MSO 201a: Probability and Statistics
2019-20-II Semester
Assignment-V
Instructor: Neeraj Misra

1. For $\mu \in \mathbb{R}$ and $\lambda > 0$, let $X_{\mu,\lambda}$ be a random variable having the p.d.f.

$$f_{\mu,\sigma}(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}, & \text{if } x \geq \mu \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find $C_r(\mu, \lambda) = E((X-\mu)^r), r \in \{1, 2, \ldots\}$ and $\mu_r'(\mu, \lambda) = E(X_{\mu,\lambda}^r), r \in \{1, 2\}$;

(b) For $p \in (0, 1)$, find the p-th quantile $\xi_p \equiv \xi_p(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$ ($F_{\mu,\lambda}(\xi_p) = p$, where $F_{\mu,\lambda}$ is the distribution function of $X_{\mu,\lambda}$);

(c) Find the lower quartile $q_1(\mu, \lambda)$, the median $m(\mu, \lambda)$ and the upper quartile $q_3(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

(d) Find the mode $m_0(\mu, \lambda)$ of the distribution of $X_{\mu,\sigma}$;

(e) Find the standard deviation $\sigma(\mu, \lambda)$, the mean deviation about median $\mathrm{MD}(m(\mu, \lambda))$, the inter-quartile range $\mathrm{IQR}(\mu, \lambda)$, the quartile deviation (or semi-inter-quartile range) $\mathrm{QD}(\mu, \lambda)$, the coefficient of quartile deviation $\mathrm{CQD}(\mu, \lambda)$ and the coefficient of variation $\mathrm{CV}(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

(f) Find the coefficient of skewness $\beta_1(\mu, \lambda)$ and the Yule coefficient of skewness $\beta_2(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

(g) Find the excess kurtosis $\gamma_2(\mu, \lambda)$ of the distribution of $X_{\mu,\lambda}$;

(h) Based on values of measures of skewness and the kurtosis of the distribution of $X_{\mu,\lambda}$, comment on the shape of $f_{\mu,\sigma}$.

2. Let X be a random variable with p.d.f. $f_X(x)$ that is symmetric about μ ($\in \mathbb{R}$), i.e., $f_X(x+\mu) = f_X(\mu - x)$, $\forall\ x \in (-\infty, \infty)$.

(a) If q_1, m and q_3 are respectively the lower quartile, the median and the upper quartile of the distribution of X then show that $\mu = m = \frac{q_1+q_3}{2}$;

(b) If $E(X)$ is finite then show that $E(X) = \mu = m = \frac{q_1+q_3}{2}$.

3. A fair dice is rolled six times independently. Find the probability that on two occasions we get an upper face with 2 or 3 dots.

4. Let n (≥ 2) and $r \in \{1, 2, , n-1\}$ be fixed integers and let $p \in (0, 1)$ be a fixed real number. Using probabilistic arguments show that

$$\sum_{j=r}^{n} \binom{n}{j} p^j (1-p)^{n-j} - \sum_{j=r}^{n-1} \binom{n-1}{j} p^j (1-p)^{n-1-j} = \binom{n-1}{r} p^r (1-p)^{n-r}.$$

5. Let $X \sim \text{Bin}(n, p)$, where n is a positive integer and $p \in (0, 1)$. Find the mode of the distribution of X.

6. Eighteen balls are placed at random in seven boxes that are labeled $B_1, \ldots, B_7$. Find the probability that boxes with labels B_1, B_2 and B_3 all together contain six balls.

7. Let T be a discrete type random variable with support $S_T = \{0, 1, 2, \ldots\}$. Show that T has the lack of memory property if, and only if, $T \sim \text{Ge}(p)$, for some $p \in (0, 1)$.

8. A person repeatedly rolls a fair dice independently until an upper face with two or three dots is observed twice. Find the probability that the person would require eights rolls to achieve this.

9. (a) Consider a sequence of independent Bernoulli trials with probability of success in each trial being $p \in (0, 1)$. Let Z denote the number of trials required to get the r-th success, where r is a given positive integer. Let $X = Z - r$. Find the marginal probability distributions of X and Z. For $r = 1$, show that $P(Z > m+n) = P(Z > m)P(Z > n)$, $m, n \in \{0, 1, \ldots\}$ (or equivalently $P(Z > m + n|Z > m) = P(Z > n)$, $m, n \in \{0, 1, \ldots\}$; this property is also known as the lack of memory property).

   (b) Two teams (say Team A and Team B) play a series of games until one team wins 5 games. If the probability of Team A (Team B) winning any game is 0.7 (0.3), find the probability that the series will end in 8 games.

10. **(Relation between binomial and negative binomial probabilities)** Let n be a positive integer, $r \in \{1, 2, \ldots, n\}$ and let $p \in (0, 1)$. Using probabilistic arguments and also otherwise show that

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = p^r \sum_{k=0}^{n-r} \cdot \binom{r+k-1}{k} (1-p)^k,$$

    i.e., for $r \in \{1, 2, \ldots, n\}$, $P(\text{Bin}(n, p) \geq r) = P(\text{NB}(r, p) \leq n - r)$.

11. A mathematician carries at all times two match boxes, one in his left pocket and one in his right pocket. To begin with each match box contains n matches. Each

time the mathematician needs a match he is equally likely to take it from either pocket. Consider the moment when the mathematician for the first time discovers that one of the match boxes is empty. Find the probability that at that moment the other box contains exactly k matches, where $k \in \{0, 1, \dots, n\}$.

