MSO 201a: Probability and Statistics

2019-20-II Semester

Assignment-I
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1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B be two events with $P(A) = 0.2$, $P(B) = 0.4$ and $P(A \cap B) = 0.1$. Find the probability that:
(a) exactly one of the events A or B will occur; (b) at least one of the events A or B will occur; (c) none of A and B will occur.

2. Suppose that n (≥ 3) persons $P_1, \ldots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, \ldots, n-2\}$.

3. Three numbers are chosen at random from the set $\{1, 2, \ldots, 50\}$. Find the probability that the chosen numbers are in geometric progression.

4 **(Matching Problem)** A secretary types n letters and the n corresponding envelopes. In a hurry, she places at random one letter in each envelope. What is the probability that at least one letter is in the correct envelope? Find an approximation of this probability for $n = 50$.

5. In a probability space $(\Omega, \mathcal{F}, P)$, let $\{E_n\}_{n\geq 1}$ be a sequence of events.

   (a) If $\{E_n\}_{n\geq 1}$ is an increasing sequence (written as $E_n \uparrow$), i.e., $E_n \subseteq E_{n+1}$, $n = 1, 2, \ldots$, then show that

$$P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n).$$

   (b) If $\{E_n\}_{n\geq 1}$ is a decreasing sequence (written as $E_n \downarrow$), i.e., $E_{n+1} \subseteq E_n$, $n = 1, 2, \ldots$, then show that

$$P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n).$$

6. (a) **(Generalized Boole's Inequality)** For a sequence $\{E_k\}_{k\geq 1}$ of events, in a probability space $(\Omega, \mathcal{F}, P)$, show that

$$P(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} P(E_k).$$

Hint: Use Problem 5 and Boole's inequality.

(b). Let $\{E_\alpha : \alpha \in \Lambda\}$ be a countable collection of events. Show that:

(i) $P(E_\alpha) = 0, \forall \alpha \in \Lambda \ \Leftrightarrow\ P(\cup_{\alpha\in\Lambda} E_\alpha) = 0$;

(ii) $P(E_\alpha) = 1, \forall \alpha \in \Lambda \ \Leftrightarrow\ P(\cap_{\alpha\in\Lambda} E_\alpha) = 1$.

Hint: Use (a) and monotonicity of probability measures.

7. Consider four coding machines M_1, M_2, M_3 and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1} ($k = 1, 2, 3$) which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3 and M_4 change the code with probability $\frac{3}{4}$. Given that the machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

8. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that the probabilities of the student clearing examinations in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$, respectively. Assuming that the performances of the student in four subjects are independent, find the probability that the student will clear examination(s) of

(a) all the subjects; (b) no subject; (c) exactly one subject; (d) exactly two subjects; (e) at least one subject.

9. Let $\{E_k\}_{k\geq 1}$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, P)$.

(a) Suppose that $\sum_{n=1}^{\infty} P(E_n) < \infty$. Show that $P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 0$. Hence conclude that if $\sum_{n=1}^{\infty} P(E_n) < \infty$ then, with probability one, only finitely many E_n's will occur.

Hint: Use Problem 5 and Boole's inequality.

(b) If $E_1, \ldots, E_n$ are independent, show that $P(\cap_{i=1}^{n} E_i^c) \leq e^{-\sum_{i=1}^{n} P(E_i)}$;

Hint: $e^{-x} \geq 1 - x, \forall x \in \mathbb{R}$.

(c) If $E_1, E_2, \ldots$ are independent, show that $P(\cap_{i=1}^{\infty} E_i^c) \leq e^{-\sum_{i=1}^{\infty} P(E_i)}$;

Hint: Use (b) and Problem 5.

(d) Suppose that $E_1, E_2, \ldots$ are independent and $\sum_{n=1}^{\infty} P(E_n) = \infty$. Show that $P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 1$. Hence conclude that if $E_1, E_2, \ldots$ are independent and $\sum_{n=1}^{\infty} P(E_n) = \infty$ then, with probability one, infinitely many E_n's will occur.

