MSO201a: Probability and Statistics
2019-20-II Semester
Assignment No. 6
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1. Let $X_1, X_2, \ldots$ be a sequence of r.v.s, such that X_n, $n = 1, 2, \ldots$, has the d.f.: $F_n(x) = 0$, if $x < -n$, $= \frac{x+n}{2n}$, if $-n \le x < n$, and $= 1$, if $x \ge n$. Does $F_n(\cdot)$ converge to a d.f., as $n \to \infty$?

2. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s and let $X_{1:n} = \min\{X_1, \ldots, X_n\}$ and $Y_n = nX_{1:n}$, $n = 1, 2, \ldots$. Find the limiting distributions of $X_{1:n}$ and Y_n (as $n \to \infty$) when (a) $X_1 \sim \mathrm{U}(0, \theta)$, $\theta > 0$; (b) $X_1 \sim \mathrm{Exp}(\theta)$, $\theta > 0$.

3. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $P(X_n = x) = \frac{1}{2}$, if $x = -n^{\frac{1}{4}}$, $n^{\frac{1}{4}}$, and $= 0$, otherwise. Show that $\overline{X}_n \xrightarrow{P} 0$, as $n \to \infty$.

4. (a) If $X_n \xrightarrow{P} a$ and $X_n \xrightarrow{P} b$, then show that $a = b$.
(b) Let a and $r > 0$ be real numbers. If $E(|X_n - a|^r) \to 0$, as $n \to \infty$, then show that $X_n \xrightarrow{P} a$.

5. (a) For $r > 0$ and $t > 0$, show that $E(\frac{|X|^r}{1+|X|^r}) - \frac{t^r}{1+t^r} \le P(|X| \ge t) \le \frac{1+t^r}{t^r} E(\frac{|X|^r}{1+|X|^r})$.
(b) Show that $X_n \xrightarrow{P} 0 \;\Leftrightarrow\; E(\frac{|X_n|^r}{1+|X_n|^r}) \to 0$, for some $r > 0$.

6. (a) If $\{X_n\}_{n \ge 1}$ are identically distributed and $a_n \to 0$, then show that $a_n X_n \xrightarrow{P} 0$.
(b) If $Y_n \le X_n \le Z_n$, $n = 1, 2, \ldots$, $Y_n \xrightarrow{P} a$ and $Z_n \xrightarrow{P} a$, then show that $X_n \xrightarrow{P} a$.
(c) If $X_n \xrightarrow{P} c$ and $a_n \to a$, then show that, as $n \to \infty$, $X_n + a_n \xrightarrow{P} c + a$ and $a_n X_n \xrightarrow{P} ac$.
(d) Let $X_n = \min(|Y_n|, a)$, $n = 1, 2, \ldots$, where a is a positive constant. Show that $X_n \xrightarrow{P} 0 \Leftrightarrow Y_n \xrightarrow{P} 0$.

7. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s with mean μ and finite variance. Show that:
(a) $\frac{2}{n(n+1)} \sum_{i=1}^{n} i X_i \xrightarrow{P} \mu$;
(b) $\frac{6}{n(n+1)(2n+1)} \sum_{i=1}^{n} i^2 X_i \xrightarrow{P} \mu$.

8. Let X_n, $n = 1, 2, \ldots$, have a negative binomial distribution with parameters n and $p_n = 1 - \frac{\theta}{n}$, i.e., X_n has the p.m.f. $P(X_n = x) = \binom{n+x-1}{x} p_n^n (1-p_n)^x$, $x = 0, 1, 2, \ldots$; $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{d} X \sim \mathrm{Poisson}(\theta)$.

9. (a) Let $X_n \sim \text{Gamma}(\frac{1}{n}, n)$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{P} 1$.
(b) Let $X_n \sim N(\frac{1}{n}, 1 - \frac{1}{n})$, $n = 1, 2, \ldots$. Show that $X_n \xrightarrow{d} Z \sim N(0, 1)$.

10. (a) Let $f(x) = \frac{1}{x^2}$, if $1 \leq x < \infty$, and $= 0$, elsewhere, be the p.d.f. of a r.v. X. Consider the random sample of size 72 from the distribution having p.d.f. $f(\cdot)$. Compute, approximately, the possibility that more than 50 of the items of the random sample are less than 3.
(b) Let $X_1, X_2, \ldots$ be a random sample from Poisson(3) distribution and let $Y = \sum_{i=1}^{100} X_i$. Find, approximately, $P(100 \leq Y \leq 200)$.
(c) Let $X \sim \text{Bin}(25, 0.6)$. Find, approximately, $P(10 \leq X \leq 16)$. What is the exact value of this probability?

11. (a) Show that $\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}$.
(b) Show that $\lim_{n\to\infty} 2^{-n} \sum_{k=0}^{r_n} \binom{n}{k} = \frac{1}{2}$, where r_n is the largest integer $\leq \frac{n}{2}$.

12. (a) If $T_n = \max(|X_1|, \ldots, |X_n|) \xrightarrow{P} 0$, as $n \to \infty$, then show that $\overline{X}_n \xrightarrow{P} 0$. Is the conclusion true if only $S_n = \max(X_1, \ldots, X_n) \xrightarrow{P} 0$.
(b) If $\{X_n\}_{n\geq 1}$ are i.i.d. $U(0,1)$ r.v.s. and $Z_n = (\prod_{i=1}^{n} X_i)^{\frac{1}{n}}$, $n = 1, 2, \ldots$. Find a real α such that $Z_n \xrightarrow{P} \alpha$.

