# MSO 201a: Probability and Statistics

## 2019-20-II Semester

## Assignment-II

**Instructor: Neeraj Misra**

1. Let $D$ be the set of discontinuity points of a distribution function $F$. For each $n \in \{1, 2, \ldots\}$, define $D_n = \{x \in \mathbb{R} : F(x) - F(x-) \geq \frac{1}{n}\}$. Show that each $D_n$ $(n = 1, 2, \ldots)$ is finite. Hence show that a distribution function can not have uncountable number of discontinuities.

2. Do the following functions define distribution functions?

$$(i) \quad F_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2} \end{cases}; \qquad (ii) \quad F_2(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \geq 0 \end{cases};$$

and (iii) $F_3(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}, -\infty < x < \infty$.

3. Let $X$ be a random variable with distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{2}{3}, & \text{if } 0 \leq x < 1 \\ \frac{7-6c}{6}, & \text{if } 1 \leq x < 2 \\ \frac{4c^2-9c+6}{4}, & \text{if } 2 \leq x \leq 3 \\ 1, & \text{if } x > 3 \end{cases},$$

where $c$ is a real constant.

(i) Find the value of constant $c$;

(ii) Using the distribution function $F$, find $P(1 < X < 2)$; $P(2 \leq X < 3)$; $P(0 < X \leq 1)$; $P(1 \leq X \leq 2)$; $P(X \geq 3)$; $P(X = \frac{5}{2})$ and $P(X = 2)$;

(iii) Find the conditional probabilities $P(X = 1 | 1 \leq X \leq 2)$ and $P(1 \leq X < 2 | X > 1)$.

(iv) Show that $X$ is a discrete r.v.. Find the support and the p.m.f. of $X$.

4. Let $X$ be a random variable with distribution function $F(\cdot)$. In each of the following cases determine whether $X$ is a discrete r.v. or a continuous r.v.. Also find the

p.d.f./p.m.f. of $X$:

$$\text{(i) } F(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{3}, & \text{if } -2 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 5 \\ \frac{3}{4}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases} ; \quad \text{(ii) } F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}.$$

5. Let the random variable $X$ have the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \le x < 1 \\ \frac{2}{3}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

(i) Show that $X$ is neither a discrete r.v. nor a continuous r.v.;
(ii) Evaluate $P(X = 1)$, $P(X = 2)$, $P(X = 1.5)$ and $P(1 < X < 2)$;
(iii) Evaluate the conditional probability $P(1 \le X < 2 | 1 \le X \le 2)$.

6. A random variable $X$ has the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{2}{3}, & \text{if } 2 \le x < 5 \\ \frac{7-6k}{6}, & \text{if } 5 \le x < 9 \\ \frac{3k^2-6k+7}{6}, & \text{if } 9 \le x < 14 \\ \frac{16k^2-16k+19}{16}, & \text{if } 14 \le x \le 20 \\ 1, & \text{if } x > 20 \end{cases},$$

where $k \in \mathbb{R}$.
(i) Find the value of constant $k$;
(ii) Show that $X$ is a discrete r.v. and find its support;
(iii) Find the p.m.f. of $X$.

7. A discrete random variable $X$ has the p.m.f.

$$f(x) = \begin{cases} \frac{c}{(2x-1)(2x+1)}, & \text{if } x \in \{1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases},$$

where $c \in \mathbb{R}$.
(i) Find the value of constant $c$;
(ii) For positive integers $m$ and $n$ such that $m < n$, using the p.m.f. evaluate

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Assignment - II

Solutions

**Problem No. 1** **Claim I** Each $D_n$ $(n = 1, 2, \ldots)$ is finite.

On contrary suppose that some $D_{n_0}$ is infinite. Then $D_{n_0}$ contains a countably infinite set (every infinite set has a countably infinite subset), say $E = \{a_1, a_2, \ldots\}$ (every countably infinite set can be represented by a sequence $\{a_n\}_{n \geq 1}$).

For $x \in E$ $(\subseteq D_{n_0})$, $F(x) - F(x-) \geq \frac{1}{n_0}$,

$$P(X \in D_{n_0}) \geq P(X \in E) = \sum_{x \in E} P(X = x) = \sum_{x \in E} [F(x) - F(x-)] \geq \sum_{x \in E} \frac{1}{n_0} = \infty$$

$\rightarrow$ Contradiction

**Claim II** $D$ (= set of discontinuity points of $F$) is countable.

$$D = \{x \in \mathbb{R} : F(x) - F(x-) > 0\} = \bigcup_{n=1}^{\infty} \{x \in \mathbb{R} : F(x) - F(x-) \geq \tfrac{1}{n}\} = \bigcup_{n=1}^{\infty} D_n$$

$\rightarrow$ Countable (Countable union of countable sets is countable).