10. Let $\Omega = \{1,2,3,4\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$ (power set of Ω). Consider the probability space $(\Omega, \mathcal{F}, P)$, where $P(\{i\}) = \frac{1}{4}, i = 1,2,3,4$. Let $A = \{1,4\}, B = \{2,4\}$ and $C = \{3,4\}$.
(a) Are A, B and C pairwise independent?; (b) Are A, B and C independent?; (c) Interpret the findings of (a) and (b) above.

11. Let A, B and C be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following:
(a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$; (b) (**Berkson's Paradox**) $P(A \cap B|C) = P(A|C)P(B|C)$ if A and B are independent events; (c) Interpret the finding of (b) above.

12. (**Simpson's Paradox:** Trends observed within different groups may disappear or reverse when groups are combined) In a probability space $(\Omega, \mathcal{F}, P)$, let A, B and D be three events. Construct an example to illustrate that it is possible to have $P(A|B \cap D) < P(A|B^c \cap D)$ and $P(A|B \cap D^c) < P(A|B^c \cap D^c)$ but $P(A|B) > P(A|B^c)$. (Read the famous example of UC Berkeley's admission data: https://www.geeksforgeeks.org/probability-and-statistics-simpsons-paradox-uc-berkeleys-lawsuit/)

13. (**Monty Hall Problem**) There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being the host of the game, knows what is behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal goat behind it (note that at least one of the other two doors has a goat behind it and Monty Hall knows the door having goat behind it). Monty Hall offers to trade the door that contestant has chosen for the other door that is closed. Should the contestant switch doors if his goal is to win the car? (This problem is based on the American television game show "Let's Make a Deal" hosted by Monty Hall.)

14. (**Gambler's Ruin Problem**) Two gamblers A and B have initial capitals of Rs. i and $N - i$, respectively, for some positive integer i. The two gamblers bet on successive and independent flips of a coin that, on each flip, results in a head with probability $p \in (0,1)$ and a tail with probability $q = 1 - p$. On each flip if heads shows up A wins Rs.1 from B and if tails shows up then B wins Rs.1 from A. The game continues until one of the players is bankrupt (ruined of all the capital he/she has). (a) Find the probability that A ends up with all the Rs.N. (b) Show that the probability that either A or B will end up with all the money

is 1 (i.e., the probability that the game will continue indefinitely is 0). (c) For $(i, N, p) = (10, 20, 0.49), (50, 100, 0.49), (100, 200, 0.49), (5, 15, 0.5), (5, 15, 0.6)$, find the probabilities that A will end up with all the money. Interpret your findings in terms of casino business.

MSO 201a: Probability and Statistics

Assignment - I

Solutions

Problem No. 1

(a) Required probability $= P((A \cap B^c) + (A^c \cap B))$

$$= P(A \cap B^c) + P(A^c \cap B)$$

$$= (P(A) - P(A \cap B)) + (P(B) - P(A \cap B))$$

$$= P(A) + P(B) - 2P(A \cap B)$$

$$= 0.2 + 0.4 - 2 \times 0.1 = 0.4$$

(b) Required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B) = 0.5$$

(c) Required probability $= P(A^c \cap B^c)$

$$= P((A \cup B)^c)$$

$$= 1 - P(A \cup B) = 1 - 0.5 \quad \text{(using (b))}$$

$$= 0.5$$

Problem No. 2

Total number of ways in which $P_1, \ldots, P_n$ can stand in a row $= \underline{|n}$

Total number of possible positions for P_1 and P_2 such that there are exactly r positions between P_1 and P_2

$$= \underline{|2} \times (n-r-1)$$

$\underline{|2}$: Corresponds to permutation of positions for P_1 and P_2

$(n-r-1)$: (Corresponds to positions $\{1, r+2\}$, $\{2, r+3\}, \ldots, \{n-r-1, n\}$)

Thus total number of ways in which $P_1, \ldots, P_n$ can stand in a row such that there are exactly r persons between P_1 and P_2

$$= (\underline{|2} \times (n-r-1)) \times \underline{|n-2}$$

$\underline{|n-2}$ → Corresponds to permutations of $(n-2)$ persons other than P_1 and P_2

Required probability $= \dfrac{(\underline{|2} \times (n-r-1)) \times \underline{|n-2}}{\underline{|n}} = \dfrac{2(n-r-1)}{n(n-1)}$.

