ESO 209: Probability and Statistics
2019-2020-II Semester
Assignment No. 7

1. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. In each of the following situations, find the M.M.E. and the M.L.E.. Also verify if they are consistent estimators of $g(\underline{\theta})$.

   (a) $f(x|\theta) = \theta(1-\theta)^{x-1}$, if $x = 1, 2, \ldots$, and $= 0$, otherwise; $\Theta = (0, 1)$; $g(\theta) = \theta$.

   (b) $X_1 \sim \text{Poisson}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = e^{\theta}$.

   (c) $f(x|\underline{\theta}) = \theta_1$, if $x = 1$, $= \frac{1-\theta_1}{\theta_2 - 1}$, if $x = 2, 3, \ldots, \theta_2$, and $= 0$, otherwise; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : 0 < z_1 < 1, z_2 \in \{2, 3, \ldots\}\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

   (d) $f(x|\theta) = K(\theta)x^{\theta}(1-x)$, if $0 \leq x \leq 1$, and $= 0$, otherwise; $\Theta = (-1, \infty)$; $g(\theta) = \theta$; here $K(\theta)$ is the normalizing factor.

   (e) $X_1 \sim \text{Gamma}(\alpha, \mu)$; $\underline{\theta} = (\alpha, \mu)$; $\Theta = (0, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\alpha, \mu)$.

   (f) $f(x|\underline{\theta}) = (\sigma\sqrt{2\pi})^{-1}x^{-1}\exp(-\frac{1}{2\sigma^2}(\ln x - \mu)^2)$, if $x > 0$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

   (g) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_{\theta}(X_1 \leq 1)$.

   (h) $X_1 \sim \text{Poisson}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = P_{\theta}(X_1 + X_2 + X_3 = 0)$.

   (i) $X_1 \sim U(-\frac{\theta}{2}, \frac{\theta}{2})$; $\Theta = (0, \infty)$; $g(\theta) = (1+\theta)^{-1}$.

   (j) $X_1 \sim N(\mu, \sigma^2)$; $\underline{\theta} = (\mu, \sigma^2)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \frac{\mu^2}{\sigma^2}$.

   (k) $f(x|\underline{\theta}) = \sigma^{-1}\exp(-\frac{x-\mu}{\sigma})$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = (\mu, \sigma)$.

   (l) $X_1 \sim U(\theta_1, \theta_2)$; $\underline{\theta} = (\theta_1, \theta_2)$; $\Theta = \{(z_1, z_2) : -\infty < z_1 < z_2 < \infty\}$; $g(\underline{\theta}) = (\theta_1, \theta_2)$.

2. Suppose a randomly selected sample of size five from the distribution having p.m.f. given in Problem 1 (a) gives the following data: $x_1 = 2$, $x_2 = 7$, $x_3 = 6$, $x_4 = 5$ and $x_5 = 9$. Based on this data compute the m.l.e. of $P_{\theta}(X_1 \geq 4)$.

3. The lifetimes of a brand of a component are assumed to be exponentially distributed with mean (in hours) θ, where $\theta \in \Theta = (0, \infty)$ is unknown. Ten of these components were independently put in test. The only data recorded were the number of components that had failed in less than 100 hours versus the number that had not failed. It was found that three had failed before 100 hours. What is the m.l.e. of θ?

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\theta)$, where $\theta \in \Theta$ is an unknown parameter. In each of the following situations, find the M.L.E. of θ and verify if it is a consistent estimator of θ.

   (a) $X_1 \sim N(\theta, 1)$, $\Theta = [0, \infty)$. (b) $X_1 \sim \text{Bin}(1, \theta)$, $\Theta = [\frac{1}{4}, \frac{3}{4}]$.

5. Let $X_1, \ldots, X_n$ be a random sample from a distribution having mean μ and finite variance σ^2. Show that $\overline{X}$ and S^2 are unbiased estimators of μ and σ^2, respectively.

6. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$.

   (a) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

   (b) $n \geq 2$, $X_1 \sim N(\mu, \sigma^2)$; $\underline{\theta} = (\mu, \sigma^2)$; $\Theta = (-\infty, \infty)$; $g(\underline{\theta}) = \mu + \sigma$.

   (c) Same as (b) with $g(\underline{\theta}) = \frac{\mu}{\sigma}$.

   (d) $X_1 \sim \text{Poisson}(\theta)$; $\underline{\Theta} = (0, \infty)$; $g(\theta) = e^{\theta}$.

7. Let $X_1, \ldots, X_n$ be a random sample from a distribution having p.d.f. (or p.m.f.) $f(x|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let $g(\theta)$ be the estimand. In each of the following situations, find the M.L.E., say $\delta_M(\underline{X})$, and the unbiased estimator based on the M.L.E., say $\delta_U(\underline{X})$. Also compare the m.s.e.s of δ_M and δ_U.

   (a) $f(x|\theta) = e^{-(x-\theta)}$, if $x > \theta$, and $= 0$, otherwise; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta$.

   (b) $n \geq 2$, $f(x|\underline{\theta}) = \frac{1}{\sigma} e^{-\frac{x-\mu}{\sigma}}$, if $x > \mu$, and $= 0$, otherwise; $\underline{\theta} = (\mu, \sigma)$; $\Theta = (-\infty, \infty) \times (0, \infty)$; $g(\underline{\theta}) = \mu$.

   (c) Same as (b) with $g(\underline{\theta}) = \sigma$.

   (d) $X_1 \sim \text{Exp}(\theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta$.

   (e) $X_1 \sim U(0, \theta)$; $\Theta = (0, \infty)$; $g(\theta) = \theta^r$, for some known positive integer r.

   (f) $X_1 \sim N(\theta, 1)$; $\Theta = (-\infty, \infty)$; $g(\theta) = \theta^2$.

8. Let X_1, X_2 be a random sample from a distribution having p.d.f. (or p.m.f.) $f(\cdot|\underline{\theta})$, where $\underline{\theta} \in \Theta$ is unknown, and let the estimand be $g(\underline{\theta})$. Show that given any unbiased estimator, say $\delta(\underline{X})$, which is not permutation symmetric (i.e., $P_{\underline{\theta}}(\delta(X_1, X_2) = \delta(X_2, X_1)) < 1$, for some $\underline{\theta} \in \Theta$), there exists a permutation symmetric and unbiased estimator $\delta_U(\underline{X})$ which is better than $\delta(\cdot)$. Can you extend this result to the case when we have a random sample consisting of n (≥ 2) observations.

9. Consider a single observation X from a distribution having p.m.f. $f(x|\theta) = \theta$, if $x = -1$, $= (1-\theta)^2\theta^x$, if $x = 0, 1, 2, \ldots$, and $= 0$, otherwise, where $\theta \in \Theta = (0, 1)$ is an unknown parameter. Determine all unbiased estimators of θ.

10. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from a distribution having p.d.f.

$$f(x|\underline{\theta}) = \begin{cases} \frac{1}{\sigma} e^{-\frac{(x-\mu)}{\sigma}}, & \text{if } x > \mu \\ 0, & \text{otherwise,} \end{cases}$$

where $\underline{\theta} = (\mu, \sigma) \in \Theta = (-\infty, \infty) \times (0, \infty)$ is unknown. Let the estimand be $g(\underline{\theta}) = \mu$. Find an unbiased estimator of $g(\underline{\theta})$ which is based on the M.L.E.. Let $X_{(1)} = \min\{X_1, \ldots, X_n\}$ and let $T = \sum_{i=1}^{n}(X_i - X_{(1)})$. Among the estimators of μ, which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_c(\underline{X}) = X_{(1)} - cT, c > 0\}$, find the estimator having the smallest m.s.e., at each parametric point.

11. Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$ distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Of the two estimators, the M.M.E. and the M.L.E, of θ, which one would you prefer with respect to (a) the criterion of the bias; (b) the criterion of the m.s.e. Among the estimators of θ, which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_c(\underline{X}) = cX_{(n)}, c > 0\}$, find the estimator having the smallest m.s.e., at each parametric point.

12. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from $U(\theta - 0.5, \theta + 0.5)$ distribution, where $\theta \in \Theta = (-\infty, \infty)$ is an unknown parameter. Let the estimand be $g(\theta) = \theta$. Among the estimators which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_\alpha(\underline{X}) : \delta_\alpha(\underline{X}) = \alpha(X_{(n)} - 0.5) + (1 - \alpha)(X_{(1)} + 0.5), 0 \leq \alpha \leq 1\}$, find the estimator having the smallest m.s.e., at each parametric point.