**Problem No. 2** (i) $F_1(\frac{1}{2}+) = 1 \neq \frac{1}{2} = F_1(\frac{1}{2}) \Rightarrow F_1$ is not a d.f.

(ii) $F_2$ is non-decreasing, $F_2(-\infty) = 0$, $F_2(+\infty) = 1$ and $F_2$ is right continuous (in fact continuous) $\Rightarrow F_2$ is a d.f.

(iii) Using arguments of (ii) $F_3$ is a d.f.

**Problem No. 3** (i) $F(3) = F(3+) \Rightarrow \dfrac{4c^2 - 9c + 6}{4} = 1 \Rightarrow c = \frac{1}{4}, 2$

$F(1-) \leq F(1) \Rightarrow \frac{2}{3} \leq \dfrac{7 - 6c}{6} \Rightarrow c \leq \frac{1}{2}$

$\Rightarrow c = \frac{1}{4}$

(ii) $P(1 < X < 2) = F(2-) - F(1) = \frac{11}{12} - \frac{11}{12} = 0$

$P(2 \leq X < 3) = F(3-) - F(2-) = 1 - \frac{11}{12} = \frac{1}{12}$

$P(0 < X \leq 1) = F(1) - F(0) = \frac{11}{12} - \frac{2}{3} = \frac{1}{4}$

$P(1 \leq X \leq 2) = F(2) - F(1-) = 1 - \frac{2}{3} = \frac{1}{3}$

$P(X \geq 3) = 1 - F(3-) = 1 - 1 = 0$

$P(X = \frac{5}{2}) = F(\frac{5}{2}) - F(\frac{5}{2}-) = 0$

$P(X = 2) = F(2) - F(2-) = 1 - \frac{11}{12} = \frac{1}{12}$

(iii) $P(X=1 \mid 1 \le X \le 2) = \dfrac{P(X=1,\ 1 \le X \le 2)}{P(1 \le X \le 2)}$

$$= \frac{P(X=1)}{P(1 \le X \le 2)} = \frac{F_X(1) - F_X(1-)}{F_X(2) - F_X(1-)}$$

$$= \frac{\frac{11}{12} - \frac{2}{3}}{1 - \frac{2}{3}} = \frac{3}{4}$$

$P(1 \le X < 2 \mid X > 1) = \dfrac{P(1 \le X < 2,\ X > 1)}{P(X > 1)}$

$$= \frac{P(1 < X < 2)}{P(X > 1)} = \frac{F_X(2-) - F_X(1)}{1 - F_X(1)}$$

$$= \frac{\frac{11}{12} - \frac{11}{12}}{1 - \frac{11}{12}} = 0.$$

(iv) $D =$ set of discontinuity points of $F = \{0, 1, 2\}$

Sum of jumps $= [F_X(0) - F_X(0-)] + [F_X(1) - F_X(1-)] + [F_X(2) - F_X(2-)]$

$$= \left(\frac{2}{3} - 0\right) + \left(\frac{11}{12} - \frac{2}{3}\right) + \left(1 - \frac{11}{12}\right)$$

$$= 1$$

$\Rightarrow$ $X$ is of discrete type.

The p.m.f. of $X$ is

$$f_X(x) = P(X=x) = F_X(x) - F_X(x-)$$

$$= \begin{cases} \frac{2}{3}, & \text{if } x = 0 \\ \frac{1}{4}, & \text{if } x = 1 \\ \frac{1}{12}, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

**Problem No. 4** (i) D = set of discontinuity points of $F_X$

$= \{-2, 0, 5, 6\}$

sum of jumps $= \sum_{x \in D} [F(x) - F(x-)]$

$= \frac{1}{3} + \frac{1}{6} + \frac{1}{4} + \frac{1}{4} = 1$

$\Rightarrow$ X is of discrete type with p.m.f.

$$f_X(x) = P(X=x) = F(x) - F(x-) = \begin{cases} \frac{1}{3}, & x=-2 \\ \frac{1}{6}, & x=0 \\ \frac{1}{4}, & x=5, 6 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Clearly $F_X$ is continuous everywhere and differentiable everywhere except at $x=0$.