12. Consider a population comprising of N (≥ 2) units out of which a ($\in \{1, 2, \dots, N-1\}$) are labeled as S (success) and $N - a$ are labeled as F (failure). A sample of size n ($\in \{1, 2, \dots, N-1\}$) is drawn from this population, drawing one unit at a time and without replacing it back into the population (i.e., sampling is without replacement). Let A_i ($i = 1, 2, \dots, n$) denote the probability of obtaining success (S) in the i-th trial. Show that $P(A_i) = \frac{a}{N}$, $i = 1, \dots, n$ (i.e., probability of obtaining S remains the same in each trial). **Hint:** Use induction by first showing that $P(A_1) = P(A_2) = \frac{a}{N}$.

13. (a) (**Binomial approximation to hypergeometric distribution**) Show that a hypergeometric distributed can be approximated by a $\mathrm{Bin}(n, p)$ distribution provided N is large ($N \to \infty$), a is large ($a \to \infty$) and $a/N \to p$, as $N \to \infty$;

    (b) Out of 120 applicants for a job, only 80 are qualified for the job. Out of 120 applicants, five are selected at random for the interview. Find the probability that at least two selected applicants will be qualified for the job. Using Problem 13 (a) find an approximate value of this probability.

14. Urn U_i ($i = 1, 2$) contains N_i balls out of which r_i are red and $N_i - r_i$ are black. A sample of n ($1 \leq n \leq N_1$) balls is chosen at random (without replacement) from urn U_1 and all the balls in the selected sample are transferred to urn U_2. After the transfer two balls are drawn at random from the urn U_2. Find the probability that both the balls drawn from urn U_2 are red.

15. Let $X \sim \mathrm{Po}(\lambda)$, where $\lambda > 0$. Find the mode of the distribution of X.

16. (a) **(Poisson approximation to negative binomial distribution)** Show that $\lim_{r\to\infty} \binom{r+k-1}{k} p_r^r (1-p_r)^k = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, \dots,$ provided $\lim_{r\to\infty} p_r = 1$ and $\lim_{r\to\infty}(r(1-p_r)) = \lambda > 0$.

    (b) Consider a person who plays a series of 2500 games independently. If the probability of person winning any game is 0.002, find the probability that the person will win at least two games.

17. (a) If $X \sim \mathrm{Po}(\lambda)$, find $E((2+X)^{-1})$;

    (b) If $X \sim \mathrm{Ge}(p)$ and r is a positive integer, find $E(\min(X, r))$.

18. A person has to open a lock whose key is lost among a set of N keys. Assume that out of these N keys only one can open the lock. To open the lock the person tries keys one by one by choosing, at each attempt, one of the keys at random from the unattempted keys. The unsuccessful keys are not considered for future attempts. Let Y denote the number of attempts the person will have to make to open the lock. Show that $Y \sim U(\{1, 2, \ldots, N\})$ and hence find the mean and the variance of the r.v. Y.

19. Let $a > 0$ be a real constant. A point X is chosen at random on the interval $(0, a)$ (i.e., $X \sim U(0, a)$).
(a) If Y denotes the area of equilateral triangle having sides of length X, find the mean and variance of Y;
(b) If the point X divides the interval $(0, a)$ into subintervals $I_1 = (0, X)$ and $I_2 = [X, a)$, find the probability that the larger of these two subintervals is at least the double the size of the smaller subinterval.

20. Using a random observation $U \sim U(0, 1)$, describe a method to generate a random observation X from the distribution having
(a) probability density function

$$f(x) = \frac{e^{-|x|}}{2}, \quad -\infty < x < \infty;$$

(b) probability mass function

$$g(x) = \begin{cases} \binom{n}{x}\theta^x(1-\theta)^{n-x}, & \text{if } x \in \{0, 1, \ldots, n\} \\ 0, & \text{otherwise} \end{cases},$$

where $n \in \mathbb{N}$ and $\theta \in (0, 1)$ are real constants.

21. Let $X \sim U(0, \theta)$, where θ is a positive integer. Let $Y = X - [X]$, where $[x]$ is the largest integer $\leq x$. Show that $Y \sim U(0, 1)$.

22. Let $U \sim U(0, 1)$ and let X be the root of the equation $3t^2 - 2t^3 - U = 0$. Show that X has p.d.f. $f(x) = 6x(1 - x)$, if $0 \leq x \leq 1$, $= 0$, otherwise.

23. **(Relation between gamma and Poisson probabilities)** For $t > 0, \theta > 0$ and $n \in \{1, 2, \ldots\}$, using integration by parts, show that

$$\frac{1}{\theta^n \Gamma(n)} \int_t^\infty e^{-\frac{x}{\theta}} x^{n-1} dx = \sum_{j=0}^{n-1} \frac{e^{-\frac{t}{\theta}} (\frac{t}{\theta})^j}{j!},$$

i.e., $P(G(n, \theta) > t) = P(\mathrm{Po}(\frac{t}{\theta}) \leq n - 1)$.

24. Let Y be a random variable of continuous type with $F_Y(0) = 0$, where $F_Y(\cdot)$ is the distribution function of Y. Show that Y has the lack of memory property if, and only if, $Y \sim \text{Exp}(\theta)$, for some $\theta > 0$.

25. **(Relation between binomial and beta probabilities)** Let n be a positive integer, $k \in \{0, 1, \ldots, n\}$ and let $p \in (0, 1)$. If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Be}(k, n-k+1)$, show that $P(\{X \geq k\}) = P(\{Y \leq p\})$.

26. Let $\underline{X} = (X_1, X_2)'$ have the joint p.d.f.

$$f_{\underline{X}}(x_1, x_2) = \phi(x_1)\phi(x_2)[1+\alpha(2\Phi(x_1)-1)(2\Phi(x_2)-1)], \ -\infty < x_i < \infty, \ i = 1, 2, \ |\alpha| \leq 1.$$

(a) Verify that $f_{\underline{X}}(x_1, x_2)$ is a p.d.f.;

(b) Find the marginal p.d.f.s of X_1 and X_2;

(c) Is (X_1, X_2) jointly normal?