13. Let $X_1, \ldots, X_n$ be a random sample from the Exp(θ) distribution, where $\theta \in \Theta = (0, \infty)$ is an unknown parameter. Let the estimand be $g(\theta) = \theta^r$, for some fixed positive integer r. Among the estimators which are based on the M.L.E. and belong to the class $\mathcal{D} = \{\delta_c(\underline{X}) = c\overline{X}^r, c > 0\}$, find the estimator having the smallest m.s.e. at each parametric point. Is this estimator consistent?

MTH 201: Probability and Statistics

2019 - 2020 - II Semester

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Hints/ Solutions

Problem 1 (a) M.L.E = M.M.E = $\frac{1}{\bar{X}}$. By WLLN $\bar{X} \xrightarrow{P} E(X_1) = \frac{1}{\theta} \Rightarrow$ M.L.E = M.M.E = $\frac{1}{\bar{X}} \xrightarrow{P} \theta$, so consistent. ($g(x) = \frac{1}{x}$ is continuous for $x > 0$)

(b) M.L.E. = M.M.E = $e^{\bar{X}}$. Using W.L.L.N $\bar{X} \xrightarrow{P} \theta \Rightarrow$ M.M.E = M.L.E = $e^{\bar{X}} \xrightarrow{P} e^{\theta}$, so consistent ($g(x) = e^{x}$ is continuous).

(c) Let $S = \#$ of $X_1, \ldots, X_n$ that are one. Then

$$L_{\underset{\sim}{x}}(\theta_1, \theta_2) = \begin{cases} \dfrac{\theta_1^S (1-\theta_1)^{n-S}}{(\theta_2 - 1)^{n-S}}, & 0 < \theta_1 < 1,\ \theta_2 \geq X_{(n)},\ \theta_2 \in \{2, 3, \ldots\} \\ 0, & \text{o.w.} \end{cases}$$

$$\text{M.L.E} = \underset{\sim}{\delta}_{ML} = (\delta_{1ML}, \delta_{2ML}) = \begin{cases} \left(\frac{S}{n}, X_{(n)}\right), & \text{if } 0 \leq S < n \\ (1, 1) \text{ or } (1, 2) \text{ or } (1, 3) \text{ or } \ldots, & \text{if } S = n \end{cases}$$

M.L.E. is not unique. In particular $\left(\frac{S}{n}, X_{(n)}\right)$ is a M.L.E.

$$E_{\underset{\sim}{\theta}}(X) = 1 + \frac{\theta_2(1-\theta_1)}{2}, \quad E_{\underset{\sim}{\theta}}(X_1^2) = \theta_1 + \frac{(1-\theta_1)}{6}(2\theta_2^2 + 5\theta_2 + 6)$$

$$\Rightarrow \hat{\theta}_1 = 1 - \frac{2(A_1 - 1)}{\hat{\theta}_2}, \quad \hat{\theta}_2 = 0 \text{ or } \hat{\theta}_2 = \frac{3(A_2 - 1)}{2(A_1 - 1)} - \frac{5}{2}.$$

Thus M.M.E: $\underset{\sim}{\delta}_{MM} = (\delta_{1MM}, \delta_{2MM}) = \left(1 - \frac{2(A_1 - 1)}{\delta_{2MM}}, \frac{3(A_2 - 1)}{2(A_1 - 1)} - \frac{5}{2}\right)$,

where $A_1 = \bar{X}$ and $A_2 = \frac{(n-1)S^2}{n} + A_1^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$.

$S \sim Bin(n, \theta_1) \Rightarrow \frac{S}{n} \xrightarrow{P} \theta_1 \Rightarrow \frac{S}{n}$ is consistent for θ_1

Since $E_{\underset{\sim}{\theta}}(X_1)$ and $E_{\underset{\sim}{\theta}}(X_1^2)$ are continuous functions of (θ_1, θ_2),

δ_{1MM} and δ_{2MM} are consistent.

For fix $\varepsilon > 0$,

$$0 \leq P[|X_{(n)} - \theta_2| > \varepsilon] = P[X_{(n)} < \theta_2 - \varepsilon] \leq P[X_{(n)} \leq \theta_2 - \varepsilon]$$

$$= \begin{cases} 0, & 2 \leq \theta_2 < 1 + \varepsilon,\ \varepsilon > 1 \\ \left[\theta_1 + \frac{1-\theta_1}{\theta_2 - 1}\left(\frac{[\theta_2 - \varepsilon]([\theta_2 - \varepsilon] + 1)}{2} - 1\right)\right]^n, & \text{if } \theta_2 \geq 1 + \varepsilon \end{cases}$$

$$\longrightarrow 0, \text{ as } n \to \infty.$$

Thus δ_{2ML} is consistent for θ_2.

Note that δ_{2MM} may take non-integer values whereas θ_2 takes integer values.

(d) $K(\theta) = (\theta+1)(\theta+2)$, $\quad \delta_{ML}(\underset{\sim}{X}) = \dfrac{-\sqrt{\bar{L}^2+4} - (3\bar{L}+2)}{2\bar{L}}$, where

$\bar{L} = \frac{1}{n}\sum_{i=1}^{n} \ln x_i$.

$E_\theta(X_1) = \dfrac{\theta+1}{\theta+3}$, $\quad A_1 = \dfrac{\hat\theta+1}{\hat\theta+3} \Rightarrow \hat\theta = \dfrac{3A_1-1}{1-A_1}$

$\Rightarrow \delta_{MM}(\underset{\sim}{X}) = \dfrac{3\bar{X}-1}{1-\bar{X}}$ (M.M.E.)

Since $E_\theta(X_1)$ is a continuous function of θ, $\delta_{MM}(\underset{\sim}{X})$ is consistent.

$E(\ln X_1) = -\dfrac{(2\theta+3)}{(\theta+1)(\theta+2)} \Rightarrow \bar{L} \xrightarrow{P} -\dfrac{(2\theta+3)}{(\theta+1)(\theta+2)}$

$\Rightarrow \delta_{ML} \xrightarrow{P} \theta \Rightarrow \delta_{ML}$ is consistent

(e) $L_{\underset{\sim}{x}}(\alpha,\mu) = \left(\dfrac{1}{\Gamma\alpha\, \mu^\alpha}\right)^n e^{-\sum_{i=1}^n \frac{x_i}{\mu}} \left(\prod_{i=1}^n x_i\right)^{\alpha-1}$, $\quad \alpha>0,\ \mu>0$

$\ln L_{\underset{\sim}{x}}(\alpha,\mu) = -n\ln(\Gamma\alpha) - n\alpha\ln(\mu) - \frac{1}{\mu}\sum_{i=1}^n x_i + (\alpha-1)\sum_{i=1}^n \ln x_i$, $\quad \alpha>0\ \mu>0$

$\frac{\partial}{\partial\alpha}\ln L_{\underset{\sim}{x}}(\alpha,\mu) = 0$, $\quad \frac{\partial}{\partial\mu}\ln L_{\underset{\sim}{x}}(\alpha,\mu) = 0$

$$\Rightarrow \begin{cases} n\dfrac{\Gamma'(\alpha)}{\Gamma(\alpha)} + n\ln(\mu) - \sum_{i=1}^n \ln(x_i) = 0 \\ n\mu\alpha - \sum_{i=1}^n x_i = 0 \end{cases} \Rightarrow \begin{array}{l} \hat\mu = \dfrac{\bar{X}}{\hat\alpha} \\ n\dfrac{\Gamma'(\hat\alpha)}{\Gamma(\hat\alpha)} + n\ln\left(\dfrac{\bar X}{\hat\alpha}\right) - \sum_{i=1}^n \ln(x_i) = 0, \end{array}$$

$\Gamma'(\alpha) = \int_0^\infty (\ln x)\, e^{-x} x^{\alpha-1}\,dx$ $\qquad \longrightarrow$ to be solved numerically. (M.L.E.)

Proving consistency of M.L.E. is a difficult problem.