$\boxed{\text{Problem No. 3}}$ For a, b and c $(a<b<c)$ to be in GP we must have $b = ar$ and $c = ar^2$ for some $r>1$ and $a, b, c \in \{1, \dots, 50\}$. Thus we have

$1 \le a < ar < ar^2 \le 50$; a, ar and ar^2 are integers

$$\Rightarrow \quad 1 \le a \le \frac{50}{r^2}, \quad r>1 \quad \Rightarrow \boxed{1 \le a \le \frac{50}{r^2}, \quad 1 < r \le \sqrt{50}}$$

$$\Downarrow$$

$$(r^2 \le 50)$$

Also r is rational (as $r = \frac{b}{a}$, $a, b \in \{1 \dots 50\}$)

The following cases arise.

<u>Case I</u>. r is an integer

$$1 < r \le \sqrt{50} \Rightarrow r \in \{2, 3, \dots, 7\}$$

For each $r \in \{2, 3, \dots, 7\}$ $\quad 1 \le a \le \left[\frac{50}{r^2}\right]$

maximum integer contained in $\frac{50}{r^2}$

$\Rightarrow$ total # of favorable cases with r as an integer

$$= \sum_{r=2}^{7} \left[\frac{50}{r^2}\right] = 12 + 5 + 3 + 2 + 1 + 1 = 24$$

<u>Case II</u> $r = \frac{m}{n}$, where m and n are coprimes, $m > n > 1$.

We have

$1 \le a < \frac{m}{n} a < \frac{m^2}{n^2} a \le 50$; $a, \frac{m}{n} a$ and $\frac{m^2}{n^2} a$ are integers (m and n are coprimes)

$\Rightarrow$ a is an integer and a is a multiple of n^2

Thus for each fixed $r = \frac{m}{n}$ $(m > n > 1$; m and n coprimes) we have

$$1 \le a \le \frac{50 n^2}{m^2} \quad \text{and a is a multiple of } n^2$$

i.e., $1 \le a \le \left[\frac{50 n^2}{m^2}\right]$ and a is a multiple of n^2

$r = \frac{m}{n}$	range of a $1 \le a \le \left[\frac{50n^2}{m^2}\right]$	Possible a's that are multiple of n^2	# of cases
$\frac{3}{2}$	[1, 22]	{4, 8, 12, 16, 20}	5
$\frac{5}{2}$	[1, 8]	{4, 8}	2
$\frac{7}{2}$	[1, 4]	{4}	1
$\frac{4}{3}$	[1, 28]	{9, 18, 27}	3
$\frac{5}{3}$	[1, 18]	{9, 18}	2
$\frac{7}{3}$	[1, 9]	{9}	1
$\frac{5}{4}$	[1, 32]	{16, 32}	2
$\frac{7}{4}$	[1, 16]	{16}	1
$\frac{6}{5}$	[1, 34]	{25}	1
$\frac{7}{5}$	[1, 25]	{25}	1
$\frac{7}{6}$	[1, 36]	{36}	1
			Total = 20

Thus total # of cases with r fractional = 20

$\Rightarrow$ total # of favorable cases (Case I + Case II) = 24 + 20 = 44

Required probability = $\dfrac{44}{\binom{50}{3}}$.

Problem No. 4 Define events

E_i: i-th letter is in right envelope, $i=1,\dots,n$.

Then

Required probability $= P(\bigcup_{i=1}^{n} E_i)$

$= p_{1,n} - p_{2,n} + p_{3,n} + \dots + (-1)^{n-1} p_{n,n}$,

(Inclusion-Exclusion principle)

where

$$p_{k,n} = \sum_{1 \le i_1 < i_2 < \dots < i_k \le n} \cdots \sum P(E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k}), \quad k=1,\dots,n$$

$\hookrightarrow$ this has $\binom{n}{k}$ terms

We have, for $1 \le i_1 < i_2 < \dots < i_k \le n$

$E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} =$ letters $i_1, \dots, i_k$ go to right envelopes

