MSO 201a: Probability and Statistics
2019-20-II Semester
Assignment-I
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1. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let A and B be two events with $P(A) = 0.2, P(B) = 0.4$ and $P(A \cap B) = 0.1$. Find the probability that:
(a) exactly one of the events A or B will occur; (b) at least one of the events A or B will occur; (c) none of A and B will occur.

2. Suppose that n (≥ 3) persons $P_1, \ldots, P_n$ are made to stand in a row at random. Find the probability that there are exactly r persons between P_1 and P_2; here $r \in \{1, \ldots, n-2\}$.

3. Three numbers are chosen at random from the set $\{1, 2, \ldots, 50\}$. Find the probability that the chosen numbers are in geometric progression.

4 (**Matching Problem**) A secretary types n letters and the n corresponding envelopes. In a hurry, she places at random one letter in each envelope. What is the probability that at least one letter is in the correct envelope? Find an approximation of this probability for $n = 50$.

5. In a probability space $(\Omega, \mathcal{F}, P)$, let $\{E_n\}_{n\geq 1}$ be a sequence of events.

   (a) If $\{E_n\}_{n\geq 1}$ is an increasing sequence (written as $E_n \uparrow$), i.e., $E_n \subseteq E_{n+1},\ n = 1, 2, \ldots$, then show that
$$P(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n).$$

   (b) If $\{E_n\}_{n\geq 1}$ is a decreasing sequence (written as $E_n \downarrow$), i.e., $E_{n+1} \subseteq E_n,\ n = 1, 2, \ldots$, then show that
$$P(\bigcap_{n=1}^{\infty} E_n) = \lim_{n\to\infty} P(E_n).$$

6. (a) (**Generalized Boole's Inequality**) For a sequence $\{E_k\}_{k\geq 1}$ of events, in a probability space $(\Omega, \mathcal{F}, P)$, show that
$$P(\cup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} P(E_k).$$

Hint: Use Problem 5 and Boole's inequality.

(b). Let $\{E_\alpha : \alpha \in \Lambda\}$ be a countable collection of events. Show that:

(i) $P(E_\alpha) = 0, \forall \alpha \in \Lambda \;\Leftrightarrow\; P(\cup_{\alpha\in\Lambda} E_\alpha) = 0$;

(ii) $P(E_\alpha) = 1, \forall \alpha \in \Lambda \;\Leftrightarrow\; P(\cap_{\alpha\in\Lambda} E_\alpha) = 1$.

Hint: Use (a) and monotonicity of probability measures.

7. Consider four coding machines M_1, M_2, M_3 and M_4 producing binary codes 0 and 1. The machine M_1 produces codes 0 and 1 with respective probabilities $\frac{1}{4}$ and $\frac{3}{4}$. The code produced by machine M_k is fed into machine M_{k+1} ($k = 1, 2, 3$) which may either leave the received code unchanged or may change it. Suppose that each of the machines M_2, M_3 and M_4 change the code with probability $\frac{3}{4}$. Given that the machine M_4 has produced code 1, find the conditional probability that the machine M_1 produced code 0.

8. A student appears in the examinations of four subjects Biology, Chemistry, Physics and Mathematics. Suppose that the probabilities of the student clearing examinations in these subjects are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$, respectively. Assuming that the performances of the student in four subjects are independent, find the probability that the student will clear examination(s) of
(a) all the subjects; (b) no subject; (c) exactly one subject; (d) exactly two subjects; (e) at least one subject.

9. Let $\{E_k\}_{k\geq 1}$ be a sequence of events in the probability space $(\Omega, \mathcal{F}, P)$.

   (a) Suppose that $\sum_{n=1}^{\infty} P(E_n) < \infty$. Show that $P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 0$. Hence conclude that if $\sum_{n=1}^{\infty} P(E_n) < \infty$ then, with probability one, only finitely many E_n's will occur.
   Hint: Use Problem 5 and Boole's inequality.

