MSO 201a: Probability and Statistics

2019-20-II Semester

Assignment-II

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1. Let D be the set of discontinuity points of a distribution function F. For each $n \in \{1, 2, \ldots\}$, define $D_n = \{x \in \mathbb{R} : F(x) - F(x-) \geq \frac{1}{n}\}$. Show that each D_n $(n = 1, 2, \ldots)$ is finite. Hence show that a distribution function can not have uncountable number of discontinuities.

2. Do the following functions define distribution functions?

$$(i) \quad F_1(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1, & \text{if } x > \frac{1}{2} \end{cases}; \qquad (ii) \quad F_2(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \geq 0 \end{cases};$$

   and (iii) $F_3(x) = \frac{1}{2} + \frac{\tan^{-1}(x)}{\pi}, -\infty < x < \infty$.

3. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{2}{3}, & \text{if } 0 \leq x < 1 \\ \frac{7-6c}{6}, & \text{if } 1 \leq x < 2 \\ \frac{4c^2-9c+6}{4}, & \text{if } 2 \leq x \leq 3 \\ 1, & \text{if } x > 3 \end{cases},$$

   where c is a real constant.
   (i) Find the value of constant c;
   (ii) Using the distribution function F, find $P(1 < X < 2)$; $P(2 \leq X < 3)$; $P(0 < X \leq 1)$; $P(1 \leq X \leq 2)$; $P(X \geq 3)$; $P(X = \frac{5}{2})$ and $P(X = 2)$;
   (iii) Find the conditional probabilities $P(X = 1 | 1 \leq X \leq 2)$ and $P(1 \leq X < 2 | X > 1)$.
   (iv) Show that X is a discrete r.v.. Find the support and the p.m.f. of X.

4. Let X be a random variable with distribution function $F(\cdot)$. In each of the following cases determine whether X is a discrete r.v. or a continuous r.v.. Also find the

p.d.f./p.m.f. of X:

$$\text{(i) } F(x) = \begin{cases} 0, & \text{if } x < -2 \\ \frac{1}{3}, & \text{if } -2 \le x < 0 \\ \frac{1}{2}, & \text{if } 0 \le x < 5 \\ \frac{3}{4}, & \text{if } 5 \le x < 6 \\ 1, & \text{if } x \ge 6 \end{cases} ; \quad \text{(ii) } F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}.$$

5. Let the random variable X have the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \le x < 1 \\ \frac{2}{3}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}.$$

(i) Show that X is neither a discrete r.v. nor a continuous r.v.;
(ii) Evaluate $P(X = 1),\ P(X = 2),\ P(X = 1.5)$ and $P(1 < X < 2)$;
(iii) Evaluate the conditional probability $P(1 \le X < 2 | 1 \le X \le 2)$.

6. A random variable X has the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 2 \\ \frac{2}{3}, & \text{if } 2 \le x < 5 \\ \frac{7-6k}{6}, & \text{if } 5 \le x < 9 \\ \frac{3k^2-6k+7}{6}, & \text{if } 9 \le x < 14 \\ \frac{16k^2-16k+19}{16}, & \text{if } 14 \le x \le 20 \\ 1, & \text{if } x > 20 \end{cases},$$

where $k \in \mathbb{R}$.
(i) Find the value of constant k;
(ii) Show that X is a discrete r.v. and find its support;
(iii) Find the p.m.f. of X.

7. A discrete random variable X has the p.m.f.

$$f(x) = \begin{cases} \frac{c}{(2x-1)(2x+1)}, & \text{if } x \in \{1, 2, 3, \ldots\} \\ 0, & \text{otherwise} \end{cases},$$

where $c \in \mathbb{R}$.
(i) Find the value of constant c;
(ii) For positive integers m and n such that $m < n$, using the p.m.f. evaluate

$P(X < m+1),\ P(X \geq m),\ P(m \leq X < n)$ and $P(m < X \leq n)$;
(iii) Find the conditional probabilities $P(X > 1 | 1 \leq X < 4)$ and $P(1 < X < 6 | X \geq 3)$.
(iv) Determine the distribution function of X.

8. Let X be a random variable with distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } 0 \leq x < 1 \\ \frac{x}{2}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}.$$

(i) Show that X is a continuous r.v.;
(ii) Using the distribution function, evaluate $P(X = 1)$; $P(X = 2)$; $P(1 < X < 2)$; $P(1 \leq X < 2)$; $P(1 < X \leq 2)$; $P(1 \leq X \leq 2)$ and $P(X \geq 1)$;
(iii) Find the p.d.f. of X;
(iv) Find the lower quartile, the median and the upper quartile of F.

9. Let X be an absolutely continuous type random variable with p.d.f.

$$f(x) = \begin{cases} k - |x|, & \text{if } |x| < \frac{1}{2} \\ 0, & \text{otherwise} \end{cases},$$

where $k \in \mathbb{R}$.
(i) Find the value of constant k;
(ii) Using the p.d.f., evaluate $P(X < 0), P(X \leq 0), P(0 < X \leq \frac{1}{4}), P(0 \leq X < \frac{1}{4})$ and $P(-\frac{1}{8} \leq X \leq \frac{1}{4})$;
(iii) Find the conditional probabilities $P(X > \frac{1}{4} | |X| > \frac{2}{5})$ and $P(\frac{1}{8} < X < \frac{2}{5} | \frac{1}{10} < X < \frac{1}{5})$;
(iv) Find the distribution function F of X;
(v) Find the lower quartile, the median and the upper quartile of F.