$\Rightarrow$ # of favorable cases to $E_{i_1} \cap E_{i_2} \cap \dots \cap E_{i_k} = \lfloor n-k$

$\Rightarrow P(E_{i_1} \cap \dots \cap E_{i_k}) = \dfrac{\lfloor n-k}{\lfloor n}$

$\Rightarrow p_{k,n} = \binom{n}{k} \dfrac{\lfloor n-k}{\lfloor n} = \dfrac{1}{\lfloor k}$, $k=1,\dots,n$

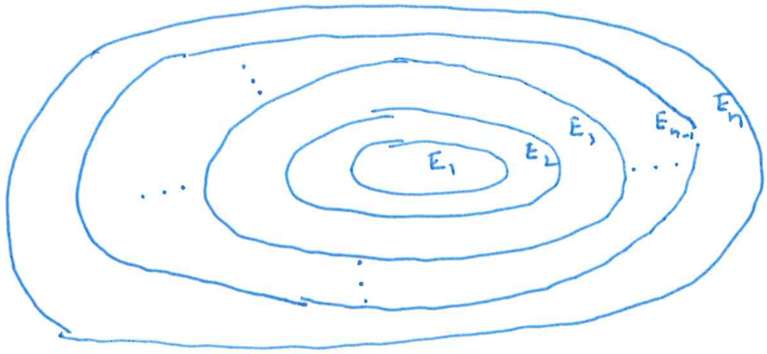
Required probability $= \dfrac{1}{\lfloor 1} - \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \dots + \dfrac{(-1)^{n-1}}{\lfloor n}$

For large n $(n \approx 50)$

Required prob. $\approx = 1 - \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \dots = 1 - e^{-1} = 0.632$

Problem No. 5

(a)



Define events

$$B_1 = E_1, \quad B_n = E_n - E_{n-1}, \quad n = 2, 3, \ldots$$

Then B_is are disjoint, $\bigcup_{k=1}^{n} B_k = E_n$, $n = 1, 2, \ldots$

$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} E_n$, and

$$P\left(\bigcup_{n=1}^{\infty} E_n\right) = P\left(\bigcup_{n=1}^{\infty} B_n\right)$$

$$= \sum_{n=1}^{\infty} P(B_n) \quad (B_n\text{'s are disjoint})$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} P(B_k)$$

$$= \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} B_k\right) = \lim_{n\to\infty} P(E_n)$$

(b) $E_n \downarrow \Rightarrow E_n^c \uparrow$. Thus, by (a)

$$P\left(\bigcup_{n=1}^{\infty} E_n^c\right) = \lim_{n\to\infty} P(E_n^c)$$

$$1 - P\left(\left(\bigcup_{n=1}^{\infty} E_n^c\right)^c\right) = \lim_{n\to\infty} [1 - P(E_n)]$$

$$P\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} P(E_n) \quad \left(\text{De-Morgan's law, } \left(\bigcup_{\alpha\in\Lambda} F_\alpha\right)^c = \bigcap_{\alpha\in\Lambda} F_\alpha^c\right)$$

Problem No. 6 (a) Define

$$B_n = \bigcup_{k=1}^{n} E_k, \quad n=1,2,\ldots$$

Then $B_n \uparrow$ and $\lim B_n = \bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} E_k = \bigcup_{k=1}^{\infty} E_k$.

By Continuity of probability measures (Problem 5)

$$P(\lim B_n) = \lim P(B_n)$$

$$P\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} E_k\right) \quad \cdots\cdots \quad (1)$$

But using Boole's inequality

$$P\left(\bigcup_{k=1}^{n} E_k\right) \le \sum_{k=1}^{n} P(E_k) \quad \forall n=1,2,\ldots$$

$$\Rightarrow \lim_{n\to\infty} P\left(\bigcup_{k=1}^{n} E_k\right) \le \lim_{n\to\infty} \sum_{k=1}^{n} P(E_k) = \sum_{k=1}^{\infty} P(E_k)$$

$$\Rightarrow P\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} P(E_k) \quad (\text{using } (1)).$$