$E_{\underset{\sim}{\theta}}(X_1) = \alpha\mu$, $E_{\underset{\sim}{\theta}}(X_1^2) = \alpha(\alpha+1)\mu^2 \Rightarrow \hat\alpha\hat\mu = A_1$, $\hat\alpha(\hat\alpha+1)\hat\mu^2 = A_2$

$\Rightarrow \hat\alpha = \dfrac{A_1^2}{A_2-A_1^2}$, $\hat\mu = \dfrac{A_2-A_1^2}{A_1}$, i.e. $\underset{\sim}{\delta}_{MM} = (\delta_{1M}, \delta_{2M})$, where

$\delta_{1M} = \dfrac{\bar{X}^2}{\frac{n-1}{n}S^2}$, $\quad \delta_{2M} = \dfrac{\left(\frac{n-1}{n}\right)S^2}{\bar{X}}$.

Since $E_{\underset{\sim}{\theta}}(X_1)$ and $E_{\underset{\sim}{\theta}}(X_1^2)$ are continuous functions of (α,μ), δ_{1M} and δ_{2M} are consistent.

(f) $\underset{\sim}{\delta}_{ML} = (\delta_{1ML}, \delta_{2ML})$, where $\delta_{1ML} = \frac{1}{n}\sum_{i=1}^n \ln x_i$, $\delta_{2ML} = \frac{1}{n}\sum_{i=1}^n (\ln x_i - \delta_{1ML})^2$.

$E(\ln X_1) = \mu$, $\mathrm{Var}(\ln X) = \sigma^2 \Rightarrow \delta_{1ML} \xrightarrow{P} \mu$, $\delta_{2ML} \xrightarrow{P} \sigma^2$,

so δ_{1ML} and δ_{2ML} are consistent estimators of μ and σ^2.

(g) $g(\theta) = P_\theta(X \le 1) = 1 - e^{-\frac{1}{\theta}}$

M.L.E of θ is $\bar{X}$ $\Rightarrow$ M.L.E of $g(\theta)$ is $\delta_{ML}(\underline{X}) = 1 - e^{-\frac{1}{\bar{X}}}$

By W.L.L.N $\bar{X} \xrightarrow{P} \theta$ and since $g(\theta)$ is a continuous function of θ

$\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} 1 - e^{-\frac{1}{\theta}} = g(\theta)$ $\Rightarrow$ $\delta_{ML}(\underline{X})$ is consistent for estimating $g(\theta)$

$E_\theta(X_1) = \theta$. So M.M.E of $g(\theta)$ is $\delta_{MM}(\underline{X}) = 1 - e^{-\frac{1}{\bar{X}}} = \delta_{ML}(\underline{X})$

(h) $g(\theta) = P_\theta(X_1 + X_2 + X_3 = 0) = e^{-3\theta}$

M.L.E of θ is $\bar{X}$ $\Rightarrow$ M.L.E. of $g(\theta)$ is $\delta_{ML}(\underline{X}) = e^{-3\bar{X}}$.

By W.L.L.N $\bar{X} \xrightarrow{P} E(X_1) = \theta$ and since $g(\theta)$ is a continuous function of θ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} e^{-3\theta} = g(\theta) \Rightarrow \delta_{ML}(\underline{X})$ is consistent for $g(\theta)$.

$E_\theta(X_1) = \theta$. So M.M.E. of $g(\theta)$ is $\delta_{MM}(\underline{X}) = e^{-3\bar{X}} = \delta_{ML}(\underline{X})$.

(i) $l_{\underline{X}}(\theta) = \begin{cases} \frac{1}{\theta^n}, & \text{if } \theta \ge 2T \\ 0, & \text{o.w} \end{cases}$, where $T = \max(|X_1|, \dots, |X_n|)$.

So M.L.E. of θ is $2T$ $\Rightarrow$ M.L.E. of $g(\theta) = (1+\theta)^{-1}$ is $\delta_{ML}(\underline{X}) = (1+2T)^{-1}$.

Fix $\varepsilon > 0$ then

$P(|2T - \theta| > \varepsilon) = P(\frac{\theta}{2} - T > \frac{\varepsilon}{2}) = P(T < \frac{\theta}{2} - \frac{\varepsilon}{2}) = \begin{cases} 0, & \text{if } \varepsilon > 2\theta \\ (1 - \frac{\varepsilon}{2\theta})^n, & \text{if } 0 < \varepsilon \le 2\theta \end{cases}$

$\to 0$, as $n \to \infty$. Thus $2T \xrightarrow{P} \theta$ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} (1+\theta)^{-1}$ (since $g(\theta) = (1+\theta)^{-1}$ is a continuous function for $\theta > 0$). So M.L.E is consistent

$E_\theta(X_1) = 0$. So method of moment for estimation fails.

But $E_\theta(X_1^2) = \frac{\theta^2}{12}$. So modified M.M.E of θ can be obtained from

$A_2 = \frac{\hat{\theta}^2}{12}$ $\Rightarrow$ $\hat{\theta} = \sqrt{12 A_2}$. So modified M.M.E. $\delta_{MM}(\underline{X}) = (1 + \sqrt{12A_2})^{-1}$, here $A_2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. By W.L.L.N $A_2 \xrightarrow{P} \frac{\theta^2}{12}$

$\Rightarrow$ $\delta_{MM}(\underline{X}) \xrightarrow{P} g(\theta) = (1+\theta)^{-1}$. So modified M.M.E $\delta_{MM}(\underline{X})$ is consistent.

(j) M.L.E. of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2) = (A_1, A_2 - A_1^2)$.

So M.L.E of $g(\underline{\theta})$ is $\delta_{ML}(\underline{X}) = \left(\frac{\bar{X}^2}{\frac{1}{n}\sum(X_i - \bar{X})^2} \right)$.

$\bar{X} \xrightarrow{P} \mu$, $\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 \xrightarrow{P} \sigma^2$ $\Rightarrow$ $\delta_{ML}(\underline{X}) \xrightarrow{P} \frac{\mu^2}{\sigma^2} = g(\underline{\theta})$

$\Rightarrow$ M.L.E. $\delta_{MM}(\underline{X})$ is consistent.

$E(X_1) = \mu$, $E(X_1^2) = \sigma^2 + \mu^2$ $\Rightarrow$

M.M.E of (μ, σ^2) is $(\hat{\mu}, \hat{\sigma}^2) = (A_1, A_2 - A_1^2)$

$\Rightarrow$ M.M.E of $g(\underline{\theta})$ is $\delta_{MM}(\underline{X}) = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \delta_{ML}(\underline{X})$.

(k) $$L^*_{\underset{\sim}{x}}(\mu,\sigma) = \ln L_{\underset{\sim}{x}}(\mu,\sigma) = \begin{cases} -n\ln\sigma - \dfrac{\sum_{i=1}^{n}(x_i-\mu)}{\sigma}, & \mu \le x_{(1)}, \sigma>0 \\ 0, & \text{o.w.} \end{cases}$$

Clearly

$$L^*_{\underset{\sim}{x}}(\mu,\sigma) \le L^*_{\underset{\sim}{x}}(x_{(1)},\sigma), \quad \forall\ \mu\le x_{(1)},\ \sigma>0.$$

$$L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) = -n\ln\sigma - \frac{1}{\sigma}\sum_{i=1}^{n}(x_i-x_{(1)}), \quad \sigma>0$$

$$\frac{\partial}{\partial\sigma} L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) = -\frac{n}{\sigma} + \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-x_{(1)})$$

$$\Rightarrow L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) \uparrow (\downarrow) \text{ if } \sigma < (>) \frac{1}{n}\sum_{i=1}^{n}(x_i-x_{(1)}) = \hat{\sigma}, \text{ say.}$$

Thus

$$L^*_{\underset{\sim}{x}}(\mu,\sigma) \le L^*_{\underset{\sim}{x}}(x_{(1)},\sigma) \le L^*_{\underset{\sim}{x}}\left(x_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(x_i-x_{(1)})\right), \quad \forall\ \mu\le x_{(1)},\ \sigma>0$$

$\Rightarrow$ M.L.E. of (μ,σ) is $\underset{\sim}{\delta}_{ML}(\underset{\sim}{X}) = (\delta_{1ML}(\underset{\sim}{X}), \delta_{2ML}(\underset{\sim}{X})) = \left(X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i-X_{(1)})\right)$.

Fix $\varepsilon>0$. Then

$$P_{\underset{\sim}{\theta}}(|X_{(1)}-\mu|>\varepsilon) = P(X_{(1)}>\mu+\varepsilon) = \prod_{i=1}^{n} P(X_i>\mu+\varepsilon) = e^{-\frac{n\varepsilon}{\sigma}} \to 0 \text{ as } n\to\infty.$$

Thus $\delta_{1ML}(\underset{\sim}{X}) = X_{(1)} \xrightarrow{P} \mu$. So δ_{1ML} is consistent for estimating μ.

