

MSO 201a: Probability and Statistics

2019-20-II Semester

Assignment-IV

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1. Let

$$F(x, y) = \begin{cases} 1, & \text{if } x + 2y \geq 1 \\ 0, & \text{if } x + 2y < 1 \end{cases}.$$

Does $F(\cdot, \cdot)$ define a d.f.?

2. Let

$$F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x + y < 1 \text{ or } y < 0 \\ 1, & \text{otherwise} \end{cases}.$$

Does $F(\cdot)$ define a d.f.?

3. Let $\underline{X} = (X_1, X_2)$ be a bivariate random vector having the d.f.

$$F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 0 \\ \frac{1+xy}{2}, & \text{if } 0 \leq x < 1, 0 \leq y < 1 \\ \frac{1+x}{2}, & \text{if } 0 \leq x < 1, y \geq 1 \\ \frac{1+y}{2}, & \text{if } x \geq 1, 0 \leq y < 1 \\ 1 & \text{if } x \geq 1, y \geq 1 \end{cases}.$$

(a) Verify that F is a d.f.; (b) Determine whether \underline{X} is of discrete or continuous; (c) Find the marginal distribution functions of X_1 and X_2 ; (d) Find $P(\frac{1}{2} \leq X_1 \leq 1, \frac{1}{4} < X_2 < \frac{1}{2})$, $P(X_1 = 1)$ and $P(X_1 \geq \frac{3}{2}, X_2 < \frac{1}{4})$; (e) Are X_1 and X_2 independent?

4. Let $\underline{X} = (X_1, X_2)$ be a bivariate random vector having the d.f.

$$F(x, y) = \begin{cases} 0, & \text{if } x < 0 \text{ or } y < 1 \\ \frac{y^2-1}{6}, & \text{if } 0 \leq x < 1, 1 \leq y < 2 \\ \frac{1}{2}, & \text{if } 0 \leq x < 1, y \geq 2 \\ \frac{y^2-1}{3}, & \text{if } x \geq 1, 1 \leq y < 2 \\ 1 & \text{if } x \geq 1, y \geq 2 \end{cases}.$$

(a) Verify that F is a d.f.; (b) Determine whether \underline{X} is a discrete or continuous r.v.; (c) Find the marginal distribution functions of X_1 and X_2 ; (d) Find $P(\frac{1}{2} \leq X_1 \leq 1, \frac{5}{4} < X_2 < \frac{3}{2})$, $P(X_1 = 1)$ and $P(X_1 \geq \frac{3}{2}, X_2 < \frac{5}{4})$; (e) Are X_1 and X_2 independent?

5. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} c(x_1 + 2x_2), & \text{if } x_1 = 1, 2, x_2 = 1, 2 \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant. (a) Find the constant c ; (b) Find marginal p.m.f.s of X_1 and X_2 ; (c) Find conditional variance of X_2 given $X_1 = x_1$, $x_1 = 1, 2$; (d) Find $P(X_1 < \frac{X_2}{3})$, $P(X_1 = X_2)$, $P(X_1 \geq \frac{X_2}{2})$ and $P(X_1 + X_2 \leq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are X_1 and X_2 independent?

6. Let the r.v. $\underline{X} = (X_1, X_2)'$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} cx_1x_2, & \text{if } x_1 = 1, 2, x_2 = 1, 2, x_1 \leq x_2 \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant. (a) Find the constant c ; (b) Find marginal p.m.f.s of X_1 and X_2 ; (c) Find conditional variance of X_2 given $X_1 = 1$; (d) Find $P(X_1 > X_2)$, $P(X_1 = X_2)$, $P(X_1 < \frac{2}{3}X_2)$ and $P(X_1 + X_2 \geq 3)$; (e) Find $\rho(X_1, X_2)$; (f) Are X_1 and X_2 independent?

7. Let (X, Y) be a random vector such that the p.d.f. of X is

$$f_X(x) = \begin{cases} 4x(1 - x^2), & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

and, for fixed $x \in (0, 1)$, the conditional p.d.f. of Y given $X = x$ is

$$f_{Y|X}(y|x) = \begin{cases} c(x)y, & \text{if } x < y < 1 \\ 0, & \text{otherwise} \end{cases},$$

where $c : (0, 1) \rightarrow \mathbb{R}$ is a given function. (a) Determine $c(x)$, $0 < x < 1$; (b) Find marginal p.d.f. of Y ; (c) Find the conditional variance of X given $Y = y$, $y \in (0, 1)$; (d) Find $P(X < \frac{Y}{2})$, $P(X + Y \geq \frac{3}{4})$ and $P(X = 2Y)$; (e) Find $\rho(X, Y)$; (f) Are X and Y independent?