(b)(i) Clearly if $P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0$ then

$$0 \le P(E_\beta) \le P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0 \quad \forall \beta\in\Delta$$

$$\left(E_\beta \subseteq \bigcup_{\alpha\in\Delta} E_\alpha \quad \forall \beta\in\Delta\right)$$

$$\Rightarrow P(E_\beta) = 0, \quad \forall \beta\in\Delta.$$

Conversely if $P(E_\beta) = 0, \ \forall \beta\in\Delta$ then using (a)

$$0 \le P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) \le \sum_{\alpha\in\Delta} P(E_\alpha) = 0$$

$$\Rightarrow P\left(\bigcup_{\alpha\in\Delta} E_\alpha\right) = 0.$$

(ii) $P(E_\beta) = 1 \ \forall \beta\in\Delta \iff P(E_\beta^c) = 0, \ \forall \beta\in\Delta$

$$\iff P\left(\bigcup_{\alpha\in\Delta} E_\alpha^c\right) = 0, \quad (\text{using } (i))$$

$$\iff P\left(\left(\bigcup_{\alpha\in\Delta} E_\alpha^c\right)^c\right) = 1$$

$$\iff P\left(\bigcap_{\alpha\in\Delta} E_\alpha\right) = 1.$$

$\boxed{6/1}$

Problem No. 7 Define events

E_i: ith machine produced Code 1 $i = 1, 2, 3, 4$

Required probability $= P(E_1^c \mid E_4) = 1 - P(E_1 \mid E_4)$

We have $P(E_1) = \frac{3}{4}$. By Baye's Theorem

$$P(E_1 \mid E_4) = \frac{P(E_4 \mid E_1)\, P(E_1)}{P(E_4 \mid E_1) P(E_1) + P(E_4 \mid E_1^c)\, P(E_1^c)}$$

$P(E_4 \mid E_1) = P($machines M_2, M_3 and M_4 either make no code change or make 2 code changes$)$

$$= \left(\frac{1}{4}\right)^3 + \binom{3}{2}\left(\frac{3}{4}\right)^2 \times \frac{1}{4} = \frac{7}{16}.$$

$P(E_4 \mid E_1^c) = P($machines M_2, M_3 and M_4 either make 1 Code Change or make 3 Code changes$)$

$$= \binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^3 = \frac{9}{16}.$$

Thus

$$\text{Required probability} = 1 - \frac{\frac{7}{16} \times \frac{3}{4}}{\frac{7}{16} \times \frac{3}{4} + \frac{9}{16} \times \frac{1}{4}} = \frac{3}{10}.$$

Problem No. 8 Define the events

B: Student clears Biology examination

C: Student clears Chemistry examination

P: Student clears Physics examination

M: Student clears Mathematics examination

Then $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{3}$, $P(P) = \frac{1}{4}$, $P(M) = \frac{1}{4}$ and

B, C, P and M are independent events.

(a) Required probability $= P(B \cap C \cap P \cap M)$

$= P(B)\, P(C)\, P(P)\, P(M)$ (independence)

$= \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} = \frac{1}{120}$

(b) Required probability $= P(B^c \cap C^c \cap P^c \cap M^c)$

$= P(B^c)\, P(C^c)\, P(P^c)\, P(M^c)$ (independence)

$= (1-\frac{1}{2})(1-\frac{1}{3})(1-\frac{1}{4})(1-\frac{1}{5}) = \frac{1}{5}$

(c) Required probability $= P(B \cap C^c \cap P^c \cap M^c) + P(B^c \cap C \cap P^c \cap M^c)$

$+ P(B^c \cap C^c \cap P \cap M^c) + P(B^c \cap C^c \cap P^c \cap M)$

$= \frac{1}{2} \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times (1-\frac{1}{5}) + (1-\frac{1}{2}) \times \frac{1}{3} \times (1-\frac{1}{4}) \times (1-\frac{1}{5})$

$+ (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times \frac{1}{4} \times (1-\frac{1}{5}) + (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times \frac{1}{5}$

(using independence)

$= \frac{5}{12}$

(d) Required probability $= P(B \cap C \cap P^c \cap M^c) + P(B \cap C^c \cap P \cap M^c)$