   (b) If $E_1, \ldots, E_n$ are independent, show that $P(\cap_{i=1}^{n} E_i^c) \leq e^{-\sum_{i=1}^{n} P(E_i)}$;
   Hint: $e^{-x} \geq 1 - x, \forall x \in \mathbb{R}$.

   (c) If $E_1, E_2, \ldots$ are independent, show that $P(\cap_{i=1}^{\infty} E_i^c) \leq e^{-\sum_{i=1}^{\infty} P(E_i)}$;
   Hint: Use (b) and Problem 5.

   (d) Suppose that $E_1, E_2, \ldots$ are independent and $\sum_{n=1}^{\infty} P(E_n) = \infty$. Show that $P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} E_k) = 1$. Hence conclude that if $E_1, E_2, \ldots$ are independent and $\sum_{n=1}^{\infty} P(E_n) = \infty$ then, with probability one, infinitely many E_n's will occur.

10. Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{F} = \mathcal{P}(\Omega)$ (power set of Ω). Consider the probability space $(\Omega, \mathcal{F}, P)$, where $P(\{i\}) = \frac{1}{4}, i = 1, 2, 3, 4$. Let $A = \{1, 4\}, B = \{2, 4\}$ and $C = \{3, 4\}$.
(a) Are A, B and C pairwise independent?; (b) Are A, B and C independent?; (c) Interpret the findings of (a) and (b) above.

11. Let A, B and C be three events such that $P(B \cap C) > 0$. Prove or disprove each of the following:
(a) $P(A \cap B|C) = P(A|B \cap C)P(B|C)$; (b) (**Berkson's Paradox**) $P(A \cap B|C) = P(A|C)P(B|C)$ if A and B are independent events; (c) Interpret the finding of (b) above.

12. (**Simpson's Paradox:** Trends observed within different groups may disappear or reverse when groups are combined) In a probability space $(\Omega, \mathcal{F}, P)$, let A, B and D be three events. Construct an example to illustrate that it is possible to have $P(A|B \cap D) < P(A|B^c \cap D)$ and $P(A|B \cap D^c) < P(A|B^c \cap D^c)$ but $P(A|B) > P(A|B^c)$. (Read the famous example of UC Berkeley's admission data: https://www.geeksforgeeks.org/probability-and-statistics-simpsons-paradox-uc-berkeleys-lawsuit/)

13. (**Monty Hall Problem**) There are 3 doors with one door having an expensive car behind it and each of the other 2 doors having a goat behind them. Monty Hall, being the host of the game, knows what is behind each door. A contestant is asked to select one of the doors and he wins the item (car or goat) behind the selected door. The contestant selects one of the doors at random, and then Monty Hall opens one of the other two doors to reveal goat behind it (note that at least one of the other two doors has a goat behind it and Monty Hall knows the door having goat behind it). Monty Hall offers to trade the door that contestant has chosen for the other door that is closed. Should the contestant switch doors if his goal is to win the car? (This problem is based on the American television game show "Let's Make a Deal" hosted by Monty Hall.)

14. (**Gambler's Ruin Problem**) Two gamblers A and B have initial capitals of Rs. i and $N - i$, respectively, for some positive integer i. The two gamblers bet on successive and independent flips of a coin that, on each flip, results in a head with probability $p \in (0, 1)$ and a tail with probability $q = 1 - p$. On each flip if heads shows up A wins Rs.1 from B and if tails shows up then B wins Rs.1 from A. The game continues until one of the players is bankrupt (ruined of all the capital he/she has). (a) Find the probability that A ends up with all the Rs.N. (b) Show that the probability that either A or B will end up with all the money

is 1 (i.e., the probability that the game will continue indefinitely is 0). (c) For $(i, N, p) = (10, 20, 0.49), (50, 100, 0.49), (100, 200, 0.49), (5, 15, 0.5), (5, 15, 0.6)$, find the probabilities that A will end up with all the money. Interpret your findings in terms of casino business.