Dialectics of Approximation of Semantics of Rough Sets

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Tone and Focus

- Rough Y Systems: Axiomatic Approach to Granules.
- Correspondences across RYS
- Less Concrete: Relation, Cover Based, Abstract RST.
- Concrete: Variations of Semantics.
- Concrete: PRAX, Tolerances, Sub Reflexive.
- More Concrete: Example Contexts/ Contexts of Problem Origin.

RBRST Context

- General approximation space: $S = \langle \underline{S}, R \rangle$.
- Rough Semantics $\mu(S, \mathscr{S})$ of various types in many Semantic Domains.
- Semantics can be difficult when R satisfies weaker forms of transitivity, etc.
- R can be approximated by quasi/partial orders and other relations.
- For quasi-orders a semantics for the set $\{(A^l, A^u);\ A \subseteq S\}$ as Nelson algebras over an algebraic lattice is known [SJ, JPR'2011].
- How well do the corresponding semantics help?

Generalised Transitive Relations

- Weakly Transitive: If whenever Rxy, Ryz and $x \neq y \neq z$ holds, then Rxz. ($(R \circ R) \setminus \Delta_S \subseteq R$)
- Transitive: whenever $Rxy \,\&\, Ryz$ holds then Rxz ($(R \circ R) \subseteq R$)
- Proto-Transitive: Whenever Rxy, Ryz, Ryx, Rzy and $x \neq y \neq z$ holds, then Rxz. Proto-transitivity of R is equivalent to $R \cap R^{-1} = \tau(R)$ being weakly transitive.

An infinite number of weakenings of transitivity is possible, but no systematic approach to handle these is known.

Definitions

- Proto Approximation Space S: $\langle \underline{S}, R \rangle$. (PRAS)
- Reflexive Proto Approximation Space: PRAX
- Successor nbd: $[x] = \{y;\, Ryx\}$ Associated Granulations $:\mathcal{G} = \{[x] : x \in S\}$
- Successor nbd: $[x]_o = \{y;\, Ryx \,\&\, Rxy\}$ Associated Granulations $:\mathcal{G}_o = \{[x] : x \in S\}$
- Upper Proto: $A^u = \bigcup_{[x]\cap A \neq \emptyset} [x]$.
- Lower Proto: $A^l = \bigcup_{[x]\subseteq A} [x]$.

Approximations

Symmetrized Upper Proto $A^{uo} = \bigcup_{[x]_o \cap A \neq \emptyset} [x]_o$.

Symmetrized Lower Proto $A^{lo} = \bigcup_{[x]_o \subseteq A} [x]_o$.

Point-wise Upper $A^{u+} = \{x : [x] \cap A \neq \emptyset\}$.

Point-wise Lower $A^{l+} = \{x : [x] \subseteq A\}$.

X**-Definite Element** a subset A satisfying $A^X = A$. $\delta_X(S)$ - Collection

Relation Between Approximations I

Theorem

$(\forall A \in \wp(S))\, A^{l+} \subseteq A^{l},\ A^{u+} \subseteq A^{u}.$

Theorem

Bi $(\forall A \in \wp(S))\, A^{ll} = A^{l} \,\&\, A^{u} \subseteq A^{uu}.$

l-Cup $(\forall A, B \in \wp(S))\, A^{l} \cup B^{l} \subseteq (A \cup B)^{l}.$

l-Cap $(\forall A, B \in \wp(S))\, (A \cap B)^{l} \subseteq A^{l} \cap B^{l}.$

u-Cup $(\forall A, B \in \wp(S))\, (A \cup B)^{u} = A^{u} \cup B^{u}$

u-Cap $(\forall A, B \in \wp(S))\, (A \cap B)^{u} \subseteq A^{u} \cap B^{u}$

Dual $(\forall A \in \wp(S))\, A^{lc} \subseteq A^{cu}.$

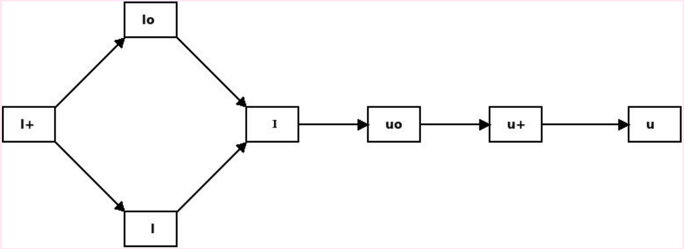
Relation Between Approximations II

Theorem

In a PRAX S, all of the following hold:

1. $(\forall A, B \in \wp(S))\, (A \cap B)^{l+} = A^{l+} \cap B^{l+}$
2. $(\forall A, B \in \wp(S))\, A^{l+} \cup B^{l+} \subseteq (A \cup B)^{l+}$
3. $(\forall A \in \wp(S))\, (A^{l+})^{c} = (A^{c})^{u+}, \,\&\, A^{l+} \subseteq A^{lo} \,\&\, A^{uo} \subseteq A^{u+} \,\&\, A^{l+} \subseteq A^{lo}.$

Relationship Diagram



Reading Help: *the $u+$- approximation of a set is included in the u-approximation of the same set.*

Types of Algebraic Semantics for PRAX

- Semantics of Definite Objects.
- Semantics of Rough Objects
- Mixed Semantics of Rough Objects
- Antichains of Rough Inclusion.
- Dialectical Semantics
- Approximation of Semantics.

