space

Categorical relationship between FBS$_n$ and Ł$_n^c$-Alg provides a relationship between Łukasiewicz n-valued logic Ł$_n^c$ with truth constants and n-valued Fuzzy Boolean Space.

Future Direction

- Finding duality in more general settings.

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- Introducing a notion of many valued geometric logic.

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- Exploring the properties of many valued geometric logic.

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- Introducing a notion of many valued geometric logic.
- Exploring the properties of many valued geometric logic.
- Introducing some notion of lattice valued fuzzy topological systems to connect existing notion of lattice valued fuzzy topological spaces(in more general settings) and finding the related algebraic structures.

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Thank You