$$F'(x) = \begin{cases} 0, & x<0 \\ e^{-x}, & x>0 \end{cases}$$

$$\int_{-\infty}^{\infty} F'(x)\,dx = \int_0^{\infty} e^{-x}\,dx = 1$$

$\Rightarrow$ X is of ~~absolutely~~ continuous type with a p.d.f.

$$f(x) = \begin{cases} e^{-x}, & \text{if } x>0 \\ 0, & \text{otherwise.} \end{cases}$$

**Problem No. 5** (i) D = set of discontinuity points of $F_X$

$= \{1, 2\} \neq \phi$

$\Rightarrow$ X is not of continuous type

sum of jumps $= [F(1) - F(1-)] + [F(2) - F(2-)]$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \neq 1$

$\Rightarrow$ X is not of discrete type.

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(ii) $P(X=1) = F(1) - F(1-) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$;

$P(X=2) = F(2) - F(2-) = 1 - \frac{2}{3} = \frac{1}{3}$;

$P(X=1.5) = F(1.5) - F(1.5-) = 0$ ($x=1.5$ is a continuity point of $F_X$)

$P(1<X<2) = F(2-) - F(1) = \frac{2}{3} - \frac{2}{3} = 0.$

(iii) $$P(1 \le X < 2 \mid 1 \le X \le 2) = \frac{P(1 \le X < 2, 1 \le X \le 2)}{P(1 \le X \le 2)}$$

$$= \frac{P(1 \le X < 2)}{P(1 \le X \le 2)} = \frac{F(2-) - F(1-)}{F(2) - F(1-)}$$

$$= \frac{\frac{2}{3} - \frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}.$$

$\boxed{\text{Problem No. 6}}$ (i) $F(20) = F(20+) \Rightarrow 16k^2 - 16k + 3 = 0$

$\Rightarrow k = \frac{1}{4}, \frac{3}{4}$ ..... (I)

$F(5-) \le F(5) \Rightarrow k \le \frac{1}{2}$. .... (II)

(I) + (II) $\Rightarrow k = \frac{1}{4}$

$$F(x) = \begin{cases} 0, & x < 2 \\ \frac{2}{3}, & 2 \le x < 5 \\ \frac{11}{12}, & 5 \le x < 9 \\ \frac{91}{96}, & 9 \le x < 14 \\ 1, & x \ge 14 \end{cases}$$

(ii) $D$ = set of discontinuity points of $F = \{2, 5, 9, 14\}$

sum of jumps $= \sum_{x \in D} [F(x) - F(x-)]$

$= (\frac{2}{3} - 0) + (\frac{11}{12} - \frac{2}{3}) + (\frac{91}{96} - \frac{11}{12}) + (1 - \frac{91}{96})$

$= 1$

$\Rightarrow X$ is of discrete type with ~~range~~ support $S_X = D_X = \{2, 5, 9, 14\}$.

(iii) The p.m.f. of X is

$$f(x) = P(X=x) = F(x) - F(x-) = \begin{cases} \frac{2}{3}, & x=2 \\ \frac{1}{4}, & x=5 \\ \frac{1}{32}, & x=9 \\ \frac{5}{96}, & x=14 \\ 0, & \text{otherwise} \end{cases}$$

**Problem No. 7**

(i) $S_X =$ support of $X = \{1, 2, 3, \ldots\}$

$$\sum_{x \in S_X} f_X(x) = 1$$

$$\Rightarrow \sum_{i=1}^{\infty} \frac{c}{(2i-1)(2i+1)} = 1$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{c}{(2i-1)(2i+1)} = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{2i-1} - \frac{1}{2i+1}\right] = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{2i-1} - \sum_{i=1}^{n} \frac{1}{2i+1}\right] = 1$$

$$\Rightarrow \frac{c}{2} \lim_{n\to\infty} \left[1 - \frac{1}{2n+1}\right] = 1 \quad \Rightarrow c = 2.$$