Definitions-1

- R is a binary relation on a set X.
- $R^o \stackrel{\partial}{=} R \cup \Delta_X$.
- Weak transitive closure of R: $R^{\#}$.
- $R^{(i)}$ is the i-times composition $\underbrace{R \circ R \ldots \circ R}_{\text{i-times}}$, then $R^{\#} = \bigcup R^{(i)}$.
- R is *acyclic* if and only if $(\forall x) \neg R^{\#} xx$.
- $R^{\cdot} ab$ if and only if $Rab \,\&\, \neg(R^{\#} ab \,\&\, R^{\#} ba)$.

Definitions-2

- $R^{\flat}ab$ if and only if $[b]_{R^o} \subset [a]_{R^o} \,\&\, [a]_{iR^o} \subset [b]_{iR^o}$.
- $R^{cyc}ab$ if and only if $R^{\#}ab$ and $R^{\#}ba$.
- $R^{h}ab$ if and only if $R^{\flat}ab$ and $R^{\cdot}ab$.

In case of PRAX, $R^o = R$, so the definition of $R^{\flat}$ would involve neighborhoods of the form $[a]$ and $[a]_i$ alone. $R^{\flat} \subset R$ and $R^{\flat}$ is a partial order.

PRAX Case

Theorem

$R^h = \emptyset$.

Proposition

All of the following hold in a PRAX S:

- $R^{\cdot}ab \longleftrightarrow (R \setminus \tau(R))ab$.
- $(\forall a, b)\neg(R^{\cdot}ab \,\&\, R^{\cdot}ba)$.
- $(\forall a, b, c)(R^{\cdot}ab \,\&\, R^{\cdot}bc \longrightarrow \neg R^{\cdot}ac)$.

Theorem

1. $R^{\#\cdot} = R^{\#} \setminus \tau(R)$.
2. $R^{\cdot\#} = (R \setminus \tau(R))^{\#}$.
3. $(R \setminus \tau(R))^{\#} \subseteq R^{\#} \setminus \tau(R)$

Possible/Desirable Properties

If $<$ is a strict partial order on S and R is a relation, then consider the conditions :

- **PO1** $(\forall a, b)(a < b \longrightarrow R^{\#}ab)$.
- **PO2** $(\forall a, b)(a < b \longrightarrow \neg R^{\#}ba)$.
- **PO3** $(\forall a, b)(R^{\flat}ab \,\&\, R^{\cdot}ab \longrightarrow a < b$.
- **PO4** If $a \equiv_R b$, then $a \equiv_< b$.
- **PO5** $(\forall a, b)(a < b \longrightarrow Rab)$.

Partial Order Approximation

- *partial order approximation* POA of R iff *PO1, PO2, PO3, PO4*.
- weak partial order approximation: WPOA *PO1, PO3, PO4*).
- *inner approximation* IPOA: *PO5*.
- R^h, $R^{\cdot\flat}$ are IPOA, while $R^{\cdot\#}$, $R^{\#\cdot}$ are POAs.
- Lean quasi order approximation $<$ of R, we will mean a quasi order satisfying *PO1* and *PO2*.
- The corresponding sets of such approximations of R will be denoted by $POA(R)$, $WPOA(R)$, $IPOA(R), IWPOA(R)$ and $LQO(R)$

Theorem

Theorem

For any $A, B \in LQO(R)$, we can define the operations $\&, \vee, \top$:

- $(\forall x, y)(A\&B)xy$ *if and only if* $(\forall x, y)Axy \,\&\, Bxy$.
- $(A \vee B) = (A \cup B)^{\#}$, $\top = R^{\#}$.

Theorem

In a PRAX, $R^{\cdot\#}\&R^{\#\cdot}xy \longleftrightarrow (R \setminus \tau(R))^{\#}xy$.

Granules-1

- $R^{\#\cdot}$: *trans ortho-completion* of R
- $[x]_{ot} = \{y\,;\, R^{\#\cdot}yx\}$. $[x]^{i}_{ot} = \{y\,;\, R^{\#\cdot}xy\}$.
- $[x]^{o}_{ot} = \{y\,;\, R^{\#\cdot}yx \,\&\, R^{\#\cdot}xy\}$.

Theorem

In a PRAX S, $(\forall x \in S)\,[x]^{o}_{ot} = \{x\}$.