8. Let $\underline{X} = (X_1, X_2, X_3)$ be a random vector with joint p.d.f.

$$f_{\underline{X}}(\underline{x}) = \begin{cases} \frac{c}{x_1x_2}, & \text{if } 0 < x_3 < x_2 < x_1 < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant. (a) Find the value of constant c ; (b) Find marginal p.d.f. of X_2 ; (c) Find the conditional variance of X_2 given $(X_1, X_3) = (x, y)$, $0 < y <$

$x < 1$; (d) Find $P(X_2 < \frac{X_1}{2})$ and $P(X_3 = 2X_2 > \frac{X_1}{2})$; (e) Find $\rho(X_1, X_2)$; (f) Are X_1, X_2, X_3 independent?.

9. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} \frac{1}{4}, & \text{if } (x_1, x_2, x_3) \in A \\ 0, & \text{otherwise} \end{cases},$$

where $A = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1)\}$. (a) Are X_1, X_2, X_3 independent?; (b) Are X_1, X_2, X_3 pairwise independent?; (c) Are $X_1 + X_2$ and X_3 independent?

10. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with joint p.d.f.

$$f_{\underline{X}}(x_1, x_2, x_3) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)} \left(1 + x_1 x_2 x_3 e^{-\frac{1}{2}(x_1^2 + x_2^2 + x_3^2)}\right), \quad -\infty < x_i < \infty,$$

$i = 1, 2, 3$. (a) Are X_1, X_2, X_3 independent?; (b) Are X_1, X_2, X_3 pairwise independent?; (c) Find the marginal p.d.f.s of (X_1, X_2) , (X_1, X_3) and (X_2, X_3) .

11. Let (X, Y, Z) have the joint p.m.f. as follows:

(x, y, z)	(1, 1, 0)	(1, 2, 1)	(1, 3, 0)	(2, 1, 1)	(2, 2, 0)	(2, 3, 1)
$f_{X,Y,Z}(x, y, z)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

and $f_{X,Y}(x, y) = 0$, elsewhere. (a) Are $X + Y$ and Z independent?; (b) Find $\rho = \text{Corr}(X+Y, Z)$.

12. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.d.f.

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} 2e^{-(x_2+2x_3)}, & \text{if } 0 < x_1 < 1, x_2 > 0, x_3 > 0 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Are X_1, X_2, X_3 independent?; (b) Are $X_1 + X_2$ and X_3 independent?; (c) Find marginal p.d.f.s of X_1, X_2 and X_3 ; (d) Find conditional p.d.f. of X_1 given $X_2 = 2$.

13. Let X_1, \dots, X_n be n r.v.s with $E(X_i) = \mu_i$, $\text{Var}(X_i) = \sigma_i^2$ and $\rho_{ij} = \text{Corr}(X_i, X_j)$, $i, j = 1, \dots, n$, $i \neq j$. For real numbers a_i, b_i , $i = 1, \dots, n$, define $Y = \sum_{i=1}^n a_i X_i$ and $Z = \sum_{i=1}^n b_i X_i$. Find $\text{Cov}(Y, Z)$.

14. Let X and Y be jointly distributed random variables with $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 2$ and $\text{Corr}(X, Y) = 1/3$. Find $\text{Corr}(\frac{X}{3} + \frac{2Y}{3}, \frac{2X}{3} + \frac{Y}{3})$.

15. Let X_1, \dots, X_n be random variables and let p_1, \dots, p_n be positive real numbers with $\sum_{i=1}^n p_i = 1$. Prove that: (a) $\sqrt{\text{Var}(\sum_{i=1}^n p_i X_i)} \leq \sum_{i=1}^n p_i \sqrt{\text{Var}(X_i)} \leq \sqrt{\sum_{i=1}^n p_i \text{Var}(X_i)}$; (b) $\text{Var}(\frac{\sum_{i=1}^n X_i}{n}) \leq \frac{1}{n} \sum_{i=1}^n \text{Var}(X_i)$.

16. Let $(x_i, y_i) \in \mathbb{R}^2, i = 1, \dots, n$ be such that $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i = 0$. Using a statistical argument show that

$$\left(\sum_{i=1}^n x_i y_i \right)^2 \leq \left(\sum_{i=1}^n x_i^2 \right) \left(\sum_{i=1}^n y_i^2 \right).$$

17. Let (X, Y) have the joint p.m.f. as follows:

(x, y)	(1, 1)	(1, 2)	(1, 3)	(2, 1)	(2, 2)	(2, 3)
$f_{X,Y}(x, y)$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{4}{15}$

and $f_{X,Y}(x, y) = 0$, elsewhere. Find $\rho = \text{Corr}(X, Y)$.

18. Let the joint m.g.f. of (Y, Z) be $M_{Y,Z}(t_1, t_2) = \frac{e^{t_1^2}}{1-2t_2}$, $t_2 < \frac{1}{2}$. (a) Find $\text{Corr}(Y, Z)$; (b) Are Y and Z independent?; (c) Find m.g.f. of $Y + Z$.