(ii) $$P(X < m+1) = P(X \le m)$$

$$= \sum_{i=1}^{m} \frac{2}{(2i-1)(2i+1)}$$

$$= \sum_{i=1}^{m} \left[\frac{1}{2i-1} - \frac{1}{2i+1}\right] = 1 - \frac{1}{2m+1} = \frac{2m}{2m+1} \quad \ldots\ldots (A)$$

$$P(X \ge m) = 1 - P(X < m)$$

$$= 1 - \frac{2(m-1)}{2(m-1)+1} \quad (\text{from } (A))$$

$$= \frac{1}{2m-1} \quad \ldots\ldots \quad (A_1)$$

$$P(m \le X < n) = P(X<n) - P(X<m)$$

$$= \frac{2(n-1)}{2n-1} - \frac{2(m-1)}{2m-1} \qquad (\text{using } (A))$$

$$= \frac{2(n-m)}{(2n-1)(2m-1)} \qquad \ldots\ldots (B)$$

$$P(m < X \le n) = P(m+1 \le X < n+1)$$

$$= \frac{2(n-m)}{(2n+1)(2m+1)} \qquad (\text{using } (B)).$$

(iii) $$P(X>1 \mid 1 \le X < 4) = \frac{P(X>1,\ 1\le X<4)}{P(1\le X<4)}$$

$$= \frac{P(1<X<4)}{P(1\le X<4)} = \frac{P(2\le X<4)}{P(1\le X<4)}$$

$$= \frac{\frac{2(4-2)}{7\times 3}}{\frac{2(4-1)}{7\times 1}} = \frac{2}{9} \quad \left(\begin{array}{c}\text{using}\\(B)\end{array}\right)$$

$$P(1<X<6 \mid X \ge 3) = \frac{P(1<X<6,\ X\ge 3)}{P(X\ge 3)}$$

$$= \frac{P(3\le X<6)}{P(X\ge 3)} = \frac{\frac{2\times 3}{11\times 5}}{\frac{1}{5}} \quad \left(\begin{array}{c}\text{using}\\(A) \text{ and}\\(B)\end{array}\right)$$

$$= \frac{6}{11}$$

(iv) Clearly, for $x<1$, $F(x)=0$. For $i \le x < i+1$, $i=1,2,3\ldots$

$$F(x) = P(X<i+1) = \frac{2i}{2i+1} \qquad (\text{using } (A))$$

$$\Rightarrow F(x) = \begin{cases} 0, & x<1 \\ \frac{2i}{2i+1}, & i\le x<i+1,\ i=1,2,3\ldots \end{cases}$$

**Problem No. 8** (i) Clearly $F_X$ is differentiable everywhere except at points 1 and 2.

$$F'(x) = \begin{cases} 0 & x<0 \\ x, & 0<x<1 \\ \frac{1}{2}, & 1<x<2 \\ 0, & x>2 \end{cases}$$

$$\int_{-\infty}^{\infty} F_X'(x)\,dx = \int_0^1 x\,dx + \int_1^2 \frac{1}{2}\,dx = 1$$

$\Rightarrow$ X is of continuous type.

(ii) $P(X=x) = F_X(x) - F_X(x^-) = 0, \ \forall x \in \mathbb{R}$ (as X is of continuous type)

$\Rightarrow P(X=1) = P(X=2) = 0$

$$P(1<X<2) = P(1\le X<2) = P(1<X\le 2) = P(1\le X\le 2)$$
$$= F(2) - F(1) = 1-\frac{1}{2} = \frac{1}{2}$$

$$P(X\ge 1) = 1-F(1) = 1-\frac{1}{2} = \frac{1}{2}.$$

(iii) From (i) the p.d.f. of X is

$$f(x) = \begin{cases} x, & 0<x<1 \\ \frac{1}{2}, & 1<x<2 \\ 0, & \text{otherwise} \end{cases}$$

(iv) $F(\xi_{\frac{1}{4}}) = \frac{1}{4}$

$\Rightarrow \xi_{\frac{1}{4}} = \frac{1}{\sqrt{2}}$

$F(\xi_{1/2}) = \frac{1}{2} \Rightarrow \xi_{\frac{1}{2}} = 1$

$F(\xi_{3/4}) = \frac{3}{4} \Rightarrow \xi_{\frac{3}{4}} = \frac{3}{2}$

**Problem No. 9** (i) Since $f$ is p.d.f.

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad\Rightarrow\quad \int_{-\frac{1}{2}}^{\frac{1}{2}} (k-|x|)\,dx = 1 \quad\Rightarrow k = \frac{5}{4}.$$