Also $E_{\underset{\sim}{\theta}}(X_1) = \mu+\sigma$. So by W.L.L.N $A_1 = \frac{1}{n}\sum_{i=1}^{n}X_i \xrightarrow{P} \mu+\sigma$

$$\Rightarrow \delta_{2ML}(\underset{\sim}{X}) = \frac{1}{n}\sum_{i=1}^{n}X_i - X_{(1)} \xrightarrow{P} \mu+\sigma-\mu = \sigma.$$

$\Rightarrow$ δ_{2ML} is consistent for σ.

$E_{\underset{\sim}{\theta}}(X_1) = \mu+\sigma$, $E_{\underset{\sim}{\theta}}(X_1^2) = (\mu+\sigma)^2+\sigma^2$.

So M.M.E. $\delta_{MM}(\underset{\sim}{X}) = (\delta_{1MM}(\underset{\sim}{X}), \delta_{2MM}(\underset{\sim}{X}))$ is given by,

$$A_1 = \delta_{1MM}+\delta_{2MM}, \quad A_2 = (\delta_{1MM}+\delta_{2MM})^2 + \delta_{2MM}^2$$

$$\Rightarrow \delta_{2MM} = \sqrt{\frac{n-1}{n}}\,S \quad \text{and} \quad \delta_{1MM} = \bar{X} - \sqrt{\frac{n-1}{n}}\,S.$$

$S^2 \xrightarrow{P} V(X_1) = \sigma^2$, $\bar{X} \xrightarrow{P} \mu+\sigma \Rightarrow \delta_{1MM} \xrightarrow{P} \mu+\sigma-\sqrt{\sigma^2} = \mu$

$\delta_{2MM} \xrightarrow{P} \sigma \Rightarrow \delta_{1MM}$ and δ_{2MM} are consistent for μ and σ, respectively.

(l) $$L_{\underset{\sim}{x}}(\theta_1,\theta_2) = \begin{cases} \dfrac{1}{(\theta_2-\theta_1)^n}, & \text{if } \theta_1\le x_{(1)} \text{ and } \theta_2 \ge x_{(n)} \\ 0, & \text{o.w.} \end{cases}$$

Clearly M.L.E. of $g(\underset{\sim}{\theta}) = (\theta_1,\theta_2)$ is $\delta_{ML}(\underset{\sim}{X}) = (\delta_{1ML}(\underset{\sim}{X}), \delta_{2ML}(\underset{\sim}{X})) = (X_{(1)}, X_{(n)})$

Fix $\varepsilon > 0$. Then

$$P_{\underline{\theta}}(|X_{(1)}-\theta_1| > \varepsilon) = P_{\underline{\theta}}(X_{(1)} > \theta_1+\varepsilon) = \prod_{i=1}^{n} P_{\underline{\theta}}(X_i > \theta_1+\varepsilon) = \begin{cases} 0 & \text{if } \varepsilon \geq \theta_2-\theta_1 \\ \left(1-\frac{\varepsilon}{\theta_2-\theta_1}\right)^n & \text{if } 0<\varepsilon<\theta_2-\theta_1 \end{cases}$$

$\to 0$ as $n\to\infty$.

So $\delta_{1ML}(\underline{X}) = X_{(1)}$ is consistent for θ_1.

$$P_{\underline{\theta}}(|X_{(n)}-\theta_2| > \varepsilon) = P_{\underline{\theta}}(\theta_2 - X_{(n)} > \varepsilon) = P_{\underline{\theta}}(X_{(n)} < \theta_2-\varepsilon) = \prod_{i=1}^{n} P_{\underline{\theta}}(X_i < \theta_2-\varepsilon)$$

$$= \begin{cases} 0 & \text{if } \varepsilon \geq \theta_2-\theta_1 \\ \left(1-\frac{\varepsilon}{\theta_2-\theta_1}\right)^n & \text{if } 0<\varepsilon<\theta_2-\theta_1 \end{cases}$$

Thus $\to 0$ as $n\to\infty$.

$\delta_{2ML}(\underline{X}) = X_{(n)}$ is consistent for θ_2.

$$E_{\underline{\theta}}(X_1) = \frac{\theta_1+\theta_2}{2}, \quad E_{\underline{\theta}}(X_1^2) = \frac{\theta_2^2+\theta_1^2+\theta_1\theta_2}{3} = \frac{(\theta_1+\theta_2)^2-\theta_1\theta_2}{3}.$$

Thus the M.M.E. $\underset{\sim}{\delta}_{MM} = (\delta_{1MM}, \delta_{2MM})$ is given by

$$\frac{\delta_{1MM}+\delta_{2MM}}{2} = A_1, \quad \frac{(\delta_{1MM}+\delta_{2MM})^2 - \delta_{1MM}\delta_{2MM}}{3} = A_2$$

$$\Rightarrow \delta_{1MM} = A_1 - \sqrt{3(A_2-A_1^2)} = \bar{X} - \sqrt{\frac{3(n-1)}{n}}\, S,$$

$$\delta_{2MM} = A_1 + \sqrt{3(A_2-A_1^2)} = \bar{X} + \sqrt{\frac{3(n-1)}{n}}\, S.$$

Since $E_{\underline{\theta}}(X_1)$ and $E_{\underline{\theta}}(X_1^2)$ are continuous functions of (θ_1, θ_2), it follows that δ_{1MM} and δ_{2MM} are consistent for estimating θ_1 and θ_2, respectively.

Problem 2

$g(\theta) = P_\theta(X_1 \geq 4) = (1-\theta)^3$

M.L.E. of θ is $\frac{1}{\bar{X}}$, so M.L.E. of $g(\theta)$ is $\hat{\delta}_{ML} = \left(1-\frac{1}{\bar{X}}\right)^3$.

$$\bar{X} = \frac{2+7+6+5+9}{5} = \frac{29}{5}.$$

So m.l.e. of $g(\theta) = \left(1-\frac{5}{29}\right)^3 = \left(\frac{24}{29}\right)^3$.

Problem 3

Let X = # of items that have failed in less than 100 hours

$X \sim \text{Bin}(10, \mu)$, where $\mu = \frac{1}{\theta}\int_0^{100} e^{-x/\theta}\,dx = 1 - e^{-\frac{100}{\theta}}$

$\Rightarrow \theta = \frac{-100}{\ln(1-\mu)}$. Given $x=3$, $\hat{\mu} = \frac{3}{10} = 0.3$ is the m.l.e. of μ

$\Rightarrow$ m.l.e. of θ is $\hat{\theta} = \frac{-100}{\ln(0.7)}$.

Problem 4 (a) $L_{\underset{\sim}{x}}(\theta) = (2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2}$, $\theta \geq 0$

$L^*_{\underset{\sim}{x}}(\theta) = \ln L_{\underset{\sim}{x}}(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{1}{2}\sum_{i=1}^{n}(x_i-\theta)^2$, $\theta \geq 0$

$\frac{d}{d\theta} L^*_{\underset{\sim}{x}}(\theta) = \sum_{i=1}^{n}(x_i-\theta) \geq 0 \iff \theta < \bar{x}$

Thus $L^*_{\underset{\sim}{x}}(\theta) \uparrow (\downarrow)$ if $\theta < \bar{x}$ $(\theta > \bar{x})$

Case I $\bar{x} < 0$

$L^*_x(\theta)$, $\theta \in [0, \infty)$, is maximized at $\theta = 0$.

Case II $\bar{x} \geq 0$

$L^*_x(\theta)$, $\theta \in [0, \infty)$, is maximized at $\theta = \bar{x}$.

Thus, the M.L.E. of θ $(\theta \in \Theta = [0, \infty))$ is $\delta_{ML}(\underset{\sim}{x}) = \max(\bar{x}, 0)$.

(b) $L_{\underset{\sim}{x}}(\theta) = \prod_{i=1}^{n}\binom{1}{x_i}\theta^{x_i}(1-\theta)^{1-x_i} = c_{\underset{\sim}{x}}\,\theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$, $\frac{1}{4} \leq \theta \leq \frac{3}{4}$,

where $c_{\underset{\sim}{x}} = \prod_{i=1}^{n}\binom{1}{x_i}$.

$L^*_{\underset{\sim}{x}}(\theta) = \ln L_{\underset{\sim}{x}}(\theta) = \ln c_{\underset{\sim}{x}} + \left(\sum_{i=1}^{n} x_i\right)\ln\theta + (n-\sum x_i)\ln(1-\theta)$.