13. Let $\{E_n\}_{n\geq 1}$ be a sequence of i.i.d. Exp(1) r.v.s.
(a) Show that $T_n \equiv \sum_{i=1}^{n} E_i \sim \text{Gamma}(n, 1)$, $n = 1, 2, \ldots$.
(b) For any real number x, show that $\lim_{n\to\infty} \int_0^{n+x\sqrt{n}} \frac{e^{-t}t^{n-1}}{\Gamma(n)} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$.
(c) For large values of n, show that an approximation (called the Stirling approximation) to the gamma function is: $\Gamma n \approx \sqrt{2\pi} e^{-n} n^{n-\frac{1}{2}}$.

14. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. r.v.s having the common Cauchy p.d.f. $f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$, $-\infty < x < \infty$.
(a) For any $\alpha \in (0, 1)$, show that $Y = \alpha X_1 + (1 - \alpha) X_2$ again has a Cauchy p.d.f. $f(\cdot)$.
(b) Note that $\overline{X}_{n+1} = \frac{n}{n+1} \overline{X}_n + \frac{1}{n+1} X_{n+1}$ and hence, using induction, conclude that $\overline{X}_n$ has the same distribution as X_1.
(c) Show that $\overline{X}_n$ does not converge in probability to any constant. (Note that $E(X_1)$ does not exist and hence the WLLN is not guaranteed).

15. Let $X_n \sim \text{Poisson}(4n)$, $n = 1, 2, \ldots$, and let $Y_n = \frac{X_n}{n}$, $n = 1, 2, \ldots$.
(a) Show that $Y_n \xrightarrow{P} 4$; (b) Show that $Y_n^2 + \sqrt{Y_n} \xrightarrow{P} 18$;
(c) Show that $\frac{n^2 Y_n^2 + n Y_n}{n Y_n + n^2} \xrightarrow{P} 16$.

16. Let $\overline{X}_n$ be the sample mean computed from a random sample of size n from a distribution with mean μ $(-\infty < \mu < \infty)$ and variance σ^2 $(0 < \sigma < \infty)$. Let $Z_n = \frac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma}$.
(a) If $Y_n \xrightarrow{P} 4$, show that: $\frac{4Z_n}{Y_n} \xrightarrow{d} Z \sim N(0, 1)$; $\frac{16 Z_n^2}{Y_n^2} \xrightarrow{d} U \sim \chi_1^2$; and $\frac{(4n + Y_n) Z_n}{(n Y_n + Y_n^2)} \xrightarrow{d}$

$Z \sim N(0,1)$.

(b) If $\sigma = 1$ and $\mu > 0$, show that: $\sqrt{n}(\ln \overline{X}_n - \ln \mu) \xrightarrow{d} V \sim N(0, \frac{1}{\mu^2})$;

(c) Show that $\frac{n^\delta(\overline{X}_n - \mu)}{\sigma} \xrightarrow{P} 0$, for any $\delta < 0.5$.

(d) Find the asymptotic distributions of: (i) $\sqrt{n}(\overline{X}_n^2 - \mu^2)$; (ii) $n(\overline{X}_n - \mu)^2$ and (iii) $\sqrt{n}(\overline{X}_n - \mu)^2$.

17. Let $X_1, X_2, \ldots$ be i.i.d. r.v.s having $\text{Exp}(\theta)$ $(\theta > 0)$ distribution and let $\overline{X}_n = \frac{\sum_{i=1}^n X_i}{n}$, $n = 1, 2, \ldots$. Show that: $\sqrt{n}(\frac{1}{\overline{X}_n} - \frac{1}{\theta}) \xrightarrow{d} N(0, \frac{1}{\theta^2})$, as $n \to \infty$.

18. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. $U(0,1)$ r.v.s. For the sequence of geometric means $G_n = (\prod_{i=1}^n X_i)^{\frac{1}{n}}$, $n = 1, 2, \ldots$, show that $\sqrt{n}(G_n - \frac{1}{e}) \xrightarrow{d} N(0, \sigma^2)$, for some $\sigma^2 > 0$. Find σ^2.

19. Let $(X_1, Y_1), (X_2, Y_2), \ldots$ be a sequence of independent bivariate random vectors having the same joint p.d.f. Let $E(X_1) = \mu$, $E(Y_1) = \nu$, $\text{Var}(X_1) = \sigma^2$, $\text{Var}(Y_1) = \tau^2$ and $\text{Corr}(X_1, Y_1) = \rho$. Let $Q_n = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{n-1}$, $S_n^2 = \frac{\sum_{i=1}^n (X_i - \overline{X}_n)^2}{n-1}$, $T_n^2 = \frac{\sum_{i=1}^n (Y_i - \overline{Y}_n)^2}{n-1}$ and $R_n = \frac{Q_n}{S_n T_n}$.

(a) Show that $Q_n \xrightarrow{P} \rho\tau\sigma$ and $R_n \xrightarrow{P} \rho$.

(b) Let $\delta = \frac{E((X_1-\mu)^2(Y_1-\nu)^2)}{\sigma^2\tau^2}$. Show that $\sqrt{n}(Q_n - \rho\sigma\tau) \xrightarrow{d} N(0, (\delta - \rho^2)\sigma^2\tau^2)$.