(ii) $$P(X<0) = P(X\le 0) = \int_{-\infty}^{0} f(x)\,dx = \int_{-\frac{1}{2}}^{0}\left(\frac{5}{4}+x\right)dx = \frac{1}{2}$$

$$P(0<X\le\tfrac{1}{4}) = P(0\le X<\tfrac{1}{4}) = \int_0^{1/4} f(x)\,dx = \int_0^{1/4}\left(\frac{5}{4}-x\right)dx = \frac{9}{32}$$

$$P\left(-\tfrac{1}{8} \le x \le \tfrac{1}{4}\right) = \int_{-\frac{1}{8}}^{\frac{1}{4}} f(x)\,dx = \int_{-\frac{1}{8}}^{0} \left(\tfrac{5}{4}+x\right)dx + \int_{0}^{1/4} \left(\tfrac{5}{4}-x\right)dx = \frac{55}{128}$$

$$\text{(iii)}\ P\left(x > \tfrac{1}{4} \,\middle|\, |x| > \tfrac{2}{5}\right) = \frac{P\left(x > \frac{1}{4},\ |x| > \frac{2}{5}\right)}{P\left(|x| > \frac{2}{5}\right)} = \frac{P\left(x > \frac{2}{5}\right)}{P\left(|x| > \frac{2}{5}\right)}$$

$$P\left(x > \tfrac{2}{5}\right) = \int_{2/5}^{1/2} \left(\tfrac{5}{4}-x\right)dx = \frac{2}{25}$$

$$P\left(|x| > \tfrac{2}{5}\right) = P\left(x < -\tfrac{2}{5} \text{ or } x > \tfrac{2}{5}\right) = P\left(x < -\tfrac{2}{5}\right) + P\left(x > \tfrac{2}{5}\right) = \int_{-\frac{1}{2}}^{-2/5} \left(\tfrac{5}{4}+x\right)dx + \frac{2}{25} = \frac{4}{25}$$

$$\Rightarrow P\left(x > \tfrac{1}{4} \,\middle|\, |x| > \tfrac{2}{5}\right) = \frac{\frac{2}{25}}{\frac{4}{25}} = \frac{1}{2}$$

$$P\left(\tfrac{1}{8} < x < \tfrac{2}{5} \,\middle|\, \tfrac{1}{10} < x < \tfrac{1}{5}\right) = \frac{P\left(\frac{1}{8} < x < \frac{2}{5},\ \frac{1}{10} < x < \frac{1}{5}\right)}{P\left(\frac{1}{10} < x < \frac{1}{5}\right)} = \frac{P\left(\frac{1}{8} < x < \frac{1}{5}\right)}{P\left(\frac{1}{10} < x < \frac{1}{5}\right)}$$

$$= \frac{\int_{\frac{1}{8}}^{\frac{1}{5}} \left(\frac{5}{4}-x\right)dx}{\int_{\frac{1}{10}}^{\frac{1}{5}} \left(\frac{5}{4}-x\right)dx} = \frac{261}{352}$$

$$\text{(iv)}\ F(x) = \int_{-\infty}^{x} f(t)\,dt = \begin{cases} 0, & x < -\frac{1}{2} \\ \int_{-\frac{1}{2}}^{x} \left(\frac{5}{4}+t\right)dt, & -\frac{1}{2} \le x < 0 \\ \int_{-\frac{1}{2}}^{0} \left(\frac{5}{4}+t\right)dt + \int_{0}^{x} \left(\frac{5}{4}-t\right)dt, & 0 \le x < \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

$$= \begin{cases} 0, & x < -\frac{1}{2} \\ \frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}, & -\frac{1}{2} \le x < 0 \\ -\frac{x^2}{2} + \frac{5}{4}x + \frac{1}{2}, & 0 \le x < \frac{1}{2} \\ 1, & x \ge \frac{1}{2} \end{cases}$$

(V)

$F(\xi_{1/4}) = \frac{1}{4}$

$\Rightarrow \frac{\xi_{1/4}^2}{2} + \frac{5}{4}\xi_{1/4} + \frac{1}{2} = \frac{1}{4}$

$2\xi_{1/4}^2 + 5\xi_{1/4} + 1 = 0$

$\xi_{1/4} = \frac{-5+\sqrt{17}}{4}$

$F(\xi_{1/2}) = \frac{1}{2}$

$\Rightarrow \xi_{1/2} = 0$

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$F(\xi_{3/4}) = \frac{3}{4}$

$\Rightarrow -\frac{\xi_{3/4}^2}{2} + \frac{5}{4}\xi_{3/4} + \frac{1}{2} = \frac{3}{4}$

$\Rightarrow 2\xi_{3/4}^2 - 5\xi_{3/4} + 1 = 0$

$\xi_{3/4} = \frac{5-\sqrt{17}}{4}$