ESO 209: Probability and Statistics
2019-2020-II Semester
Assignment No. 7

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. In each of the following situations, find the M.M.E. and the M.L.E.. Also verify if they are consistent estimators of $g(\underline{\theta})$.

   (a) $f(x|\theta) = \theta(1-\theta)^{x-1}$, if $x = 1, 2, \ldots$, and $= 0$, otherwise; $\Theta = (0, 1)$; $g(\theta) = \theta$.

   (b) $X_1 \sim \text{Poisson}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = e^{\theta}$.

   (c) $f(x|\underline{\theta}) = \theta_1$, if $x = 1$, $= \frac{1-\theta_1}{\theta_2 - 1}$, if $x = 2, 3, \ldots, \theta_2$, and $= 0$, otherwise; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : 0 < z_1 < 1, z_2 \in \{2, 3, \ldots\}\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

   (d) $f(x|\theta) = K(\theta)x^{\theta}(1-x)$, if $0 \leq x \leq 1$, and $= 0$, otherwise; $\Theta = (-1, \infty)$; $g(\theta) = \theta$; here $K(\theta)$ is the normalizing factor.

   (e) $X_1 \sim \text{Gamma}(\alpha, \mu)$; $\underline{\theta} = (\alpha, \mu)$; $\Theta = (0, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\alpha, \mu)$.

   (f) $f(x|\underline{\theta}) = (\sigma\sqrt{2\pi})^{-1}x^{-1}\exp(-\frac{1}{2\sigma^2}(\ln x - \mu)^2)$, if $x > 0$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

   (g) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_{\theta}(X_1 \leq 1)$.

   (h) $X_1 \sim \text{Poisson}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_{\theta}(X_1 + X_2 + X_3 = 0)$.

   (i) $X_1 \sim U(-\frac{\theta}{2}, \frac{\theta}{2})$; $\Theta = (0, \infty)$; $g(\theta) = (1+\theta)^{-1}$.

   (j) $X_1 \sim N(\mu, \sigma^2)$; $\underline{\theta} = (\mu, \sigma^2)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \frac{\mu^2}{\sigma^2}$.

   (k) $f(x|\underline{\theta}) = \sigma^{-1}\exp(-\frac{x-\mu}{\sigma})$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

   (l) $X_1 \sim U(\theta_1, \theta_2)$; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : -\infty < z_1 < z_2 < \infty\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

2. Suppose a randomly selected sample of size five from the distribution having p.m.f. given in Problem 1 (a) gives the following data: $x_1 = 2$, $x_2 = 7$, $x_3 = 6$, $x_4 = 5$ and $x_5 = 9$. Based on this data compute the m.l.e. of $P_{\theta}(X_1 \geq 4)$.

3. The lifetimes of a brand of a component are assumed to be exponentially distributed with mean (in hours) θ, where $\theta \in \Theta = (0, \infty)$ is unknown. Ten of these components were independently put in test. The only data recorded were the number of components that had failed in less than 100 hours versus the number that had not failed. It was found that three had failed before 100 hours. What is the m.l.e. of θ?

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\theta)$, where $\theta \in \Theta$ is an unknown parameter. In each of the following situations, find the M.L.E. of θ and verify if it is a consistent estimator of θ.

   (a) $X_1 \sim N(\theta, 1)$, $\Theta = [0, \infty)$. (b) $X_1 \sim \text{Bin}(1, \theta)$, $\Theta = [\frac{1}{4}, \frac{3}{4}]$.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution having mean μ and finite variance σ^2. Show that $\overline{X}$ and S^2 are unbiased estimators of μ and σ^2, respectively.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$.

   (a) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

   (b) $n \geq 2$, $X_1 \sim N(\mu, \sigma^2)$; $\underline{\theta} = (\mu, \sigma^2)$; $\Theta = (-\infty, \infty)$; $g(\underline{\theta}) = \mu + \sigma$.

   (c) Same as (b) with $g(\underline{\theta}) = \frac{\mu}{\sigma}$.

   (d) $X_1 \sim \text{Poisson}(\theta)$; $\underline{\Theta} = (0, \infty)$; $g(\theta) = e^\theta$.

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$. Also compare the m.s.e.s of δ_M and δ_U.

   (a) $f(x|\theta) = e^{-(x-\theta)}$, if $x > \theta$, and $= 0$, otherwise; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta$.

   (b) $n \geq 2$, $f(x|\underline{\theta}) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \mu$.

   (c) Same as (b) with $g(\underline{\theta}) = \sigma$.

   (d) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta$.

   (e) $X_1 \sim U(0, \theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

   (f) $X_1 \sim N(\theta, 1)$; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta^2$.

8. Let X_1, X_2 be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. Show that given any unbiased estimator, say $\delta(\underline{X})$, which is not permutation symmetric (i.e., $P_{\underline{\theta}}(\delta(X_1, X_2) = \delta(X_2, X_1)) < 1$, for some $\underline{\theta} \in \Theta$), there exists a permutation symmetric and unbiased estimator $\delta_U(\underline{X})$ which is better than $\delta(\cdot)$. Can you extend this result to the case when we have a random sample consisting of n (≥ 2) observations.

9. Consider a single observation X from a distribution having p.m.f. $f(x|\theta) = \theta$, if $x = -1$, $= (1-\theta)^2\theta^x$, if $x = 0, 1, 2, \ldots$, and $= 0$, otherwise, where $\theta \in \Theta = (0, 1)$ is an unknown parameter. Determine all unbiased estimators of θ.

10. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution having p.d.f.

$$f(x|\underline{\theta}) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, & \text{if } x > \mu \\ 0, & \text{otherwise,} \end{cases}$$

where $\underline{\theta} = (\mu, \sigma) \in \Theta = (-\infty, \infty) \times (0, \infty)$ is unknown. Let the estimand be $g(\underline{\theta}) = \mu$. Find an unbiased estimator of $g(\underline{\theta})$ which is based on the M.L.E.. Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and let $T = \sum_{i=1}^{n}(X_i - X_{(1)})$. Among the estimators of μ, which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_c(\underline{X}) = X_{(1)} - cT, c > 0\}$, find the estimator having the smallest m.s.e., at each parametric point.

11. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$ distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Of the two estimators, the M.M.E. and the M.L.E, of θ, which one would you prefer with respect to (a) the criterion of the bias; (b) the criterion of the m.s.e. Among the estimators of θ, which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_c(\underline{X}) = cX_{(n)}, c > 0\}$, find the estimator having the smallest m.s.e., at each parametric point.

12. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from $U(\theta - 0.5, \theta + 0.5)$ distribution, where $\theta \in \Theta = (-\infty, \infty)$ is an unknown parameter. Let the estimand be $g(\theta) = \theta$. Among the estimators which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_\alpha(\underline{X}) = \alpha(X_{(n)} - 0.5) + (1-\alpha)(X_{(1)} + 0.5), 0 \leq \alpha \leq 1\}$, find the estimator having the smallest m.s.e., at each parametric point.

13. Let $X_1, \ldots, X_n$ be a random sample from the Exp(θ) distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Let the estimand be $g(\theta) = \theta^r$, for some fixed positive integer r. Among the estimators which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_c(\underline{X}) = c\overline{X}^r, c > 0\}$, find the estimator having the smallest m.s.e. at each parametric point. Is this estimator consistent?