Continuous First-Order Logic

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Theorem (Shelah's Main Gap Theorem)

If T is a first-order theory and is stable and . . . , then the class of models looks like . . . . Otherwise, there's no hope.

Example

Let T be the theory of vector spaces over some infinite field. Then for each uncountable cardinal κ, there is exactly one model of T, up to isomorphism, with cardinality κ. Moreover, there are only countably many countable models, and we know what they are.

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3. For all $x \in V$ and all $\lambda \in \mathbb{R}$ we have $|\lambda x| = |\lambda| \cdot |x|$.
4. For all $x, y \in V$, we have $|x + y| \leq |x| + |y|$.

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Then how can we possibly think about Hilbert Spaces?

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We define $L^p(\mathbb{R})$ to be the set of all functions $f : \mathbb{R} \to \mathbb{R}$ such that $||f||_p < \infty$, with pointwise addition and scaling and the norm $||\cdot||_p$. Each space $L^p(\mathbb{R})$ is a Banach space.

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Define $\langle f, g \rangle := \left(\int fg\right)^{\frac{1}{2}}$. Then L^2, with this inner product, is a Hilbert space.

Several themes join together here:

1. Need a logic that describes ultraproduct and nonstandard hull constructions in functional analysis.
   1. Ideally, it would use natural analytic language.
   2. Ideally, it would make at least some familiar structures stable.
2. How do we reason about probability?

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- Quantifiers are sup and inf.

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A *continuous signature* is an object of the form $\mathcal{L} = (\mathcal{R}, \mathcal{F}, \mathcal{G}, n, d)$ where

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2. n is a function associating to each member of $\mathcal{R} \cup \mathcal{F}$ its arity
3. $\mathcal{G}$ has the form $\{\delta_{s,i} : (0,1] \to (0,1] : s \in \mathcal{R} \cup \mathcal{F}$ and $i < n_s\}$

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3. The function ρ assigns d to a pseudo-metric $d^{\mathfrak{M}} : M \times M \to [0, 1]$

4. For each $f \in \mathcal{F}$ for each $i < n_f$, and for each $\epsilon \in (0, 1]$, we have

$$\forall \bar{a}, \bar{b}, c, e \left[d^{\mathfrak{M}}(c, e) < \delta_{f,i} \Rightarrow d^{\mathfrak{M}}\left(f^{\mathfrak{M}}(\bar{a}, c, \bar{b}), f^{\mathfrak{M}}(\bar{a}, e, \bar{b}) \right) \leq \epsilon \right]$$

where $lh(\bar{a}) = i$ and $lh(\bar{a}) + lh(\bar{b}) = n_f - 1$

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5. For each $P \in \mathcal{R}$ for each $i < n_P$, and for each $\epsilon \in (0,1]$, we have

$$\forall \bar{a}, \bar{b}, c, e \left[d^{\mathfrak{M}}(c,e) < \delta_{f,i} \Rightarrow |P^{\mathfrak{M}}(\bar{a}, c, \bar{b}) - P^{\mathfrak{M}}(\bar{a}, e, \bar{b})| \leq \epsilon \right]$$

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A *continuous $\mathcal{L}$-structure* is a continuous $\mathcal{L}$-pre-structure such that ρ assigns to d a *complete* metric.

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This is an Erdős-Renyi random graph.

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3. We write $(\mathfrak{M}, \sigma) \models \varphi$ exactly when $\mathfrak{M}(\varphi, \sigma) = 0$.

Definition

Let $\mathcal{S}_0$ be a set of distinct propositional symbols. Let $\mathcal{S}$ be freely generated from $\mathcal{S}_0$ by the formal binary operation $\dot{-}$ and the unary operations $\neg$ and $\frac{1}{2}$. Then $\mathcal{S}$ is said to be a *continuous propositional logic*.

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   We say that v is the *truth assignment* defined by v_0.
2. We write $v \models \Sigma$ for some $\Sigma \subseteq \mathcal{S}$ whenever $v(\varphi) = 0$ for all $\varphi \in \Sigma$.