$+ P(B \cap C^c \cap P^c \cap M) + P(B^c \cap C \cap P \cap M^c)$

$+ P(B^c \cap C \cap P^c \cap M) + P(B^c \cap C^c \cap P \cap M)$

$= \frac{1}{2} \times \frac{1}{3} \times (1-\frac{1}{4}) \times (1-\frac{1}{5}) + \frac{1}{2} \times (1-\frac{1}{3}) \times \frac{1}{4} \times (1-\frac{1}{5})$

$+ \frac{1}{2} \times (1-\frac{1}{3}) \times (1-\frac{1}{4}) \times \frac{1}{5} + (1-\frac{1}{2}) \times \frac{1}{3} \times \frac{1}{4} \times (1-\frac{1}{5})$

$+ (1-\frac{1}{2}) \times \frac{1}{3} \times (1-\frac{1}{4}) \times \frac{1}{5} + (1-\frac{1}{2}) \times (1-\frac{1}{3}) \times \frac{1}{4} \times \frac{1}{5}$

$= \frac{7}{24}.$

(e) Required probability $= 1 - P(\text{no subject is cleared})$

$= 1 - \frac{1}{5} = \frac{4}{5}$ (using (b))

Problem No. 9

(a) Let $B_n = \bigcup_{k=n}^{\infty} E_k$, $n = 1, 2, \ldots$. Then $B_n \downarrow$,

$$\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim P(B_n) \quad \text{(Problem 5 (b))}$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = \lim P\left(\bigcup_{k=n}^{\infty} E_k\right)$$

$$\leq \lim_{n\to\infty} \sum_{k=n}^{\infty} P(E_k) \quad \text{(Boole's Inequality of problem 6)}$$

$$= 0 \quad \left(\text{since } \sum_{k=1}^{\infty} P(E_k) < \infty\right)$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 0$$

$$\Rightarrow P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right)^c\right) = 1$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 1$$

Note that

$\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c = \{\omega \in \Omega : \text{there exists an } n \geq 1 \text{ such that } \omega \notin E_k \ \forall k \geq n\}$

$= \{\omega \in \Omega : \omega \text{ belongs to only finitely many } E_k\text{'s}\}$.

(b)
$$P\left(\bigcap_{i=1}^{n} E_i^c\right) = \prod_{i=1}^{n} P(E_i^c) \quad \text{(Independence)}$$

$$= \prod_{i=1}^{n} (1 - P(E_i))$$

$$\leq \prod_{i=1}^{n} e^{-P(E_i)} \quad \left(e^{-x} \geq 1 - x \ \ \forall x \in \mathbb{R}\right)$$

$$= e^{-\sum_{i=1}^{n} P(E_i)}$$

(c) Let $B_n = \bigcap_{i=1}^{n} E_i^c$, $n = 1, 2, \ldots$ Then $B_n \downarrow$

$$\bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{n} E_i^c = \bigcap_{i=1}^{\infty} E_i^c$$

$$P\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n) \quad \text{(Problem 5 (b))}$$

$$\Rightarrow P\left(\bigcap_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i^c\right)$$

$$\leq \lim_{n\to\infty} e^{-\sum_{i=1}^{n} P(E_i)} \quad \text{(using (b))}$$

$$= e^{-\sum_{i=1}^{\infty} P(E_i)}$$

(d) We will show that $P\left(\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right)^c\right) = 0$, i.e.

$$P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 0.$$

Let $D_n = \bigcap_{k=n}^{\infty} E_k^c$, $n = 1, 2, \ldots$ Then $D_n \uparrow$, $\bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c$ and

$$P\left(\bigcup_{n=1}^{\infty} D_n\right) = \lim_{n\to\infty} P(D_n) \quad \text{(Problem 5(a))}$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = \lim_{n\to\infty} P\left(\bigcap_{k=n}^{\infty} E_k^c\right)$$

$$\leq \lim_{n\to\infty} e^{-\sum_{k=n}^{\infty} P(E_k)}$$

$$= 0 \quad \left(\text{as } \sum_{k=n}^{\infty} P(E_k) = \infty \;\; \forall n \geq 1\right)$$

$$\Rightarrow P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 0$$

$$\Rightarrow P\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) = 1 - P\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c\right) = 1$$

