

MSO 201a: Probability and Statistics

2019-20-II Semester

Assignment-III

Instructor: Neeraj Misra

1. Let X be a random variable with p.m.f.

$$f_X(x) = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^x, & \text{if } x \in \{0, 1, 2, \dots\} \\ 0, & \text{otherwise} \end{cases}.$$

- (a) Find the distribution function of $Y = X/(X + 1)$ and hence determine the p.m.f. of Y ;
- (b) Find the p.m.f. of $Y = X/(X + 1)$ and hence determine the distribution function of Y ;
- (c) Find the mean and the variance of X .

2. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -2 < x < -1 \\ \frac{1}{6}, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

- (a) Find the distribution function of $Y = X^2$ and hence determine the p.d.f. of Y ;
- (b) Find the probability density function of $Y = X^2$ and hence determine the distribution function of Y ;
- (c) Find the mean and the variance of X .

3. (a) Give an example of a discrete random variable X for which $E(X)$ is finite but $E(X^2)$ is not finite;
- (b) Give an example of a continuous random variable X for which $E(X)$ is finite but $E(X^2)$ is not finite.

4. Let X be a random variable with

$$P(X = -2) = \frac{1}{21}, \quad P(X = -1) = \frac{2}{21}, \quad P(X = 0) = \frac{1}{7}, \\ P(X = 1) = \frac{4}{21}, \quad P(X = 2) = \frac{5}{21}, \quad P(X = 3) = \frac{2}{7}.$$

Find the p.m.f. and distribution function of $Y = X^2$.

5. Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Find the p.d.f.s of the following random variables: (a) $Y_1 = \sqrt{X}$; (b) $Y_2 = X^2$; (c) $Y_3 = 2X + 3$; (d) $Y_4 = -\ln X$.

6. Let the random variable X have the p.d.f.

$$f_X(x) = \begin{cases} 6x(1-x), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = X^2(3 - 2X)$.

- (a) Find the distribution function of Y and hence find its p.d.f.;
 - (b) Find the p.d.f. of Y directly (i.e., without finding the distribution function);
 - (c) Find the mean and the variance of Y .
7. (a) From a box containing N identical tickets, numbered, $1, 2, \dots, N$, n ($\leq N$) tickets are drawn at random with replacement. Let $X =$ largest number drawn. Find $E(X)$.
- (b) Find the expected number of throws of a fair die required to obtain a 6.
8. Consider a target comprising of three concentric circles of radii $1/\sqrt{3}$, 1 and $\sqrt{3}$ feet. Shots within the inner circle earn 4 points, within the next ring 3 points and within the outer ring 2 points. Shots outside the target do not earn any point. Let X denote the distance (in feet) of the hit from the centre and suppose that X has the p.d.f.

$$f_X(x) = \begin{cases} \frac{2}{\pi(1+x^2)}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Find the expected score in a single shot.

9. (a) Let X be a random variable with p.d.f.

$$f_X(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and let $Y = \min(X, 1/2)$. Examine whether or not Y is a discrete or a continuous random variable. (**Note:** Function of a continuous random variable may neither be discrete nor continuous).

(b) Let the random variable X have the p.d.f.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty,$$

and let

$$Y = \begin{cases} -1, & \text{if } X < 0 \\ \frac{1}{2}, & \text{if } X = 0 \\ 1, & \text{if } X > 0 \end{cases}.$$

Examine whether Y is discrete or continuous random variable. (**Note:** Function of a continuous random variable may be a discrete random variable.)

10. (a) Let $E(|X|^\beta) < \infty$, for some $\beta > 0$. Then show that $E(|X|^\alpha) < \infty, \forall \alpha \in (0, \beta]$;
- (b) Let X be a random variable with finite expectation. Show that $\lim_{x \rightarrow -\infty} xF_X(x) = \lim_{x \rightarrow \infty} [x(1 - F_X(x))] = 0$, where F_X is the distribution function of X ;
- (c) Let X be a random variable with $\lim_{x \rightarrow \infty} [x^\alpha P(|X| > x)] = 0$, for some $\alpha > 0$. Show that $E(|X|^\beta) < \infty, \forall \beta \in (0, \alpha)$. What about $E(|X|^\alpha)$?
11. (a) Find the moments of the random variable that has the m.g.f. $M(t) = (1-t)^{-3}$, $t < 1$;
- (b) Let the random variable X have the m.g.f.

$$M(t) = \frac{e^{-t}}{8} + \frac{e^t}{4} + \frac{e^{2t}}{8} + \frac{e^{3t}}{2}, \quad t \in \mathbb{R}.$$

Find the distribution function of X and find $P(X^2 = 1)$.

(c) If the m.g.f. of a random variable X is

$$M(t) = \frac{e^t - e^{-2t}}{3t}, \quad \text{for } t \neq 0,$$

find the p.d.f. of $Y = X^2$.

12. Let $p \in (0, 1)$ and let X_p be a random variable with p.m.f.

$$f_{X_p}(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & \text{if } x \in \{0, 1, \dots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where n is a given positive integer and $q = 1 - p$.