Proposition (Ben Yaacov–Berenstein–Henson–Usvyatsov)

Let $f(\bar{x}) : [0,1]^n \to [0,1]$ be continuous. Then f can be approximated by something generated from $\dot{-}, \neg$, and $\frac{1}{2}$.

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Theorem (Ben Yaacov–Berenstein–Henson–Usvyatsov)

The following CFO theories are stable

- *Hilbert Spaces*
- *Atomless Probability Spaces*

Theorem (Compactness)

Let T be a CFO theory, and $\mathcal{C}$ a class of structures. Assume that T is finitely satisfiable in C. Then there is an ultraproduct of structures from $\mathcal{C}$ that is a model of T.

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Theorem

Let κ be an infinite cardinal, and L a signature of smaller size. Let $\mathcal{M}$ be an L-structure, and $A \subseteq M$ a set with density character at most κ. Then there is an elementary substructure $\mathcal{N}$ of $\mathcal{M}$ which contains A and has density character at most κ.

Theorem (Ben Yaacov)

Let T be a countable CFO theory. If T is categorical in some uncountable cardinal, then it is categorical in all uncountable cardinals.

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3. A "program" which instructs the head, given the memory state and the state of its cell what it should do next.

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6. So I probably should.

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1. A *probabilistic Turing machine* is a Turing machine equipped with an oracle for an element of 2^ω, called the *random bits*, with output in $\{0, 1\}$.
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3. We say that a probabilistic Turing machine M *rejects* n *with probability* p if and only if $P\{x \in 2^\omega : M^x(n) \downarrow = 1\} = p$.

Definition

A set $S \subseteq \mathbb{N}$ is of class BPP iff there is a probabilistic Turing machine which runs in polynomial time and gives the right answer to "$n \in S$?" at least $\frac{3}{4}$ of the time.

Definition

Let $\mathcal{L}$ be a computable continuous signature. Let $\mathfrak{M}$ be a continuous $\mathcal{L}$-structure. Let $\mathcal{L}(\mathfrak{M})$ be the expansion of $\mathcal{L}$ by a constant c_m for each $m \in M$ (i.e. a unary predicate $c_m \in \mathcal{R}$ where $c_m^{\mathfrak{M}}(x) := d(x, m)$).

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Definition

We treat a first-order structure $\mathcal{M}$ as the set of Gödel codes for sentences in its atomic diagram. In particular, $\mathcal{M}$ is said to be computable if and only if that set is computable.

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Definition

We say that a continuous pre-structure $\mathfrak{M}$ is *probabilistically computable* (respectively, *probabilistically decidable*) if and only if there is some probabilistic Turing machine T such that, for every pair $(\varphi, p) \in D(\mathfrak{M})$ (respectively, $D^*(\mathfrak{M})$) the machine T accepts φ with probability p.

Example

Let $p \in (0, 1)$. Take a continuous signature with a single binary predicate, and make the metric discrete.

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With a random real oracle, it is a classical Erdős-Renyi random graph.

Lemma (No Derandomization Lemma)

There is a probabilistically computable weak structure $\mathfrak{M}$ *such that the set* $\{(\varphi, p) \in D(\mathfrak{M}) : p \in \mathcal{D}\}$ *is not classically computable.*

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Proof.

Let U be a computably enumerable set, and let S be the complement of U. We construct a probabilistically computable function f such that

$$P(f^{\sigma}(x) = 0) = \frac{1}{2}$$

if and only if $x \in S$. □

Proposition

For any probabilistically computable pre-structure $\mathfrak{M}$,

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1. *There is some (classically) computable function f, monotonically increasing in the second variable, and*
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such that for any pair $(\varphi, p) \in D(\mathfrak{M})$, we have $\lim_{s\to\infty} f(\varphi, s) = p$ and $\lim_{s\to\infty} g(\varphi, s) = p$.

Definition

A first-order theory is said to be decidable iff there is a Turing machine which, given any sentence φ will determine whether $T \vdash \varphi$.

Definition

A real number $x = \sum_{i=1}^{\infty} x_i 10^{-i}$ is said to be computable if and only if the sequence of digits x_i is computable.