19. Let the joint m.g.f. of (Y, Z) be $M_{Y,Z}(t_1, t_2) = e^{\frac{t_1^2 + t_2^2 + t_1 t_2}{2}}$, $(t_1, t_2) \in \mathbb{R}^2$. (a) Find $\text{Corr}(Y, Z)$; (b) Are Y and Z independent?; (c) Find m.g.f. of $Y - Z$.

20. Let $\underline{X} = (X_1, X_2)$ have the joint p.m.f.

$$f_{\underline{X}}(x_1, x_2) = \begin{cases} \left(\frac{2}{3}\right)^{x_1+x_2} \left(\frac{1}{3}\right)^{2-x_1-x_2}, & \text{if } (x_1, x_2) = (0, 0), (0, 1), (1, 0), (1, 1) \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the joint p.m.f. of $Y_1 = X_1 - X_2$ and $Y_2 = X_1 + X_2$; (b) Find the marginal p.m.f.s of Y_1 and Y_2 ; (c) Find $\text{Var}(Y_2)$ and $\text{Cov}(Y_1, Y_2)$; (d) Are Y_1 and Y_2 independent?

21. Let X_1, \dots, X_n be a random sample of continuous random variables and let $X_{1:n} < X_{2:n} \dots < X_{n:n}$ be the corresponding order statistics. If the expectation of X_1 is finite and the distribution of X_1 is symmetric about $\mu \in (-\infty, \infty)$, show that: (a) $X_{r:n} - \mu \stackrel{d}{=} \mu - X_{n-r+1:n}$, $r = 1, \dots, n$; (b) $E(X_{r:n} + X_{n-r+1:n}) = 2\mu$; (c) $E(X_{\frac{n+1}{2}:n}) = \mu$, if n is odd; (d) $P(X_{\frac{n+1}{2}:n} > \mu) = 0.5$, if n is odd.

22. (a) Let X_1, \dots, X_n denote a random sample, where $P(X_1 > 0) = 1$. Show that

$$E\left(\frac{X_1 + X_2 + \dots + X_k}{X_1 + X_2 + \dots + X_n}\right) = \frac{k}{n}, \quad k = 1, 2, \dots, n.$$

(b) Let X_1, \dots, X_n be a random sample and let $E(X_1)$ be finite. Find the conditional expectation $E(X_1 | X_1 + \dots + X_n = t)$, where $t \in \mathbb{R}$ is such that the conditional expectation is defined.

(c) Let X_1, \dots, X_n be a random sample of random variables. Find $P(X_1 < X_2 < \dots < X_r)$, $r = 2, 3, \dots, n$.

23. Let X_1 and X_2 be independent and identically distributed random variables with common p.m.f.

$$f(x) = \begin{cases} \theta(1 - \theta)^{x-1}, & \text{if } x = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, 1)$. Let $Y_1 = \min\{X_1, X_2\}$ and $Y_2 = \max\{X_1, X_2\} - \min\{X_1, X_2\}$. (a) Find the marginal p.m.f. of Y_1 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (b) Find the marginal p.m.f. of Y_2 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (c) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (d) Are Y_1 and Y_2 independent; (e) Using (c), find the marginal p.m.f.s of Y_1 and Y_2 .

24. Let $\underline{X} = (X_1, X_2, X_3)'$ be a random vector with p.m.f.

$$f_{X_1, X_2, X_3}(x_1, x_2, x_3) = \begin{cases} \frac{2}{9}, & \text{if } (x_1, x_2, x_3) = (1, 1, 0), (1, 0, 1), (0, 1, 1) \\ \frac{1}{3}, & \text{if } (x_1, x_2, x_3) = (1, 1, 1) \\ 0, & \text{otherwise} \end{cases}.$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_2 + X_3$. (a) Find the marginal p.m.f. of Y_1 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (b) Find the marginal p.m.f. of Y_2 without finding the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (c) Find the joint p.m.f. of $\underline{Y} = (Y_1, Y_2)$; (d) Are Y_1 and Y_2 independent; (e) Using (c), find the marginal p.m.f.s of Y_1 and Y_2 .

25. Let X_1 and X_2 be independent random variables with p.d.f.s

$$f_1(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 1, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases},$$

respectively. Let $Y = X_1 + X_2$ and $Z = X_1 - X_2$. (a) Find the d.f. of Y and hence find its p.d.f.; (b) Find the joint p.d.f. of (Y, Z) and hence find the marginal p.d.f.s of Y and Z ; (c) Are Y and Z independent?

26. Let X_1 and X_2 be i.i.d. random variables with common p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}.$$

Let $Y = |X_1| + X_2$ and $Z = X_2$. (a) Find the d.f. of Y and hence find its p.d.f.; (b) Find the joint p.d.f. of (Y, Z) and hence find the marginal p.d.f.s of Y and Z ; (c) Are Y and Z independent?