27. Let $\underline{X} = (X_1, X_2)' \sim N_2(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$ and, for constants a_1, a_2, a_3 and a_4 ($a_i \neq 0$, $i = 1, 2, 3, 4$, $a_1a_4 \neq a_2a_3$), let $Y = a_1X_1 + a_2X_2$ and $Z = a_3X_1 + a_4X_2$.

(a) Find the joint p.d.f. of (Y, Z);

(b) Find the marginal p.d.f.s. of Y and Z.

28. (a) Let $(X, Y)' \sim N_2(5, 8, 16, 9, 0.6)$. Find $P(\{5 < Y < 11\}|\{X = 2\})$, $P(\{4 < X < 6\})$ and $P(\{7 < Y < 9\})$;

(b) Let $(X, Y)' \sim N_2(5, 10, 1, 25, \rho)$, where $\rho > 0$. If $P(\{4 < Y < 16\}|\{X = 5\}) = 0.954$, determine ρ.

29. Let X and Y be independent and identically distributed $N(0, \sigma^2)$ r.v.s.

(a) Find the joint p.d.f. of (U, V), where $U = aX + bY$ and $V = bX - aY$ ($a \neq 0, b \neq 0$);

(b) Show that U and V are independent;

(c) Show that $\frac{X+Y}{\sqrt{2}}$ and $\frac{X-Y}{\sqrt{2}}$ are independent $N(0, \sigma^2)$ random variables.

30. Let $\underline{X} = (X_1, X_2)' \sim N_2(0, 0, 1, 1, \rho)$.

(a) Find the m.g.f. of $Y = X_1X_2$;

(b) Using (a), find $E(X_1^2X_2^2)$;

(c) Using conditional expectation, find $E(X_1^2X_2^2)$.

31. Let $\underline{X} = (X_1, X_2)'$ have the joint p.d.f.

$$f(x,y) = \begin{cases} \frac{1}{\pi} e^{-\frac{1}{2}(x^2+y^2)}, & \text{if } xy > 0 \\ 0, & \text{otherwise} \end{cases}.$$

Show that marginals of $f(\cdot,\cdot)$ are each $N(0,1)$ but $f(\cdot,\cdot)$ is not the p.d.f. of a bivariate normal distribution.

32. Let $f_r(\cdot,\cdot)$, $-1 < r < 1$, denote the pdf of $N_2(0,0,1,1,r)$ and, for a fixed $\rho \in (-1,1)$, let the random variable (X,Y) have the joint p.d.f.

$$g_\rho(x,y) = \frac{1}{2}[f_\rho(x,y) + f_{-\rho}(x,y)].$$

Show that X and Y are normally distributed but the distribution of (X,Y) is not bivariate normal.

33. Consider the random vector (X,Y) as defined in Problem 32.

(a) Find $\mathrm{Corr}(X,Y)$;

(b) Are X and Y independent?

34. (a) Suppose that $\underline{X} = (X_1, X_2, X_3, X_4) \sim \mathrm{Mult}(30, \theta_1, \theta_2, \theta_3, \theta_4)$. Find the conditional probability mass function of (X_1, X_2, X_3, X_4) given that $X_5 = 2$, where $X_5 = 30 - \sum_{i=1}^{4} X_i$.
(b) Consider 7 independent casts of a pair of fair dice. Find the probability that once we get a sum of 12 and twice we get a sum of 8.

35. (a) Let $X \sim F_{n_1,n_2}$. Show that $Y = \frac{n_2}{n_2+n_1X} \sim \mathrm{Be}(\frac{n_1}{2}, \frac{n_2}{2})$;
(b) Let Z_1 and Z_2 be i.i.d. $N(0,1)$ random variables. Show that $\frac{Z_1}{|Z_2|} \sim t_1$ and $\frac{Z_1}{Z_2} \sim t_1$ (Cauchy distribution);
(c) Let X_1 and X_2 be i.i.d. $\mathrm{Exp}(\theta)$ random variables, where $\theta > 0$. Show that $Z = \frac{X_1}{X_2}$ has a F-distribution;
(d) Let X_1, X_2 and X_3 be independent random variables with $X_i \sim \chi^2_{n_i}, i = 1,2,3$. Show that $\frac{X_1/n_1}{X_2/n_2}$ and $\frac{X_3/n_3}{(X_1+X_2)/(n_1+n_2)}$ are independent F-variables.

36. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{Exp}(\theta)$ distribution, where $\theta > 0$. Let $X_{1:n} \le \cdots \le X_{n:n}$ denote the order statistics of $X_1, \ldots, X_n$. Define $Z_1 = nX_{1:n},\ Z_i = (n-i+1)(X_{i:n} - X_{i-1:n}),\ i = 2, \ldots, n$. Show that $Z_1, \ldots, Z_n$ are independent and identically distributed $\mathrm{Exp}(\theta)$ random variables. Hence find the mean and variance of $X_{r:n},\ r = 1, \ldots, n$. Also, for $1 \le r < s \le n$, find $\mathrm{Cov}(X_{r:n}, X_{s:n})$.