Definition (Ben Yaacov–Pedersen)

Let $\mathcal{L}$ be a continuous signature and Γ a set of formulas of $\mathcal{L}$.

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Let $\mathcal{L}$ be a continuous signature and Γ a set of formulas of $\mathcal{L}$.

1. We define

$$\varphi^\circ_\Gamma := \sup\{\mathfrak{M}(\varphi,\sigma) : (\mathfrak{M},\sigma) \models \Gamma\}.$$

2. If T is a complete continuous first-order theory, we say that T is *decidable* if and only if there is a (classically) computable function f such that $f(\varphi)$ is an index for a computable real number equal to φ°_T.

Theorem (Ben Yaacov–Pedersen)

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A continuous first-order theory is consistent if and only if there is a metric structure which satisfies it.

Proof.

1. A consistent CFO theory must be satisfied by some pre-structure
2. Get a metric structure with exactly the same satisfaction properties

□

Theorem

Let T be a decidable continuous first-order theory. Then there is a probabilistically decidable continuous weak structure $\mathfrak{M}$ such that $\mathfrak{M} \models T$.

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1. Pass to a Henkin complete theory (witnesses for all sup's).
2. Pass to a maximal consistent theory.
3. Build the model.

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How to build the model:
If we have a proof of $\varphi \dot{-} \frac{k}{2^n}$, we make sure to accept φ with probability at least $1 - \frac{k}{2^n}$, by assigning some initial segments of the random string to accept φ.

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If we have a proof of $\varphi \dot{-} \frac{k}{2^n}$, we make sure to accept φ with probability at least $1 - \frac{k}{2^n}$, by assigning some initial segments of the random string to accept φ.
If we have a proof of $\frac{k}{2^n} \dot{-} \varphi$, then we do the opposite.

Lemma

If we have proofs of both $\varphi \dot{-} \frac{k_1}{2^n}$ *and* $\frac{k_2}{2^n} \dot{-} \varphi$*, then we have*
$\left(1 - \frac{k_1}{2^n}\right) + \frac{k_2}{2^n} \leq 1$.

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Proof.

If not, then $\frac{k_2}{2^n} \dot{-} \frac{k_1}{2^n} = 0$ so that $k_1 \geq k_2$.
But also $2^n - k_1 + k_2 > 1$, so that $k_2 \gneq k_1$. □

Definition

Let $\Omega = (\Omega, \mu)$ be a measure space. Then a $[0,1]$-valued random variable on Ω is a measurable function $\Omega \to [0,1]$.

Definition

Let $A \subseteq [0,1]$, and X a random variable. Then $P(X \in A) = \mu\{x : f(x) \in A\}$.

We'd like to have a unified computational and model-theoretic way to talk about random variables.

Definition

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1. A *name* for a real number x is a decreasing sequence of closed intervals with rational endpoints whose intersection is x.
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The real numbers aren't that special here.

Theorem

Let Ω be a measure space that doesn't foil our efforts ($[0, 1]$ is good). Let X be a $[0, 1]$-valued random variable on Ω. Then the following are equivalent:

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2. *The distribution of X is computable.*
3. *If $\mathcal{M}$ is a probabilistically computable structure containing a quantifier-free copy of a dense subspace $\Omega' \subseteq (\Omega, \mu)$ and of the computable real numbers, then there is a probabilistically computable expansion of $\mathcal{M}$ by $X \restriction_{\Omega'}$.*

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4. *There is a probabilistically computable structure in which a copy of $X \restriction_{\Omega'}$ is quantifier-free definable, for some dense $\Omega' \subseteq \Omega$.*

Example

There is a uniform collection of independent computable standard normal random variables.

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Example

There are computable Bernoulli, Binomial, Geometric, and Poisson random variables.

Example

There is a computable Wiener process.

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Example

Stochastic integration in the sense of Itô (or of Stratonovich) is a computable operator.

Continuous First-Order Logic

Wesley Calvert

Calcutta Logic Circle
4 September 2011