(a) Find the m.g.f. of X_p and hence find the mean and variance of X_p , $p \in (0, 1)$;

- (b) Let $Y_p = n - X_p, p \in (0, 1)$. Using the m.g.f. of X_p show that the p.m.f. of Y_p is

$$f_{Y_p}(y) = \begin{cases} \binom{n}{y} q^y (1-q)^{n-y}, & \text{if } y \in \{0, 1, \dots, n\} \\ 0, & \text{otherwise} \end{cases}.$$

13. (a) For any random variable X having the mean μ and finite second moment, show that $E((X - \mu)^2) \leq E((X - c)^2), \forall c \in \mathbb{R}$;
- (b) Let X be a continuous random variable with distribution function F_X that is strictly increasing on its support. Let m be the median of (distribution of) X . Show that $E(|X - m|) \leq E(|X - c|), \forall c \in (-\infty, \infty)$.
14. (a) Let X be a non-negative continuous random variable (i.e., $P(X \geq 0) = 1$) and let h be a real-valued function defined on $(0, \infty)$. Define $\psi(x) = \int_0^x h(t) dt, x \geq 0$, and suppose that $h(x) \geq 0, \forall x \geq 0$. Show that

$$E(\psi(X)) = \int_0^\infty h(y) P(X > y) dy;$$

- (b) Let α be a positive real number. Under the assumptions of (a), show that

$$E(X^\alpha) = \alpha \int_0^\infty x^{\alpha-1} P(X > x) dx;$$

- (c) Let $F(0) = G(0) = 0$ and let $F(t) \geq G(t), \forall t > 0$, where F and G are distribution functions of continuous random variables X and Y , respectively. Show that $E(X^k) \leq E(Y^k), \forall k > 0$, provided the expectations exist.
15. (a) Let X be a random variable such that $P(X \leq 0) = 0$ and let $\mu = E(X)$ be finite. Show that $P(X \geq 2\mu) \leq 0.5$;
- (b) If X is a random variable such that $E(X) = 3$ and $E(X^2) = 13$, then determine a lower bound for $P(-2 < X < 8)$.
16. (a) An enquiry office receives, on an average, 25,000 telephone calls a day. What can you say about the probability that this office will receive at least 30,000 telephone calls tomorrow?
- (b) An enquiry office receives, on an average, 20,000 telephone calls per day with a variance of 2,500 calls. What can be said about the probability that this office will receive between 19,900 and 20,100 telephone calls tomorrow? What can you say about the probability that this office will receive more than 20,200 telephone calls tomorrow?
17. Let X be a random variable with m.g.f. $M(t), -h < t < h$.

- (a) Prove that $P(X \geq a) \leq e^{-at}M(t)$, $0 < t < h$;
 (b) Prove that $P(X \leq a) \leq e^{-at}M(t)$, $-h < t < 0$;
 (c) Suppose that $M(t) = \frac{1}{4} \left(1 - \frac{t}{3}\right)^{-1} + \frac{3}{4} \left(1 - \frac{t}{2}\right)^{-1}$, $t < \frac{1}{3}$. Find $P(X > 1)$.
18. Let $\mu \in \mathbb{R}$ and $\sigma > 0$ be real constants and let $X_{\mu,\sigma}$ be a random variable having p.d.f.

$$f_{X_{\mu,\sigma}}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty.$$

- (a) Show that $f_{X_{\mu,\sigma}}$ is a p.d.f.;
 (b) Show that the probability distribution function of $X_{\mu,\sigma}$ is symmetric about μ . Hence find $E(X_{\mu,\sigma})$;
 (c) Find the m.g.f. of $X_{\mu,\sigma}$ and hence find the mean and variance of $X_{\mu,\sigma}$;
 (d) Let $Y_{\mu,\sigma} = aX_{\mu,\sigma} + b$, where $a \neq 0$ and b are real constants. Using the m.g.f. of $X_{\mu,\sigma}$, show that the p.d.f. of $Y_{\mu,\sigma}$ is

$$f_{Y_{\mu,\sigma}}(y) = \frac{1}{|a|\sigma\sqrt{2\pi}} e^{-\frac{(y-(a\mu+b))^2}{2a^2\sigma^2}}, \quad -\infty < y < \infty.$$

19. Let X be a random variable with p.d.f.

$$f(x) = \begin{cases} \frac{1}{\pi} \frac{1}{\sqrt{x(1-x)}}, & \text{if } 0 < x < 1, \\ 0, & \text{otherwise} \end{cases}.$$

Show that the distribution of X is symmetric about a point μ . Find this point μ . Also find $E(X)$ and $P(X > \mu)$.

20. Let X be a random variable with p.d.f.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty.$$

Show that $X \stackrel{d}{=} -X$. Hence find $E(X^3)$ and, $P(X > 0)$.

21. (a) Let X be a random variable with $E(X) = 1$. Show that $E(e^{-X}) \geq \frac{1}{3}$;
 (b) For pairs of positive real numbers (a_i, b_i) , $i = 1, \dots, n$ and $r \geq 1$, show that

$$\left(\sum_{i=1}^n a_i^r b_i \right) \left(\sum_{i=1}^n b_i \right)^{r-1} \geq \left(\sum_{i=1}^n a_i b_i \right)^r.$$

Hence show that, for any positive real number m ,

$$\left(\sum_{i=1}^n a_i^{2m+1} \right) \left(\sum_{i=1}^n a_i \right) \geq \left(\sum_{i=1}^n a_i^{m+1} \right)^2.$$

22. Let X be a random variable such that $P(X > 0) = 1$. Show that:

(a) $E(X^{2m+1}) \geq (E(X))^{2m+1}$, $m \in \{1, 2, \dots\}$;

(b) $E(Xe^X) + e^{E(X)} \geq E(X)e^{E(X)} + E(e^X)$,

provided the involved expectations are finite.