MSO 201A / ESO 209: Probability and Statistics

2014 - 2015 - II Semester

Assignment - V

Solutions

Problem No. 1 (a) For $r \in \{1, 2, \ldots\}$

$$\boxed{C_r(\mu,\lambda) = E\left((X-\mu)^r\right) = \int_\mu^\infty (x-\mu)^r \frac{1}{\lambda} e^{-\frac{(x-\mu)}{\lambda}}\,dx = \lambda^r \int_0^\infty t^r e^{-t}\,dt = \lambda^r\, r!\,.}$$

$E(X-\mu) = \lambda \Rightarrow E(X) = \mu+\lambda$, i.e. $\boxed{\mu_1'(\mu,\lambda) = \mu+\lambda}$

$E((X-\mu)^2) = 2\lambda^2 \Rightarrow E(X^2) - 2\mu E(X) + \mu^2 = 2\lambda^2 \Rightarrow \mu_2'(\mu,\lambda) = E(X^2) = 2\lambda^2 + 2\mu\lambda + \mu^2$

$\Rightarrow \boxed{\mu_2'(\mu,\lambda) = 2\lambda^2 + 2\mu\lambda + \mu^2}$

(b) $F_{\mu,\lambda}(\xi_p) = p \Rightarrow \int_\mu^{\xi_p} \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}\,dx = p \Rightarrow 1 - e^{-\frac{\xi_p-\mu}{\lambda}} = p \Rightarrow \boxed{\xi_p = \mu - \lambda \ln(1-p)}$

(c) $\boxed{q_1(\mu,\lambda) = \xi_{1/4} = \mu - \lambda \ln \frac{3}{4}}$, $\boxed{m(\mu,\lambda) = \xi_{1/2} = \mu - \lambda \ln\left(\frac{1}{2}\right)}$

$\boxed{q_3(\mu,\lambda) = \xi_{3/4} = \mu - \lambda \ln \frac{1}{4}}$. (lower quartile, median and upper quartile)

(d) Clearly $f_{\mu,\sigma}(x) \downarrow$ on $[\mu, \infty) \Rightarrow \sup\{f_{\mu,\sigma}(x) : x \in \mathbb{R}\} = f_{\mu,\sigma}(\mu) = \frac{1}{\lambda}$

$\Rightarrow \boxed{m_o(\mu,\lambda) = \mu.}$ (Mode)

(e) $\sigma(\mu,\lambda) = \sqrt{\mu_2} = \boxed{\sqrt{2}\,\lambda}$ (from (a)) (Standard deviation)

For $d>0$: $MD(m(\mu,\lambda)) = E(|X-\mu+\lambda \ln \frac{1}{2}|) = E(|X-\mu-\lambda \ln 2|)$

$E(|X-\mu-d\lambda|) = \int_\mu^\infty |x-\mu-d\lambda| \frac{1}{\lambda} e^{-\frac{x-\mu}{\lambda}}\,dx = \lambda \int_0^\infty |z-d| e^{-z}\,dz$

$= \lambda(d-1+2e^{-d}) \Rightarrow \boxed{MD(m(\mu,\lambda)) = \lambda \ln 2}$ (Mean deviation about median)

$IQR(\mu,\lambda) = q_3(\mu,\lambda) - q_1(\mu,\lambda) = \boxed{\lambda \ln 3}$ (Inter quartile range)

$CQD(\mu,\lambda) = \dfrac{q_3 - q_1}{q_3 + q_1} = \boxed{\dfrac{\lambda \ln 3}{2\mu - \lambda \ln \frac{3}{16}}}$ (Coefficient of quartile deviation)

$CV(\mu,\lambda) = \dfrac{\sigma(\mu,\lambda)}{\mu_1'(\mu,\lambda)} = \boxed{\dfrac{\sqrt{2}\,\lambda}{\mu+\lambda}}$ (Coefficient of Variation)

(f) $\mu_3 = E((X-\mu)^3) = 6\lambda^3 \Rightarrow \beta_1(\mu,\lambda) = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{36\lambda^6}{8\lambda^6} = \boxed{\dfrac{9}{2}}$ (Coeff. of Skewness)

$\beta_2(\mu,\lambda) = \dfrac{q_3 - 2m + q_1}{q_3 - q_1} = \boxed{\dfrac{\ln(4/3)}{\ln 3}}$ (Yule Coefficient of Skewness)

(g) $\mu_4 = E((X-\mu)^4) = 24\lambda^4$; $\gamma_1(\mu,\lambda) = \dfrac{\mu_4}{\mu_2^2} = \dfrac{24\lambda^4}{4\lambda^4} = \boxed{6}$ (Kurtosis)

$\gamma_2(\mu,\lambda) = \gamma_1(\mu,\lambda) - 3 = \boxed{3}$ (Excess Kurtosis)

(h) $\beta_1(\mu,\lambda) > 0$ and $\beta_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of $X_{\mu,\sigma}$ is +vely skewed

$\Rightarrow f_{\mu,\sigma}$ has longer tails on the rhs

$\gamma_2(\mu,\lambda) > 0 \Rightarrow$ Distribution of $X_{\mu,\sigma}$ is leptokurtic

$\Rightarrow f_{\mu,\sigma}$ is more peaked around μ than normal distribution.

Problem No. 2 Note that $f_X(x+\mu) = f_X(\mu - x),\ \forall x \in \mathbb{R} \Rightarrow X-\mu \overset{d}{=} \mu - X$

(a) $X-\mu \overset{d}{=} \mu - X \Rightarrow P(X-\mu \le 0) = P(\mu - X \le 0) \Rightarrow F_X(\mu) + F_X(\mu-) = 1$

$\Rightarrow F_X(\mu-) \le \frac{1}{2} \le F_X(\mu)$ (since $F_X(\mu-) \le F_X(\mu)$)

$\Rightarrow m_e = \mu$

Also

$P(X < q_3) \le \frac{3}{4} \le P(X \le q_3) \Rightarrow P(X-\mu < q_3-\mu) \le \frac{3}{4} \le P(X-\mu \le q_3-\mu)$

$\Rightarrow P(\mu - X < q_3 - \mu) \le \frac{3}{4} \le P(\mu - X \le q_3 - \mu)$ (since $X-\mu \overset{d}{=} \mu - X$)

$\Rightarrow 1 - F_X(2\mu - q_3) \le \frac{3}{4} \le 1 - F_X((2\mu - q_3)-)$

$\Rightarrow F_X((2\mu-q_3)-) \le \frac{1}{4} \le F_X(2\mu - q_3) \Rightarrow q_1 = 2\mu - q_3$

$\Rightarrow \mu = m_e = \dfrac{q_1+q_3}{2}$

(b) $X-\mu \overset{d}{=} \mu - X \Rightarrow E(X-\mu) = E(\mu - X) \Rightarrow E(X) = \mu = \dfrac{q_1+q_3}{2}$

Problem No. 3 Consider getting an upper face with 2 or 3 dots as success (S). Then $P(S) = \frac{1}{3}$

$X =$ # of successes in 6 Bernoulli trials $\sim \text{Bin}(6, \frac{1}{3})$

Required probability $= P(X=2) = \binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4 = \dfrac{80}{243}$.