$\frac{d}{d\theta} L^*_{\underset{\sim}{x}}(\theta) = \frac{\sum_{i=1}^{n} x_i}{\theta} - \frac{(n-\sum_{i=1}^{n} x_i)}{1-\theta} \geq 0 \iff \theta < \bar{x}$.

Thus $L^*_{\underset{\sim}{x}}(\theta) \uparrow (\downarrow)$ if $\theta < \bar{x}$ $(\theta > \bar{x})$.

Case I $0 \leq \bar{x} \leq \frac{1}{4}$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = \frac{1}{4}$.

Case II. $\frac{1}{4} \leq \bar{x} \leq \frac{3}{4}$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = \bar{x}$

Case III $\bar{x} \geq 3/4$

$L^*_{\underset{\sim}{x}}(\theta)$ is maximized at $\theta = 3/4$.

Thus, the M.L.E. of θ $(\theta \in \Theta = [\frac{1}{4}, \frac{3}{4}])$ is

$$\delta_{ML}(\underset{\sim}{x}) = \begin{cases} \frac{1}{4}, & \text{if } 0 \leq \bar{x} < \frac{1}{4} \\ \bar{x} & \text{if } \frac{1}{4} \leq \bar{x} \leq 3/4 \\ 3/4 & \text{if } \bar{x} \geq 3/4 \end{cases} = \psi(\bar{x}), \text{ say.}$$

By W.L.L.N $\bar{X} \xrightarrow{P} \theta \Rightarrow \psi(\bar{X}) \xrightarrow{P} \begin{cases} \frac{1}{4}, & \text{if } 0 \le \theta < 1/4 \\ \theta, & \text{if } \frac{1}{4} \le \theta \le 3/4 \\ 3/4, & \text{if } \theta > 3/4 \end{cases}$ (Since

$\psi(\theta)$ is a continuous function of θ). Thus $\delta_{ML}(\underline{X}) \xrightarrow{P} \theta$, $\forall\, \theta \in \Theta = [\frac{1}{4}, \frac{3}{4}] \Rightarrow \delta_{ML}$ is consistent for estimating θ ($\theta \in \Theta = [\frac{1}{4}, \frac{3}{4}]$).

$\boxed{\text{Problem 5}}$ Let $\underline{\theta} = (\mu, \sigma^2)$. Then

$$E_{\underline{\theta}}(\bar{X}) = E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i) = \frac{1}{n}\sum_{i=1}^{n}\mu = \mu$$

$$E_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right) = \frac{1}{n}\sum_{i=1}^{n} E_{\underline{\theta}}(X_i^2) = E_{\underline{\theta}}(X_1^2) = V_{\underline{\theta}}(X_1) + \left(E_{\underline{\theta}}(X_1)\right)^2$$
$$= \sigma^2 + \mu^2$$

$$\Rightarrow E_{\underline{\theta}}(S^2) = E_{\underline{\theta}}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = \frac{n}{n-1} E_{\underline{\theta}}\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2\right].$$

$$V_{\underline{\theta}}(\bar{X}) = V_{\underline{\theta}}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} V_{\underline{\theta}}(X_i) = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 = \frac{\sigma^2}{n}.$$

$$\Rightarrow E_{\underline{\theta}}(\bar{X}^2) = V_{\underline{\theta}}(\bar{X}) + (E_{\underline{\theta}}(\bar{X}))^2 = \frac{\sigma^2}{n} + \mu^2$$

Thus

$$E_{\underline{\theta}}(S^2) = \frac{n}{n-1}\left[\sigma^2 + \mu^2 - \frac{\sigma^2}{n} - \mu^2\right] = \sigma^2.$$

$\boxed{\text{Problem 6}}$ (a) M.L.E of θ is $\bar{X}$. So M.L.E of $g(\theta)$ is $\delta_{M}(\underline{X}) = \bar{X}^r = \frac{T^r}{n^r}$,

where $T = \sum_{i=1}^{n} X_i \sim$ Gamma(n, θ)

$$E_\theta(T^r) = n(n+1)\cdots(n+r-1)\,\theta^r$$

$$\Rightarrow E_\theta\left(\frac{T^r}{n(n+1)\cdots(n+r-1)}\right) = \theta^r$$

$$\Rightarrow \delta_U(\underline{X}) = \frac{T^r}{n(n+1)\cdots(n+r-1)} = \frac{n^r}{n(n+1)\cdots(n+r-1)}\,\bar{X}^r.$$

(b) M.L.E. of (μ, σ^2) is $(\bar{X}, \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2)$. So M.L.E. of $g(\underline{\theta})$ is

$$\delta_M(\underline{X}) = \bar{X} + \frac{T}{\sqrt{n}}, \quad \text{where } T = \sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2}.$$

$$\frac{T^2}{\sigma^2} \sim \chi^2_{n-1} \Rightarrow E_{\underline{\theta}}\left(\frac{T}{\sigma}\right) = \frac{2^{n/2}\,\Gamma\frac{n}{2}}{2^{\frac{n-1}{2}}\,\Gamma\frac{n-1}{2}}$$

$$\Rightarrow E_{\underline{\theta}}\left(\frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\,\Gamma\frac{n}{2}}\,T\right) = \sigma$$

$$\Rightarrow \delta_U(\underline{X}) = \bar{X} + \frac{\Gamma\frac{n-1}{2}}{\sqrt{2}\,\Gamma\frac{n}{2}}\,T.$$

(c) M.L.E. of θ is $\bar{X}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = e^{\bar{X}} = e^{\frac{T}{n}}$,

where $T = \sum_{i=1}^{n} X_i \sim \text{Poisson}(n\theta)$.

We need to find $h(T)$ (a function of T or equivalently of $\bar{X}$) s.t.,

$$E_\theta(h(T)) = e^\theta, \quad \forall \theta > 0$$

$$\Rightarrow \sum_{j=0}^{\infty} h(j) \frac{e^{-n\theta}(n\theta)^j}{j!} = e^\theta \quad \forall \theta > 0$$

$$\Rightarrow \sum_{j=0}^{\infty} \frac{h(j)\, n^j}{j!} \theta^j = e^{(n+1)\theta}, \quad \forall \theta > 0$$

$$= \sum_{j=0}^{\infty} \frac{(n+1)^j \theta^j}{j!}, \quad \forall \theta > 0$$

Since the two power series (on L.H.S and R.H.S.) match $\forall \theta > 0$, the coefficients of θ^j in two power series are same, i.e.

$$\frac{h(j)\, n^j}{j!} = \frac{(n+1)^j}{j!}, \quad j = 0, 1, 2, \ldots$$

$$\Rightarrow h(j) = \left(1 + \frac{1}{n}\right)^j, \quad j = 0, 1, 2, \ldots$$

$$\Rightarrow h(T) = \left(1 + \frac{1}{n}\right)^T = \left(1 + \frac{1}{n}\right)^{n\bar{X}}$$

$$\Rightarrow \delta_U(\underline{X}) = \left(1 + \frac{1}{n}\right)^{n\bar{X}}$$

$\boxed{\text{Problem 7}}$ (a) M.L.E. of $g(\theta) = \theta$ is $\delta_M(\underline{X}) = X_{(1)}$.

$$f_{X_{(1)}}(x) = \begin{cases} e^{-n(x-\theta)}, & \text{if } x > 0 \\ 0, & \text{o.w.} \end{cases}$$

$E_\theta(X_{(1)}) = \theta + \frac{1}{n}, \quad \forall \theta \in (-\infty, \infty) \quad \Rightarrow \quad E_\theta\left(X_{(1)} - \frac{1}{n}\right) = \theta \quad \forall \theta \in (-\infty, \infty)$

$$\Rightarrow \delta_U(\underline{X}) = X_{(1)} - \frac{1}{n}.$$

$$M_{\delta_M}(\underset{\sim}{\theta}) - M_{\delta_U}(\underset{\sim}{\theta}) = E_{\underset{\sim}{\theta}}[(X_{(1)} - \theta)^2] - E_{\underset{\sim}{\theta}}[(X_{(1)} - \tfrac{1}{n} - \theta)^2]$$

$$= +\frac{2}{n} E_\theta(X_{(1)} - \theta) - \frac{1}{n^2} = \frac{1}{n^2} > 0, \quad \forall \theta \in (-\infty, \infty)$$

Thus, in terms of M.S.E., δ_U is better than δ_M.