Note that

$\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k = \{\omega \in \Omega : \forall n \geq 1, \exists k \geq n$ such that $\omega \in E_k\}$

$= \{\omega \in \Omega : \omega$ belongs to infinitely many E_n's$\}$

Problem No. 10 (a) $P(A) = P(B) = P(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

$$P(A \cap B) = P(A \cap C) = P(B \cap C) = P(\{4\}) = \frac{1}{4}$$

Clearly $P(A \cap B) = P(A)\,P(B)$, $P(A \cap C) = P(A)\,P(C)$ and $P(B \cap C) = P(B)\,P(C)$

$\Rightarrow$ A, B and C are pairwise independent.

(b)

$$P(A \cap B \cap C) = P(\{4\}) = \frac{1}{4} \neq P(A)\,P(B)\,P(C) = \frac{1}{8}$$

$\Rightarrow$ A, B and C are not independent.

(c) pairwise independence $\not\Rightarrow$ independence.

Problem No. 11 (a) $P(A \cap B \mid C) = \dfrac{P(A \cap B \cap C)}{P(C)} = P(A \mid B \cap C)\,\dfrac{P(B \cap C)}{P(C)}$

$$= P(A \mid B \cap C)\,P(B \mid C)$$

(b) Consider the example of Problem 10.

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$P(B \mid C) = \frac{P(B \cap C)}{P(C)} = \frac{1/4}{1/2} = \frac{1}{2}$$

Clearly

$$P(A \cap B \mid C) \neq P(A \mid C)\,P(B \mid C).$$

(c) Events A and B may be independent in pair but they may not be independent given any other event C

Problem No. 12

Let $P(A|B\cap D) = \alpha_1$, $P(A|B^c\cap D) = \beta_1$, $P(A|B\cap D^c) = \alpha_2$, $P(A|B^c\cap D^c) = \beta_2$, $P(D|B) = p_1$ and $P(D|B^c) = p_2$. Then

$$P(A|B) = P(A\cap D|B) + P(A\cap D^c|B)$$
$$= P(A|D\cap B)\,P(D|B) + P(A|D^c\cap B)\,P(D^c|B)$$
$$= p_1\alpha_1 + (1-p_1)\alpha_2.$$

$$P(A|B^c) = p_2\beta_1 + (1-p_2)\beta_2.$$

We have to choose real numbers $\alpha_1, \alpha_2, \beta_1, \beta_2, p_1$ and p_2 such that $0<\alpha_1<\beta_1<1$, $0<\alpha_2<\beta_2<1$, $0<p_1<1$, $0<p_2<1$ and

$$p_1\alpha_1 + (1-p_1)\alpha_2 > p_2\beta_1 + (1-p_2)\beta_2.$$

i.e. $$\alpha_2 + (\alpha_1-\alpha_2)p_1 > \beta_2 + (\beta_1-\beta_2)p_2.$$

Let us take $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\beta_1 = 0.4$ and $\beta_2 = 0.8$. Then

$$0.6 - 0.4p_1 > 0.8 - 0.4p_2 \Rightarrow p_2 - p_1 > \frac{1}{2}$$

Thus one may take, for example, $\alpha_1 = 0.2$, $\alpha_2 = 0.6$, $\beta_1 = 0.4$, $\beta_2 = 0.8$, $p_1 = 0.2$ and $p_2 = 0.8$.

Problem No. 13 Let the doors be numbered 1 to N and WLOG assume that car is behind door No. 1. Define events,

D_i: Contestant chooses door No. i, $i = 1, \dots, N$

W: 'A' wins the car.

Case I: Contestant decides to switch

$P(W \mid D_1) = 0$, $P(W \mid D_i) = 1$ $i = 2, 3, \dots, N$.

$$P(W) = \sum_{i=1}^{N} P(W \mid D_i) P(D_i) \qquad \text{(Theorem of Total Probability)}$$

$$= \frac{1}{N}[0 + N-1] = \frac{N-1}{N}$$

For $N=3$, $P(W) = \frac{2}{3}$

Case II: Contestant decides not to switch

$P(W \mid D_1) = 1$, $P(W \mid D_i) = 0$, $i = 2, 3, \dots N$

$$P(W) = \sum_{i=1}^{N} P(W \mid D_i) P(D_i)$$

$$= \frac{1}{N}.$$

For $N=3$, $P(W) = \frac{1}{3}$

Thus the Contestant should ~~~~ switch the door as the probability of win doubles by doing so in case of $N=3$ and increases $N-1$ times in case of N doors.