Problem No. 4 Consider a sequence of Bernoulli trials with success probability in each trial as p. Define

X_n = # of successes in first n trials
X_{n-1} = # of successes in first $(n-1)$ trials

Then $X_n \sim Bin(n, p)$ and $X_{n-1} \sim Bin(n-1, p)$. Also

$X_n \overset{d}{=} X_{n-1} + Y_n$, where $Y_n \sim Bin(1, p)$ and X_{n-1} and Y_n are indep t.

Thus

$$
\begin{aligned}
P(X_n \geq r) &= P(X_{n-1}+Y_n \geq r) = P(X_{n-1}+Y_n \geq r, Y_n=0) + P(X_{n-1}+Y_n \geq r, Y_n=1) \\
&= P(X_{n-1} \geq r, Y_n=0) + P(X_{n-1} \geq r-1, Y_n=1) \\
&= P(X_{n-1} \geq r)\, P(Y_n=0) + P(X_{n-1} \geq r-1)\, P(Y_n=1) \\
&= P(X_{n-1} \geq r)(1-p) + P(X_{n-1} \geq r-1)\, p \\
&= P(X_{n-1} \geq r) + p\,\big(P(X_{n-1} \geq r-1) - P(X_{n-1} \geq r) \big) \\
&= P(X_{n-1} \geq r) + p\, P(X_{n-1} = r-1) \\
&= P(X_{n-1} \geq r) + \left\{ \binom{n-1}{r-1} p^{r-1} (1-p)^{n-r} \right\} p
\end{aligned}
$$

Problem No. 5 $f_X(x) = P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$

$$f_X(x+1) > f_X(x) \iff \frac{\lfloor n}{\lfloor x+1 \, \lfloor n-x-1} p^{x+1} (1-p)^{n-x-1} > \frac{\lfloor n}{\lfloor x \, \lfloor n-x} p^x (1-p)^{n-x}$$

$$\iff x < (n+1)p - 1 \quad \ldots\ldots (A)$$

$$f_X(x+1) < f_X(x) \iff x > (n+1)p-1 \quad \ldots\ldots (B)$$

Case I $(n+1)p$ is an integer

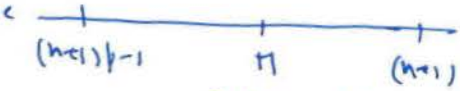
We have from (A) and (B) (along with the fact that $P(X=(n+1)p-1) = P(X=(n+1)p)$)

$$P(X=0) < P(X=1) < \cdots < P(X=(n+1)p-1) = P(X=(n+1)p) > P(X=(n+1)p+1) > \cdots > P(X=n)$$

$\Rightarrow$ We have two modes $(n+1)p-1$ and $(n+1)p$.

Case II $(n+1)p$ is not an integer.

Let $M = \lceil (n+1)p \rceil$. Then we have from (A) and (B)



$$P(X=0) < P(X=1) < \cdots < P(X=M) > P(X=M+1) > \cdots > P(X=n)$$

$= \lceil (n+1)p \rceil$

$\Rightarrow M_0 = \lceil (n+1)p \rceil$ is the mode.

Problem No. 6 Denote the event of a ball going into any of the boxes B_1 B_2 and B_3 as success, so that $P(S) = 3/7$

X = # of balls in boxes B_1, B_2 and B_3 taken together $\sim Bin(18, \frac{3}{7})$.

Required probability $= P(X=6) = \binom{18}{6}\left(\frac{3}{7}\right)^6\left(\frac{4}{7}\right)^{12}$.

Problem No. 7 Suppose that $T \sim Ge(p)$. Then $P(T \geq j) = (1-p)^j$, $j = 0, 1, 2, \ldots$

$$P(T \geq j+k) = p^{j+k} = p^j p^k = P(T \geq j)\, P(T \geq k),\ \forall\, j, k \in S_T.$$

Conversely suppose that T has LOM property i.e.

$$P(T \geq j+k) = P(T \geq j)\, P(T \geq k) \quad \forall\, j, k \in S_T$$

$$\Rightarrow P(T \geq j+1) = P(T \geq j)\, P(T \geq 1)$$
$$= P(T \geq j-1)\, (P(T \geq 1))^2$$
$$\vdots$$
$$= P(T \geq 0)\, (P(T \geq 1))^{j+1}$$
$$= (1-p)^{j+1},$$

where $p = P(T=0) \in (0, 1)$.

Thus, for $k \in \{0, 1, 2, \ldots\}$

$$P(T=k) = P(T \geq k) - P(T \geq k+1)$$
$$= p^k - p^{k+1} = p(1-p)^k$$

$\Rightarrow T \sim Ge(p)$.

Problem No. 8 In each trial, label the outcome of observing an upper face with two or three dots as success and observing any other outcomes as failure. Then $P(S) = \frac{1}{3}$.