(b) M.L.E. of $\underline{\theta} = (\mu, \sigma)$ is $\left(X_{(1)}, \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})\right)$. So the M.L.E. of $g(\underline{\theta})$ is $\delta_M(\underline{X}) = X_{(1)}$.

Let $T = \sum_{i=1}^{n}(X_i - X_{(1)})$. Then $X_{(1)}$ and T are independent (see the Problem 4 of Mid Sem Exam-II), with

$$f_{X_{(1)}}(x) = \begin{cases} \frac{n}{\sigma} e^{-\frac{n}{\sigma}(x-\mu)}, & \text{if } x \geq \mu \\ 0, & \text{o.w.} \end{cases}$$

$$f_T(t) = \frac{e^{-\frac{t}{\sigma}} t^{n-2}}{\Gamma(n-1)\, \sigma^{n-1}}, \quad t > 0.$$

$$E_{\underline{\theta}}(\delta_m) = \mu + \frac{\sigma}{n}, \quad E_\theta(T) = (n-1)\sigma \Rightarrow E_\theta\left(\frac{T}{n-1}\right) = \sigma, \ \forall \underset{\sim}{\theta}.$$

$$\Rightarrow \delta_U(\underline{x}) = \delta_m - \frac{T}{n(n-1)} = x_{(1)} - \frac{1}{n(n-1)} \sum_{i=1}^{n} (x_i - x_{(1)})$$

$$M_{\delta_m}(\underline{\theta}) - M_{\delta_U}(\underline{\theta}) = E_{\underline{\theta}}[(x_{(1)}-\mu)^2] - E_\theta[(x_{(1)} - \frac{T}{n(n-1)} - \mu)^2]$$

$$= \frac{2}{n(n-1)} E_\theta[(x_{(1)}-\mu)T] - \frac{1}{n^2(n-1)^2} E_{\underline{\theta}}[T^2]$$

$$= \frac{2}{n(n-1)} E_\theta[(x_{(1)}-\mu)] E_\theta[T] - \frac{1}{n^2(n-1)^2} E_\theta(T^2)$$

$$E_\theta[(x_{(1)}-\mu)] = \frac{\sigma}{n} \quad E_\theta(T) = (n-1)\sigma, \quad E_\theta(T^2) = n(n-1)\sigma^2$$

$$M_{\delta_m}(\underline{\theta}) - M_{\delta_U}(\underline{\theta}) = \frac{2}{n(n-1)} \times \frac{\sigma}{n} \times (n-1)\sigma - \frac{1}{n^2(n-1)^2} \times n(n-1)\sigma^2$$

$$= \frac{2}{n^2}\sigma^2 - \frac{\sigma^2}{n(n-1)} > 0,$$

$\Rightarrow$ Under the M.S.E criterion, δ_U is better than δ_m.

(c) M.L.E of $g(\underline{\theta})$ is $\delta_m(\underline{x}) = \frac{1}{n}\sum_{i=1}^{n}(x_i - x_{(1)}) = \frac{T}{n}$, Now,

$$E_{\underline{\theta}}(\delta_m(\underline{x})) = E_\theta\left(\frac{T}{n}\right) = \frac{n-1}{n}\sigma, \quad \forall \underset{\sim}{\theta}$$

$$\Rightarrow E_\theta\left(\frac{T}{n-1}\right) = \sigma, \quad \forall \underset{\sim}{\theta} \in \Theta$$

$$\Rightarrow \delta_U(\underline{x}) = \frac{\sum_{i=1}^{n}(x_i - x_{(1)})}{n-1}.$$

$$M_{\delta_m}(\underline{\theta}) - M_{\delta_U}(\theta) = E_{\underline{\theta}}\left[\left(\frac{T}{n} - \sigma\right)^2\right] - E_\theta\left[\left(\frac{T}{n-1} - \sigma\right)^2\right]$$

$$= \left(\frac{1}{n^2} - \frac{1}{(n-1)^2}\right) E_{\underline{\theta}}(T^2) - 2\sigma[E_\theta(T)]\left[\frac{1}{n} - \frac{1}{n-1}\right]$$

$$= \frac{-2n+1}{n^2(n-1)^2} \times n(n-1)\sigma^2 + 2\sigma^2 \times \frac{n-1}{n(n-1)}$$

$$= -\frac{2n-1}{n(n-1)}\sigma^2 + \frac{2\sigma^2}{n} < 0 \Rightarrow$$ δ_m is better than δ_U under the M.S.E. criterion

(d) M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = \overline{X} = \frac{T}{n}$, where $T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

$E_\theta(T) = n\theta$, $\forall\, \theta > 0 \Rightarrow E_\theta\left(\frac{T}{n}\right) = \theta$, $\forall\, \theta \in \Theta$

$\Rightarrow \delta_U(\underline{X}) = \delta_M(\underline{X}) = \overline{X}$.

(e) M.L.E. of θ is $X_{(n)}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = X_{(n)}^r$.

$$f_{X_{(n)}}(x) = \begin{cases} \frac{n x^{n-1}}{\theta^n}, & 0 < x < \theta \\ 0, & \text{o.w.} \end{cases}$$

$E_\theta(X_{(n)}^r) = \frac{n}{n+r}\theta^r$, $\forall\, \theta \in \Theta \Rightarrow E_\theta\left(\frac{n+r}{n} X_{(n)}^r\right) = \theta^r$, $\theta > 0$.

$\Rightarrow \delta_U(\underline{X}) = \frac{n+r}{n} X_{(n)}^r$.

$$M_{\delta_M}(\theta) - M_{\delta_U}(\theta) = E_\theta\left[(X_{(n)}^r - \theta^r)^2\right] - E_\theta\left[\left(\frac{n+r}{n} X_{(n)}^r - \theta^r\right)^2\right]$$

$$= \left[1 - \frac{(n+r)^2}{n^2}\right] E_\theta(X_{(n)}^{2r}) - 2\theta^r\left[1 - \frac{n+r}{n}\right] E_\theta(X_{(n)}^r)$$

$$= \left[1 - \frac{(n+r)^2}{n^2}\right] \cdot \frac{n}{n+2r}\theta^{2r} - 2\theta^r\left(1 - \frac{n+r}{n}\right)\frac{n}{n+r}\theta^r$$

$$= \frac{2r\,\theta^{2r}}{n+r} - \frac{2nr + r^2}{n(n+2r)}\theta^{2r}$$

$$= \frac{r^2(n-r)}{n(n+r)(n+2r)} > 0, \text{ if } n > r, \quad = 0 \text{ if } n = r, \quad < 0 \text{ if } n < r.$$

Thus for $n > r$, δ_U is better than δ_M (under m.s.e.), for $n < r$, δ_M is better than δ_U and for $n = r$ δ_M and δ_U have the same m.s.e.

(f) M.L.E. of θ is $\overline{X}$. So M.L.E. of $g(\theta)$ is $\delta_M(\underline{X}) = \overline{X}^2$.

$E_\theta(\overline{X}^2) = \frac{1}{n} + \theta^2 \Rightarrow E_\theta\left(\overline{X}^2 - \frac{1}{n}\right) = \theta^2$, $\forall\, \theta$

$\Rightarrow \delta_U(\underline{X}) = \overline{X}^2 - \frac{1}{n}$.

$$M_{\delta_M}(\theta) - M_{\delta_U}(\theta) = E_\theta\left[(\overline{X}^2 - \theta^2)^2\right] - E_\theta\left[\left(\overline{X}^2 - \frac{1}{n} - \theta^2\right)^2\right]$$

$$= \frac{2}{n} E_\theta\left[\overline{X}^2 - \theta^2\right] - \frac{1}{n^2}$$

$$= \frac{1}{n^2} > 0$$

$\Rightarrow$ δ_U is better than δ_M under the m.s.e. criterion.

Problem 8 Suppose that $\delta(X_1, X_2)$ is unbiased, i.e. $E_{\underline{\theta}}(\delta(X_1, X_2)) = g(\underline{\theta})$ $\forall \underline{\theta} \in \Theta$, and $P_{\underline{\theta}_0}(\delta(X_1, X_2) = \delta(X_2, X_1)) < 1$, for some $\underline{\theta}_0 \in \Theta$.

