A logic for reasoning about incomplete knowledge

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1. MEL
2. Semantics and completeness of MEL
3. MEL as a basis for reasoning about uncertainty

Language

$\mathcal{L}$: propositional language with $\top$

Main idea: to *encapsulate* PL inside a language equipped with a modality $\Box$; to separate propositions in $\mathcal{L}$ referring to the real world, and propositions that refer to an agent's 'epistemic state', where the symbol $\Box$ appears.

At: Atoms of MEL, of the form $\Box\alpha$, $\alpha \in \mathcal{L}$; get a propositional language:

$$\Box\alpha, \alpha \in \mathcal{L} | \neg\phi | \phi \wedge \psi$$

Define $\Diamond$ as usual

No nesting of modalities.

MEL: *meta-epistemic* logic – we take an imperfect external point of view on the agent knowledge (e.g. Aucher)

Also 'minimal' epistemic logic: basic representation of incomplete information common to uncertainty theories

Interpretations

Incomplete knowledge about the real world possessed by an agent will be represented just by a non-empty subset of interpretations, one and only one of which this agent believes is true – an 'epistemic state'

All that is known about the agent's epistemic state stems from what this agent sincerely reports. So we have incomplete knowledge about this epistemic state – consider families of epistemic states ('meta-epistemic states'), one of which is the agent's state.

– This kind of representation of higher order incomplete knowledge already exists in uncertainty theories. In Shafer's theory of evidence, a belief function is represented by a probability distribution over epistemic states.

Axioms

$\phi, \psi, \mu \in MEL$, and $\alpha, \beta \in \mathcal{L}$.

(PL) : (i) $\phi \rightarrow (\psi \rightarrow \phi)$; (ii) $(\phi \rightarrow (\psi \rightarrow \mu)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \mu))$; (iii) $(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$.

(K) : $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$.

(N) : $\Box\alpha$, whenever $\vdash_{PL} \alpha$.

(D) : $\Box\alpha \rightarrow \Diamond\alpha$.

Rule:

(MP) : If $\phi, \phi \rightarrow \psi$ then ψ.

- $\Gamma \vdash_{MEL} \phi \iff \Gamma \cup \{K, N, D\} \vdash_{PL} \phi$.
- Boolean version of axioms of the fuzzy logic of necessities (Hájek)

Satisfaction of MEL-formulae

Finite set of proposition variables;
$\alpha \in \mathcal{L}$, $\phi, \psi \in MEL$, $E(\neq \emptyset) \subseteq \mathcal{V}$, the set of all propositional valuations, $[\alpha]:=\{w \in \mathcal{V} : w \models \alpha\}$

- $E \models \Box\alpha$, if and only if $E \subseteq [\alpha]$.
- $E \models \neg\phi$, if and only if $E \not\models \phi$.
- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$.

No reference to Kripke frames / accessibility relations

The encapsulation of PL into MEL:

$\Box\mathcal{B} \vdash_{MEL} \Box\alpha$, if and only if $\mathcal{B} \vdash_{PL} \alpha$, for any $\alpha \in \mathcal{L}$.

A necessity measure

[A set-function $\mathcal{N}$ with range on the unit interval that satisfies $\mathcal{N}(\emptyset) = 0, \mathcal{N}(\mathcal{V}) = 1, \mathcal{N}(A \cap B) = \min(\mathcal{N}(A), \mathcal{N}(B))$.]
For any valuation $v : At \to \{0, 1\}$, define a Boolean set-function $g_v : 2^{\mathcal{V}} \to \{0, 1\}$:

$$g_v([\alpha]) := v(\Box\alpha), \Box\alpha \in At.$$

- If v satisfies K,D,N, g_v is a (Boolean) necessity measure.
- Conversely, given a Boolean necessity measure $\mathcal{N}$ on $2^{\mathcal{V}}$, the valuation v defined by $v(\Box\alpha) := \mathcal{N}([\alpha])$, for all $\alpha \in \mathcal{L}$, satisfies all instances of axioms K, N, D.

Propositional MEL-valuations – epistemic states

For a $\{K, D, N\}$-model $v : At \longrightarrow \{0, 1\}$, define an associated epistemic state

$$E_v := \{w \in \mathcal{V} : g_v(\mathcal{V} \setminus \{w\}) = 0\}.$$

(i) Since g_v is a necessity measure, E_v is unique and non-empty.

(ii) $v \models \phi$ (in the propositional semantics) if and only if $E_v \models \phi$ (in the epistemic semantics).

Conversely, given an epistemic state E, we can define a valuation of the modal language of MEL as follows:

$$v_E(\Box\alpha) := \begin{cases} 1 \text{ if } E \subseteq [\alpha]; \\ 0 \text{ otherwise.} \end{cases}$$

By construction, $v_E \models \phi$ (in the propositional semantics) if and only if $E \models \phi$ (in MEL semantics).

Use the constructs to get completeness.