X = # of failures preceding the 2^{nd} success $\sim NB(2, \frac{1}{3})$

Required prob $= P(X+2 = 8)$

$$= P(X=6) = \binom{7}{1}\left(\frac{1}{3}\right)^2\left(1-\frac{1}{3}\right)^6 = \frac{448}{6561}.$$

$\boxed{\text{Problem No. 9}}$ (a) $X = Z - r$ denotes the number of failures preceding the r-th success $\sim NB(r, p)$.

$$f_Z(z) = P(Z = z) = P(X = z - r) = \begin{cases} \binom{z-1}{r-1} p^r (1-p)^{z-r}, & z = r, r+1, \ldots \\ 0, & \text{otherwise} \end{cases}$$

For $r = 1$, $X \sim NB(1, p)$. Thus X has LOM property and

$$P(Z > m+n) = P(X > m+n-1) = P(X \geq m+n) = P(X \geq m)\, P(X \geq n)$$
$$= P(Z \geq m+1)\, P(Z \geq n+1)$$
$$= P(Z > m)\, P(Z > n).$$

(b) X_1 = # of games team A will have to play to secure 5^{th} win

X_2 = # of games team B will have to play to secure 5^{th} win

Required prob $= p = P(X_1 = 8) + P(X_2 = 8)$

$$= P(Y_1 = 3) + P(Y_2 = 3),$$

where $Y_1 = X_1 - 5$ ($Y_2 = X_2 - 5$) is the number of failures preceding the 5^{th} success for Team A (Team B); here success for a team is win. Then $Y_1 \sim NB(5, 0.7)$ and $Y_2 \sim NB(5, 0.3)$.

$$P(X_1 = 8) = P(Y_1 = 3) = \binom{7}{4} (0.7)^5 (1 - 0.7)^3 \approx .1588$$

$$P(X_2 = 8) = P(Y_2 = 3) = \binom{7}{4} (0.3)^5 (1 - 0.3)^3 \approx 0.0292$$

Required prob $= P(X_1 = 8) + P(X_2 = 8) = 0.188$ (approx.).

$\boxed{\text{Problem No. 10}}$ Consider a sequence of independent Bernoulli trials with probability of success in each trial being p. Then

$$\sum_{k=r}^{n} \binom{n}{k} p^k (1-p)^{n-k} = P(\text{at least } r \text{ successes in } n \text{ Bernoulli trials})$$

$$= P\Big(\sum_{k=0}^{n-r} \{r\text{th success in } (r+k)\text{-th trial}\}\Big)$$

$$= \sum_{k=0}^{n-r} P(r\text{th success in } (r+k)\text{-th trial})$$

$$= \sum_{k=0}^{n-r} \underbrace{\binom{r+k-1}{r-1} p^{r-1} (1-p)^k}_{(r-1) \text{ successes in first } (r+k-1) \text{ trials}} \underbrace{p}_{\text{success in } (r+k)\text{-th trial}} = \text{RHS}$$

$\boxed{5/5}$

Direct Method: Let $q = 1-p$. We need to show that

$$\sum_{k=r}^{n}\binom{n}{k}(1-q)^{k-r}q^{n-k} = \sum_{k=0}^{n-r}\binom{r+k-1}{k}q^k$$

LHS = polynomial in q of degree $(n-r)$ = $\sum_{k=0}^{n-r} c_k q^k$

It is enough to show that $c_k = \binom{r+k-1}{k}$, $k = 0, 1, 2, \ldots, n-r$.

For $k \in \{0, 1, 2, \ldots, n-r\}$

$$c_k = \text{coefficient of } q^k \text{ in } \sum_{j=r}^{n}\binom{n}{j}(1-q)^{j-r}q^{n-j} = \sum_{j=0}^{n-r}\underbrace{\binom{n}{n-j}}_{=\binom{n}{j}}(1-q)^{n-j-r}q^j$$

$$= \text{Coefficient of } q^k \text{ in } \sum_{j=0}^{k}\binom{n}{j}(-1)^{k-j}\binom{n-j-r}{k-j}$$

$$\hookrightarrow = \sum_{j=0}^{k}\binom{n}{k-j}(-1)^j\binom{n-k-r+j}{j}$$

But

$$(-1)^j\binom{n-k-r+j}{j} = (-1)^j\frac{(n-k-r+j)(n-k-r+j-1)\cdots(n-k-r+1)}{j!}$$

$$= \frac{(-n+k+r-j)(-n+k+r-j+1)\cdots(-n+k+r-1)}{j!}$$

$$= \binom{-n+k+r-j}{j}$$

$$\Rightarrow c_k = \sum_{j=0}^{k}\binom{n}{k-j}\binom{-n+k+r-j}{j} = \binom{k+r-1}{k}.$$

$\boxed{\text{Problem No. 11}}$ Let us call the event of choosing Box 1 as success and that of choosing Box 2 as failure. Then we have a sequence of independent Bernoulli trials with probability of success in each trial being p.

Required prob. = P(when box 1 is found empty, box 2 has k matches)

+ P(when box 2 is found empty, box 1 has k matches)

= P($(n-k)$ failures precede the $(n+1)$th success)

+ P($(n-k)$ successes precede the $(n+1)$th failure)

$$= \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^{n-k}\times\tfrac{1}{2} + \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^{n-k}\times\tfrac{1}{2}$$

$$= \binom{2n-k}{n}\left(\tfrac{1}{2}\right)^{2n-k} \qquad k = 0, 1, 2, \ldots, n.$$

$\boxed{\text{Problem No. 12}}$ Clearly $P(A_1) = \frac{a}{N}$ and

$$P(A_2) = P(A_1^c \cap A_2) + P(A_1 \cap A_2)$$
$$= P(A_1^c)\, P(A_2 | A_1^c) + P(A_1)\, P(A_2 | A_1)$$
$$= \frac{N-a}{N} \cdot \frac{a}{N-1} + \frac{a}{N} \cdot \frac{a-1}{N-1} = \frac{a}{N}$$

Now suppose that $P(A_m) = \frac{a}{N}$, for some $m \in \{1, 2, \dots, N-1\}$

Then

$$P(A_{m+1}) = \sum_{k=0}^{m} P(A_{m+1} \cap \{X_{a,m,N} = k\})$$
$$= \sum_{k=0}^{m} f_{a,m,N}(k)\; P(A_{m+1} | X_{a,m,N} = k)$$
$$= \sum_{k=0}^{m} f_{a,m,N}(k)\; \frac{a-k}{N-m},$$

where $f_{a,m,N}$ is the p.m.f. of $X_{a,m,N}$.