Remark: In light of new additional information it is advisable to update prior probabilities

$\longrightarrow$ Bayesian Approach.

Problem No. 14 (a) Let

$p_i = P(A \text{ wins all the money}), \quad i = 1, 2, \ldots, N$

Here the probability of win depends on the initial capital 'i' available with 'A'. Then by theorem of total probability (on the result of first flip)

$p_i = P(A \text{ wins all the money} \mid \text{first flip is head}) \times p + P(A \text{ wins all the money} \mid \text{first flip is tail}) \times (1-p)$

$$\Rightarrow \quad p_i = p\,p_{i+1} + q\,p_{i-1}, \quad i = 2, 3, \ldots, N-1 \quad \ldots\ldots (I)$$

$$p_1 = p\,p_2 \Rightarrow p_2 - p_1 = \frac{q}{p} p_1$$

$$p_N = 1$$

$$p_{i+1} - p_i = \frac{q}{p}(p_i - p_{i-1}), \quad i = 2, 3, \ldots N-1 \quad (\text{using } (I))$$

$$p_2 - p_1 = \frac{q}{p} p_1$$

$$p_3 - p_2 = \frac{q}{p}(p_2 - p_1) = \left(\frac{q}{p}\right)^2 p_1$$

$$p_4 - p_3 = \frac{q}{p}(p_3 - p_2) = \left(\frac{q}{p}\right)^3 p_1$$

$$\vdots$$

$$p_i - p_{i-1} = \left(\frac{q}{p}\right)^{i-1} p_1$$

Summing the last $(i-1)$ rows we get

$$p_i - p_1 = \left[\frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right]$$

$$p_i = p_1\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \cdots + \left(\frac{q}{p}\right)^{i-1}\right] = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-q/p}\, p_1, & \text{if } \frac{q}{p} \neq 1 \\ i\,p_1, & \text{if } \frac{q}{p} = 1 \end{cases}$$

We have $p_N = 1$. Thus

$$p_i = \begin{cases} \dfrac{1-(q/p)^i}{1-(q/p)^N}, & \text{if } p \neq \frac{1}{2} \\ \dfrac{i}{N}, & \text{if } p = \frac{1}{2} \end{cases}, \quad i = 1, \ldots, N$$

$$\left(\frac{q}{p} \neq (=)\, 1 \iff p \neq (=)\, \frac{1}{2}\right)$$

(b) Let q_i be the probability that B will win all the money.

By symmetry

$$q_i = \begin{cases} \dfrac{1-(p/q)^{N-i}}{1-p/q}, & \text{if } q \neq \frac{1}{2} \ (\text{or } p \neq \frac{1}{2}) \\ \dfrac{N-i}{N}, & \text{if } q = \frac{1}{2} \ (\text{or } p = \frac{1}{2}) \end{cases}$$

Clearly $p_i + q_i = 1 \quad \forall \ i = 1, \dots, N$.

(c) For $i = 10$, $N = 20$ & $p = 0.49$, $p_i = 0.4$, $q_i = 0.6$

For $i = 50$, $N = 100$ & $p = 0.49$, $p_i = 0.12$, $q_i = 0.88$

For $i = 100$, $N = 200$ & $p = 0.49$, $p_i = 0.02$, $q_i = 0.98$

In Casino even if the game may look fair ($p \approx 0.49$), the gambler is bound to be ruined.

For $i = 5$, $N = 15$ and $p = 0.5$, $p_i = \frac{1}{3}$, $q_i = \frac{2}{3}$

For $i = 5$, $N = 15$ and $p = 0.6$, $p_i = 0.87$, $q_i = 0.13$

Small variations in p effect p_i significantly