Symmetric Center of R

- Definition: $K_R = \bigcup e_i(\tau(R) \setminus \Delta_S)$.
- K_R can be used to partially categorize subsets of S based on intersection.
- Prop1: $(\forall x)\, [x] \triangle [x]_{ot} \neq \emptyset$ as
- Prop2: $x \notin K_R \longrightarrow [x] \subset [x]_{ot}$.
- Prop3: $x \in K_R \longrightarrow [x] \nsubseteq [x]_{ot} \,\&\, \{x\} \subset [x] \cap [x]_{ot}$.
- Prop4: $(R \setminus \tau(R))^{\#} \cup \tau(R)$ is not necessarily a quasi order.

Proposition

$((R \setminus \tau(R))^{\#} \cup \tau(R))^{\#} = R^{\#}$.

Relation Between Semantics

- Perspective-1: The definite or rough objects most closely related to the difference of lower approximations and those related to the difference of upper approximations can be expected to be related in a nice way.
- We prove that nice does not have a rough evolution - anyway it is a semantics that involves that of [JPR'2011, SJ].
- Perspective-2: Starting from sets of the form $A^* = (A^l \setminus A^{l_\#}) \cup (A^{u_\#} \setminus A^u)$ and taking their lower ($l_\#$) and upper ($u_\#$) approximations - the resulting structure would be a partial algebra derived from a Nelson algebra over an algebraic lattice ([AM'2012C]).

Perspective-1

Proposition

In a PRAX S, (we use $\#$ subscripts for neighborhoods, approximation operators and rough equalities of the weak transitive completion):

- *Nbd:* $(\forall x \in S)\, [x]_R \subseteq [x]_{R^\#}$.
- *App:* $(\forall A \subseteq S)\, A^l \subseteq A^{l_\#} \,\&\, A^u \subseteq A^{u_\#}$.
- *REq:* $(\forall A \subseteq S)(\forall B \in [A]_{\approx})(\forall C \in [A]_{\approx_\#})\, B^l \subseteq C^{l_\#} \,\&\, B^u \subseteq C^{u_\#}$.

A more general partial order: $\preceq$ *over* $\wp(\wp(S))$ *via* $A \preceq B$ *if and only if* $(\forall C \in A)(\forall E \in B)\, C^l \subseteq E^{l_\#} \,\&\, C^u \subseteq E^{u_\#}$.

Definition

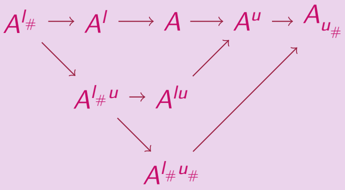
- *l-scedastic approximation*: $A^{\hat{l}} = (A^l \setminus A^{l_\#})^l$.
- *u-scedastic approximation*: $A^{\hat{u}} = (A^{u_\#} \setminus A^u)^{u_\#}$.
- These are the best possible from closeness to properties of rough approximations.

Scedasticity-1

Theorem

For an arbitrary subset $A \subseteq S$ of a PRAX S,the following statements and diagram of inclusion ($\longrightarrow$) hold:

- $A^{l_\# l} = A^{l_\#} = A^{l l_\#} = A^{l_\# l_\#}$
- *If* $A^u \subset A^{u_\#}$ *then* $A^{u u_\#} \subseteq A^{u_\# u_\#}$.



Scedasticity-2

Theorem

For an arbitrary subset $A \subseteq S$ *of a* PRAX S,

$$(A^{l} \setminus A^{l_{\#}})^{l} \nsubseteq (A^{u_{\#}} \setminus A^{u})^{u_{\#}} \longrightarrow A^{u_{\#}} = A^{u}.$$

$$A^{u_{\#}} \neq A^{u} \longrightarrow A^{l} \setminus A^{l_{\#}})^{l} \subseteq (A^{u_{\#}} \setminus A^{u})^{u_{\#}}.$$

Theorem

Key properties of the scedastic approximations follow:

1. $(\forall B \in \wp(S))(B^{\hat{l}} = B \nrightarrow B^{\hat{u}} = B)$.
2. $(\forall B \in \wp(S))(B^{\hat{u}} = B \rightarrow B^{\hat{l}} = B)$.
3. $(\forall B \in \wp(S))\, B^{\hat{l}\hat{l}} = B^{\hat{l}}$.
4. $(\forall B \in \wp(S))\, B^{\hat{u}\hat{u}} \neq B^{\hat{u}}$.
5. *It is possible that* $(\exists B \in \wp(S)\, B^{\hat{u}\hat{u}} \subset B^{\hat{u}})$.

An interesting problem can be given A for which $A^{u_{\#}} \neq A^{u}$, when does there exist a B such that

$$B^{l} = (A^{l} \setminus A^{l_{\#}})^{l} = A^{\hat{l}} \;\&\; B^{u} = (A^{u_{\#}} \setminus A^{u})^{u_{\#}} = A^{\hat{u}}?$$

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THANK YOU !