Define

$$\delta_U(X_1, X_2) = \frac{1}{2}[\delta(X_1, X_2) + \delta(X_2, X_1)]. \text{ Then } \delta_U(X_1, X_2) = \delta_U(X_2, X_1), \text{ w.p. 1.}$$

X_1, X_2 is a random sample $\Rightarrow (X_1, X_2) \stackrel{d}{=} (X_2, X_1) \Rightarrow E_{\underline{\theta}}[\delta(X_2, X_1)] =$

$E_{\underset{\sim}{\theta}}(\delta(X_1, X_2)) = g(\underline{\theta})$, $\forall \underline{\theta} \in \Theta$.

Thus,

$$E_{\underline{\theta}}(\delta_U(X_1, X_2)) = g(\underline{\theta}), \quad \forall \underline{\theta} \in \Theta.$$

Also

$$V_\theta(\delta_U(X_1, X_2)) = \frac{1}{4}[V_\theta(\delta(X_1, X_2)) + V_\theta(\delta(X_2, X_1)) + 2\,Cov_\theta(\delta(X_1, X_2), \delta(X_2, X_1))]$$

$$\leq \frac{1}{4}\left[V_\theta(\delta(X_1, X_2)) + V_\theta(\delta(X_2, X_1)) + 2\sqrt{V_\theta(\delta(X_1, X_2))\,V_\theta(\delta(X_2, X_1))}\right]$$

$$= V_{\underline{\theta}}(\delta(X_1, X_2)), \quad \forall \theta \in \Theta \quad \left(\text{since } (X_1, X_2) \stackrel{d}{=} (X_2, X_1) \Rightarrow V_\theta(\delta(X_1, X_2)) = V_\theta(\delta(X_2, X_1))\right)$$

and we have equality at $\theta \in \Theta$ iff

$$P_\theta\left(\frac{\delta(X_1, X_2) - g(\underline{\theta})}{\sqrt{V_\theta(\delta(X_1, X_2))}} = \frac{\delta(X_2, X_1) - g(\theta)}{\sqrt{V_\theta(\delta(X_2, X_1))}}\right) = 1$$

$$\Leftrightarrow \quad P_\theta(\delta(X_1, X_2) = \delta(X_2, X_1)) = 1$$

Since $P_{\theta_0}(\delta(X_1, X_2) = \delta(X_2, X_1)) < 1$, it follows that

$V_{\underline{\theta}}(\delta_U(X_1, X_2)) \leq V_{\underset{\sim}{\theta}}(\delta(X_1, X_2))$, $\forall \underline{\theta} \in \Theta$

and $V_{\underset{\sim}{\theta}_0}(\delta_U(X_1, X_2)) < V_{\underset{\sim}{\theta}_0}(\delta(X_1, X_2))$

$\Rightarrow$ δ_U is permutation symmetric and better than δ.

The result can be extended to a sample size of $n (\geq 3)$, by considering

$$\delta_U(X_1, X_2, \ldots, X_n) = \frac{1}{n!}\sum_{(i_1 \ldots i_n)} \cdots \sum \delta(X_{i_1}, \ldots, X_{i_n}),$$

where sum is over all $n!$ permutations $(i_1 \ldots i_n)$ of $(1, \ldots, n)$.

Problem 9 δ is an unbiased estimator of θ iff

$$E_\theta(\delta(X)) = \theta, \quad \forall \theta \in (0, 1)$$

$$\Leftrightarrow \quad \delta(-1)\theta + (1-\theta)^2 \sum_{x=0}^{\infty} \delta(x)\theta^x = \theta, \quad \forall \theta \in (0, 1)$$

$$\Leftrightarrow \sum_{x=0}^{\infty} \delta(x)\,\theta^{x} = (1-\delta(-1))\,\theta\,(1-\theta)^{-2}, \quad \forall\ 0<\theta<1$$

$$\Leftrightarrow \sum_{x=0}^{\infty} \delta(x)\,\theta^{x} = (1-\delta(-1))\,\theta\,(1+2\theta+3\theta^2+4\theta^3+\cdots), \quad \forall\ 0<\theta<1$$

$$= (1-\delta(-1))\,(\theta+2\theta^2+3\theta^3+4\theta^4+\cdots), \quad \forall\ 0<\theta<1$$

Thus we have two power series in θ matching on interval $(0,1)$

$$\Leftrightarrow \delta(0)=0, \quad \delta(x) = (1-\delta(-1))\,x, \quad x=1,2,\ldots$$

Thus the class of unbiased estimators of θ is

$$\mathcal{D}_U = \{\delta :\ \delta(-1)=a,\ \delta(k)=(1-a)k,\ k=0,1,2,\ldots,\ a\in\mathbb{R}\}.$$

$\boxed{\text{Problem 10}}$ M.L.E. of (μ,σ) is $(X_{(1)}, T)$. We ~~know~~ have ~~(see the problem given in Mid Sem-II Exam)~~

$$\left.\begin{array}{l} \dfrac{n(X_{(1)}-\mu)}{\sigma} \sim \text{Exp}(1) \\[2ex] \dfrac{T}{\sigma} \sim \text{Gamma}(n-1,1) \end{array}\right\rangle \text{independent.} \quad \text{(Show this)}$$

Thus, $\forall\ \underline{\theta}\in\Theta$,

$$E_{\underline{\theta}}\left[\frac{n(X_{(1)}-\mu)}{\sigma}\right]=1, \quad E_{\underline{\theta}}\left[\frac{T}{\sigma}\right]=n-1$$

$$\Rightarrow E_{\underline{\theta}}[X_{(1)}] = \mu+\frac{\sigma}{n}, \quad E_{\underline{\theta}}[T] = \sigma(n-1)$$

$$\Rightarrow E_{\underline{\theta}}\left[X_{(1)} - \frac{T}{n(n-1)}\right] = \mu, \quad \forall\ \underline{\theta}\in\Theta$$

$$\Rightarrow \delta_U(\underline{X}) = X_{(1)} - \frac{1}{n(n-1)}\sum_{i=1}^{n}(X_i - X_{(1)}). \quad \text{(Also see Problem 7(b))}.$$

For $\underline{\theta}\in\Theta$ and $c\in(-\infty,\infty)$

$$r_{\delta_c}(\underline{\theta}) = E_{\underline{\theta}}[(X_{(1)}-cT-\mu)^2]$$

$$= c^2 E_{\underline{\theta}}[T^2] - 2c\,E_{\underline{\theta}}[(X_{(1)}-\mu)T] + E_{\underline{\theta}}[(X_{(1)}-\mu)^2]$$

$$= c^2 E_{\underline{\theta}}[T^2] - 2c\,E_{\underline{\theta}}[X_{(1)}-\mu]\,E_{\underline{\theta}}[T] + E_{\underline{\theta}}[(X_{(1)}-\mu)^2]$$

($X_{(1)}$ and T are independent).

Fix $\underline{\theta}\in\Theta$. Then

$$\frac{\partial}{\partial c} r_{\delta_c}(\underline{\theta}) = 2c\,E_{\underline{\theta}}(T^2) - 2\,E_{\underline{\theta}}[(X_{(1)}-\mu)]\,E_{\underline{\theta}}[T]$$

$$\frac{\partial^2}{\partial c^2} r_{\delta_c}(\underline{\theta}) = 2\,E_{\underline{\theta}}(T^2) > 0.$$

Thus, for fixed $\underline{\theta} \in \Theta$, $M_{\delta_c}(\underline{\theta})$ is minimized at

$$c = \frac{E_{\underline{\theta}}[X_{(1)} - \mu]\, E_{\underline{\theta}}(T)}{E_{\underline{\theta}}(T^2)} = \frac{\frac{\sigma}{n} \times \sigma(n-1)}{n(n-1)\sigma^2}$$

$$= \frac{1}{n^2} \longrightarrow \text{does not dependent on } \underline{\theta}.$$

Thus for every $\underline{\theta} \in \Theta$, $M_{\delta_c}(\underline{\theta})$ is minimized at $c = \frac{1}{n^2} = c_0$ (say).

$$\Rightarrow \delta_{c_0}(\underline{X}) = X_{(1)} - \frac{1}{n^2} \sum_{i=1}^{n} (X_i - X_{(1)})$$

is the best estimator, with respect to the m.s.e. criterion in the class $\mathcal{D}$.