Logical representation of sets of epistemic models

$$\delta_E := \Box\alpha_E \;\wedge\; \bigwedge_{w\in E} \neg\Box\neg\alpha_w = \Box\alpha_E \;\wedge\; \bigwedge_{w\in E} \Diamond\alpha_w.$$

$[\delta_E] = \{E\}$; there is a one-to-one correspondence between MEL-valuations (satisfying MEL axioms) and formulae δ_E.
Extend to a meta-epistemic state $\mathcal{E} := \{E_1, \ldots E_n\}$ – described completely by $\delta_{\mathcal{E}} := \bigvee_{1\le i\le n} \delta_{E_i}$.

There is a bijection between the set of all sets of epistemic states and the set of all (deductively closed) belief sets of MEL, i.e. Γ such that $Con(\Gamma) = \Gamma$. For any family $\mathcal{E} \subseteq 2^{\mathcal{V}} \setminus \{\emptyset\}$, the correspondence is given by: $\mathcal{E} \mapsto Con(\delta_{\mathcal{E}})$.

Epistemic logics and Uncertainty theories

- Epistemic logics: reason about knowledge or about beliefs. Uncertainty theories also consider belief as a central notion. – But these represent different streams of thought/communities!
- MEL's set-valued semantics is in terms of epistemic states, or equivalently possibility distributions. So it is easier to relate uncertainty theories to MEL than to modal logics that extensively use accessibility relations. The latter play no role in uncertainty theories.

Möbius transform and MEL-formulae

Connection between MEL and belief functions:
the pair (Bel, Pl) can be viewed as quantitative versions of KD modalities $(\Box, \Diamond)$ (Smets), hence of MEL
Basic assignment $m(E)$ for each subset E of $\mathcal{V}$: $m(E) \geq 0$,;

$$\sum_{\emptyset \neq E \subseteq \mathcal{V}} m(E) = 1.$$

Then, given m,

$$Bel([\alpha]) = \sum_{E \models \Box\alpha} m(E).$$

The assertion of a MEL-formula $\Box\alpha$ is expressed by $Bel([\alpha]) = 1$.

Möbius transform and MEL-formulae

Conversely, given *Bel*,

$$m(E) = \sum_{A \subseteq E} (-1)^{|E \setminus A|} Bel(A),$$

the Möbius transform. In fact,

The logical rendering of the Möbius transform for computing $m(E)$ is logically equivalent to δ_E.

: similarity between the problem of reconstructing a mass assignment from the knowledge of a belief function and the problem of representing an epistemic state E in the language of MEL
So belief functions may be considered as numerical generalizations of MEL-formulae.

Möbius transform and MEL-formulae

A logic of belief functions that builds on MEL syntax and semantics:

Use graded modal propositions $\Box_r \alpha$, α in PL, $r \in [0, 1]$ a lower bound for the degree of belief of α ($Bel([\alpha]) \geq r$).
Satisfaction relation: $m \models \Box_r \alpha$ when $\sum_{E \subseteq [\alpha]} m(E) \geq r$.

To compare a belief version counterpart of MEL with the fuzzy logic of belief functions (Godo et al.)

MEL, possibilistic and probabilistic logics

- Possibilistic logic is also a two-tiered logic like MEL: it is propositional logic embedded within a multivalent logic. Uses weighted formulae (α, a) – assume only maximal weights $a = 1$, and identify $(\alpha, 1)$ with $\Box\alpha$ – possibilistic logic coincides with the fragment of MEL containing only conjunction of boxed formulae.

  Natural to consider the extension of PL to a graded MEL, for a full-fledged uncertainty logic handling certainty and partial ignorance at a syntactic level (Dubois & Prade)
- MEL can be considered as a degenerated probabilistic logic by interpreting $\Box\alpha$ as $Prob(\alpha) = 1$, and hence $\Diamond\alpha$ as $Prob(\alpha) > 0$.

Related work

- Consensus logic C for *consensus voting*: language and axiomatization identical to those of MEL, similar semantics, however set in a different context altogether. C-models are multisets of PL-valuations, while MEL-models are sets.
- Kleene logic: can be mapped to a fragment of MEL, where modal atoms $\Box\alpha$ are restricted to literals inside (i.e. are of the form $\Box p$ and $\Box\neg p$) and only conjunctions and disjunctions of such modal atoms are allowed (no negation) (Ciucci and Dubois).

Conclusions and perspectives

- Two-tiered logic
- The Boolean version of uncertainty theory logics

Possible directions

- A logic of belief functions that builds on MEL syntax and semantics
- To compare a belief version counterpart of MEL with the fuzzy logic of belief functions
- Consider extension of PL to a graded MEL, generalize MEL and possibilistic logic using (graded) multi-modalities

Conclusions and perspectives

Possible directions

- Set of formulae in MEL viewed as a testimony. Belnap's four-valued logic based on information sources – extension of MEL to the setting where several emitter agents provide information, and conflicts can be handled
- MEL and its possible extensions (to mutual or common beliefs) in the framework of multiagent systems – consider notions of belief change

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Thank you