Thus

$$P(A_{m+1}) = \sum_{k=\max\{0,\, m-N+a\}}^{\min\{m,\, a\}} \frac{\binom{a}{k}\binom{N-a}{m-k}}{\binom{N}{m}}\; \frac{a-k}{N-m}$$
$$= \frac{a}{N-m} - \frac{1}{N-m} E(X_{a,m,N})$$
$$= \frac{a}{N-m} - \frac{1}{N-m}\, m \frac{a}{N} = \frac{a}{N}.$$

Hence the result follows by induction.

$\boxed{\text{Problem No. 13}}$ (a) Let $N = a+b$. Then for $x \in \{0, 1, \dots, n\}$, $n \le a+b$ and $\max\{0, n-b\} \le x \le \min\{n, a\}$

$$f_X(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$
$$= \binom{n}{x} \frac{a(a-1)\cdots(a-x+1)\, b(b-1)\cdots(b-n+x+1)}{(a+b)(a+b-1)\cdots(a+b-n+1)}$$
$$= \binom{n}{x} \frac{\frac{a}{a+b}\left(\frac{a}{a+b} - \frac{1}{a+b}\right)\cdots\left(\frac{a}{a+b} - \frac{x-1}{a+b}\right) \frac{b}{a+b}\left(\frac{b}{a+b} - 1\right)\cdots\left(\frac{b}{a+b} - \frac{n-x-1}{a+b}\right)}{\left(1 - \frac{1}{a+b}\right)\cdots\left(1 - \frac{n-1}{a+b}\right)}$$
$$\longrightarrow \binom{n}{x} p^x (1-p)^{n-x} \quad \left(\text{since } \frac{a}{a+b} = \frac{a}{N} \to p,\ \frac{b}{a+b} = 1 - \frac{a}{a+b} \to 1-p\right)$$

(b) $N = 120$ is large; $a = 80$ is large; $n = 5$; $p = \frac{a}{N} = \frac{2}{3}$

X = # of applicants (out of 5 selected for interview) qualified for job

Using (a)

$$X \overset{\text{approx}}{\sim} \text{Bin}(5, p)$$

Required prob $= P(X \geq 2)$

$$= 1 - P(X=0) - P(X=1)$$

$$= 1 - \frac{\binom{40}{5}}{\binom{120}{5}} - \frac{\binom{80}{1}\binom{40}{4}}{\binom{120}{5}}$$

Approx. Prob $= P(X \geq 2) = 1 - P(X=0) - P(X=1)$

$$= 1 - (1-p)^n - \binom{n}{1} p (1-p)^{n-1} \quad (n=5,\ p=\tfrac{2}{3})$$

$$= 1 - \left(\tfrac{1}{3}\right)^5 - \tfrac{10}{3}\left(\tfrac{1}{3}\right)^4.$$

Problem No. 14 Let

$X =$ # of red balls among the n balls drawn from U_1

E: both the balls drawn from urn U_2 are red

Then $X \sim \text{Hyp}(r_1, n, N_1)$ and

Required prob. $= P(E)$

$$= \sum_{x=0}^{n} P(E \mid X=x)\, P(X=x)$$

$$= \sum_{x=\max\{0,\, n-N_1+r_1\}}^{\min\{n, r_1\}} P(E \mid X=x)\, P(X=x)$$

$$P(E \mid X=x) = \frac{\binom{r_2+x}{2}}{\binom{N_2+n}{2}} = \frac{r_2(r_2-1) + 2r_2 x + x(x-1)}{(N_2+n)(N_2+n-1)}$$

Thus

$$P(E) = \frac{[r_2(r_2-1) + 2r_2 E(X) + E(X(X-1))]}{(N_2+n)(N_2+n-1)}$$

$$= \frac{1}{(N_2+1)(N_2+n-1)}\left[r_2(r_2-1) + 2r_2 \frac{n r_1}{N_1} + n(n-1)\frac{r_1(r_1-1)}{N_1(N_1-1)}\right].$$

Problem No. 15 $f_X(x) = \dfrac{e^{-\lambda}\lambda^x}{\lfloor x}, \quad x = 0, 1, 2, \ldots$

$$f_X(x+1) > f_X(x) \iff x < \lambda - 1 \quad \ldots\ldots (A)$$

$$f_X(x+1) < f_X(x) \iff x > \lambda - 1 \quad \ldots\ldots (B)$$

Case I λ is an integer

Here

$$f_X(0) < f_X(1) < \dots < f_X(\lambda-2) < f_X(\lambda-1) = f_X(\lambda) > f_X(\lambda+1) > f_X(\lambda+2) > \dots$$

In this case there are two modes $\lambda-1$ and λ (if $\lambda > 1$) and One mode λ if $\lambda \in \{0,1\}$.