Problem 11 M.L.E. of θ is $\delta_{ML}(\underline{X}) = X_{(n)}$
M.M.E. of θ is $\delta_{MM}(\underline{X}) = 2\bar{X}$ (since $E(X_1) = \frac{\theta}{2}$)

$$E_{\theta}(X_{(n)}) = \frac{n}{n+1}\theta, \quad E_{\theta}(X_{(n)}^2) = \frac{n}{n+2}\theta^2, \quad \theta \in \Theta$$

$$E_{\theta}(X_1) = \frac{\theta}{2}, \quad E_{\theta}(X_1^2) = \frac{\theta^2}{3}, \quad V_{\theta}(X_1) = \frac{\theta^2}{12}, \quad \theta \in \Theta.$$

$$M_{\delta_{ML}}(\underline{X}) = E_{\theta}[(X_{(n)} - \theta)^2] = E_{\theta}[X_{(n)}^2] - 2\theta E_{\theta}(X_{(n)}) + \theta^2$$

$$= \theta^2 \left[\frac{n}{n+2} - \frac{2n}{n+1} + 1\right] = \frac{2}{(n+1)(n+2)}\theta^2.$$

$$M_{\delta_{MM}}(\underline{X}) = E_{\theta}[(2\bar{X} - \theta)]^2 = V_{\theta}(2\bar{X}) = 4V_{\theta}(\bar{X}) = 4 \times \frac{\theta^2}{12n} = \frac{\theta^2}{3n}$$

$$> \frac{2}{(n+1)(n+2)}\theta^2, \quad \forall \theta \in \Theta, \ n \geq 2.$$

Thus, for $n \geq 2$, δ_{ML} is preferable over δ_{MM}.

Fix $\theta \in \Theta$. Then, for $c \in (0, \infty)$,

$$M_{\delta_c}(\theta) = E_{\theta}[c X_{(n)} - \theta]^2 = c^2 E_{\theta}(X_{(n)}^2) - 2c\theta E_{\theta}(X_{(n)}) + \theta^2$$

$$\frac{\partial}{\partial c} M_{\delta_c}(\theta) = 2c E_{\theta}(X_{(n)}^2) - 2\theta E_{\theta}(X_{(n)})$$

$$\frac{\partial^2}{\partial c^2} M_{\delta_c}(\theta) = 2 E_{\theta}(X_{(n)}^2) > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at

$$c = \frac{\theta E_{\theta}(X_{(n)})}{E_{\theta}(X_{(n)}^2)} = \frac{n+2}{n+1} \longrightarrow \text{does not depend on } \theta.$$

Thus, for every $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at $c = \frac{n+2}{n+1} = c_0$ say.

Thus among the estimators in class $\mathcal{D}$, $\delta_{c_0}(\underline{x}) = \frac{n+2}{n+1} X_{(n)}$ has the smallest m.s.e. for each $\theta \in \Theta$

$\boxed{\text{Problem 12}}$ Fix $\theta \in \Theta$. Then, for $\alpha \in [0,1)$,

$$M_{\delta_\alpha}(\theta) = E_\theta\left[\left(\alpha\left(X_{(n)} - \tfrac{1}{2}\right) + (1-\alpha)\left(X_{(1)} + \tfrac{1}{2}\right) - \theta\right)^2\right]$$

$$= E_\theta\left[\left(\alpha\left(X_{(n)} - X_{(1)} - 1\right) + X_{(1)} + \tfrac{1}{2} - \theta\right)^2\right]$$

$$= \alpha^2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] + 2\alpha E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$+ E_\theta\left[\left(X_{(1)} + \tfrac{1}{2} - \theta\right)^2\right].$$

$$\frac{\partial}{\partial\alpha} M_{\delta_\alpha}(\theta) = 2\alpha E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] + 2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$\frac{\partial^2}{\partial\alpha^2} M_{\delta_\alpha}(\theta) = 2 E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_\alpha}(\theta)$ is minimized at

$$\alpha = \frac{E_\theta\left[\left(1 - X_{(n)} + X_{(1)}\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]}{E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right]}$$

$$f_{X_{(1)}, X_{(n)}}(x, y) = n(n-1)(y-x)^{n-2}, \quad \theta - \tfrac{1}{2} < x < y < \theta + \tfrac{1}{2}.$$

$$E_\theta\left[\left(1 - X_{(n)} + X_{(1)}\right)\left(X_{(1)} + \tfrac{1}{2} - \theta\right)\right]$$

$$= \iint_{\theta - \frac{1}{2} < x < y < \theta + \frac{1}{2}} (1 - y + x)\left(x + \tfrac{1}{2} - \theta\right) n(n-1)(y-x)^{n-2}\, dy\, dx$$

$$= n(n-1) \iint_{0 < s < t < 1} s\,(1-t)^{n-2}\, ds\, dt \qquad \left(\begin{array}{l} x + \frac{1}{2} - \theta = s \\ 1 - y + x = t \end{array}\right)$$

$$= \frac{n(n-1)}{2} B(4, n-1)$$

$$E_\theta\left[\left(X_{(n)} - X_{(1)} - 1\right)^2\right] = n(n-1) \iint_{0 < s < t < 1} t^2 (1-t)^{n-2}\, dt = n(n-1) B(4, n-1)$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_\alpha}(\theta)$ is minimised at

$$\alpha = \frac{1}{2} = \alpha_0 \text{ say}$$

$\Rightarrow$ Among the estimators in the class $\mathcal{D}$, $\delta_{\alpha_0}(X) = \frac{X_{(n)} + X_{(1)}}{2}$ has the smallest m.s.e at each parametric point.

Problem 13 $\quad T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$

$$E(T^k) = \frac{\Gamma(n+k)\,\theta^{k+n}}{\Gamma(n)\,\theta^n} = \frac{\Gamma(n+k)}{\Gamma(n)}\theta^k, \quad k>0.$$

Fix $\theta \in \Theta$. Then

$$M_{\delta_c}(\theta) = E_\theta[(\delta_c(\underline{X}) - \theta^r)^2] = E_\theta[(c\bar{X}^r - \theta^r)^2]$$

$$= c^2 E_\theta(\bar{X}^{2r}) - 2c\theta^r E_\theta(\bar{X}^r) + \theta^{2r}.$$

$$\frac{\partial}{\partial c} M_{\delta_c}(\theta) = 2c\,E_\theta(\bar{X}^{2r}) - 2\theta^r E_\theta(\bar{X}^r)$$

$$\frac{\partial^2}{\partial c^2} M_{\delta_c}(\theta) = 2E_\theta(\bar{X}^{2r}) > 0.$$

Thus, for fixed $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at

$$c = \frac{\theta^r E_\theta(\bar{X}^r)}{E_\theta(\bar{X}^{2r})} = n^r \theta^r \frac{E_\theta(T^r)}{E_\theta(T^{2r})}$$

$$= n^r \theta^r \frac{\frac{\Gamma(n+r)}{\Gamma(n)}\theta^r}{\frac{\Gamma(n+2r)}{\Gamma(n)}\theta^{2r}}$$

$$= n^r \frac{\Gamma(n+r)}{\Gamma(n+2r)} = \frac{n^r}{(n+2r-1)(n+2r-2)\cdots(n+r)} = c_0, \text{ say}$$

$\longrightarrow$ does not depend on θ

Thus, for every $\theta \in \Theta$, $M_{\delta_c}(\theta)$ is minimized at $c = c_0 = n^r \frac{\Gamma(n+r)}{\Gamma(n+2r)}$.

$\Rightarrow$ Among the estimators in the class $\mathcal{D}$, the estimator $\delta_{c_0}(\underline{X})$

$= \frac{n^r\,\Gamma(n+r)}{\Gamma(n+2r)}\left(\sum_{i=1}^{n} X_i\right)^r = n^{2r}\frac{\Gamma(n+r)}{\Gamma(n+2r)}\bar{X}^r$ has the smallest m.s.e

at each parametric point.

$$\frac{n^{2r}\,\Gamma(n+r)}{\Gamma(n+2r)} = \frac{n^{2r}}{(n+2r-1)(n+2r-2)\cdots(n+r)} = \frac{1}{[1+\frac{2r-1}{n}][1+\frac{2r-2}{n}]\cdots[1+\frac{r}{n}]}$$

$\longrightarrow 1$ as $n \to \infty$.

Also, by W.L.L.N $\bar{X}_n \xrightarrow{P} E(X_1) = \theta \Rightarrow \bar{X}_n^r \xrightarrow{P} \theta^r$. Thus

$\delta_{c_0}(\underline{X}) \xrightarrow{P} \theta^r$. $\Rightarrow \delta_{c_0}(\underline{X})$ is consistent for estimating θ.