Case II λ is not an integer

Let $m = [\lambda]$

Here

$\lambda-1 \quad m \quad \lambda$

$$f_X(0) < f_X(1) < \dots < f_X(m-1) < f_X(m) > f_X(m+1) > f_X(m+2) > \dots$$

$$\Rightarrow \text{mode} = m = [\lambda].$$

Problem No. 16 (a) We will use the Stirling approximation

$$\lim_{n\to\infty} \frac{\lfloor n}{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}} = 1, \text{ i.e. } \lfloor n \approx \sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}} \text{ for large } n.$$

Then

$$\text{LHS} = \frac{\lfloor r+k-1}{\lfloor k \; \lfloor r-1} p_r^r (1-p_r)^k$$

$$\overset{r\to\infty}{\approx} \frac{\sqrt{2\pi}\, e^{-(r+k-1)} (r+k-1)^{r+k-\frac{1}{2}}}{\lfloor k \; \sqrt{2\pi}\, e^{-(r-1)} (r-1)^{r-\frac{1}{2}}} \left(1-\frac{\lambda}{r}\right)^r \left(\frac{\lambda}{r}\right)^k$$

$$= \frac{e^{-k}\lambda^k}{\lfloor k} \left(1-\frac{\lambda}{r}\right)^r \frac{\left(1+\frac{k-1}{r}\right)^{r+k-\frac{1}{2}}}{\left(1-\frac{1}{r}\right)^{r-\frac{1}{2}}}$$

$$\approx \frac{e^{-k}\lambda^k}{\lfloor k} e^{-\lambda} \frac{e^{k-1}}{e^{-1}} = \frac{e^{-\lambda}\lambda^k}{\lfloor k}, \quad k = 0, 1, 2, \dots$$

(b) Let us label winning of the game by a person as success and his/her losing the game as failure.

Then we have a sequence of $n = 2500$ independent Bernoulli trials with probability of success in each trial as $p = 0.002$.

X = # of successes in n Bernoulli trials $\sim \text{Bin}(2500, 0.002)$

$$\text{Required prob} = P(X \geq 2) = 1 - P(X=0) - P(X=1).$$

$$= 1 - \{(1-0.002)^{2500} + 2500 \times 0.002 \times (1-0.002)^{2499}\}$$

$n = 2500$ is large, $p = 0.002$ is small. $np = 5 = \lambda$ (say).

Thus $X \overset{\text{approx}}{\sim} Po(5)$, and

$$\text{Required prob} = P(X \geq 2)$$
$$\overset{\text{approx.}}{=} P(Y \geq 2) \qquad (Y \sim Po(5))$$
$$= 1 - [P(Y=0) + P(Y=1)]$$
$$= 1 - [e^{-\lambda} + \lambda e^{-\lambda}]$$
$$= 1 - 6e^{-5} = 0.9596$$

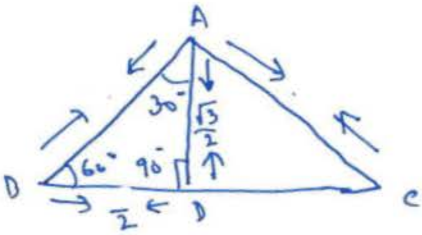
$\boxed{\text{Problem No. 17}}$ (a)
$$E\left(\frac{1}{2+X}\right) = \sum_{j=0}^{\infty} \frac{1}{2+j} \frac{e^{-\lambda}\lambda^j}{j!} = e^{-\lambda} \sum_{j=0}^{\infty} \frac{(j+1)\lambda^j}{(j+2)!}$$
$$= e^{-\lambda} \sum_{j=2}^{\infty} \frac{(j-1)\lambda^{j-2}}{j!} = \frac{e^{-\lambda}}{\lambda^2}\left[\sum_{j=0}^{\infty} \frac{(j-1)\lambda^j}{j!} + 1\right]$$
$$= \frac{1}{\lambda^2}\left[\sum_{j=0}^{\infty} (j-1) \frac{e^{-\lambda}\lambda^j}{j!} + e^{-\lambda}\right] = \frac{1}{\lambda^2}\left[E(X-1) + e^{-\lambda}\right]$$
$$= \frac{\lambda - 1 + e^{-\lambda}}{\lambda^2}$$

(b) $P(X=n) = p(1-p)^n, \ n = 0, 1, 2, \ldots$
$$E(\min(X, r)) = \sum_{n=0}^{\infty} \min(n, r)\, p(1-p)^n = p\left[\sum_{n=0}^{r} n(1-p)^n + rp \sum_{n=r+1}^{\infty} (1-p)^n\right]$$
$$= \frac{(1-p)\left[1-(1-p)^r\right]}{p} - r(1-p)^{r+1} + r(1-p)^{r+1}$$
$$= \frac{(1-p)\left[1-(1-p)^r\right]}{p}.$$

$\boxed{\text{Problem 18}}$ For $r \notin \{1, 2, \ldots, N\}$, $P(Y=r) = 0$. For $r \in \{1, 2, \ldots, N\}$
$$P(Y=r) = \frac{N-1}{N} \cdot \frac{N-2}{N-1} \cdots \frac{N-(r-1)}{N-(r-2)} \cdot \frac{1}{N-(r-1)} = \frac{1}{N}$$
$\Rightarrow Y \sim U(\{1, 2, \ldots, N\})$

$\Rightarrow E(Y) = \frac{N+1}{2}$ and $Var(Y) = \frac{N^2-1}{12}$.

$\boxed{\text{Problem 19}}$ (a) In the equilateral ΔABC



$AB = BC = X$, $BD = \frac{X}{2}$ and $AD = \frac{\sqrt{3}}{2}X$

$\Rightarrow Y = \frac{1}{2} \cdot X \cdot \frac{\sqrt{3}}{2} X = \frac{\sqrt{3}}{4} X^2$

$E(Y) = \frac{\sqrt{3}}{4} E(X^2) = \frac{\sqrt{3}}{12} a^2$

$\boxed{10/10}$

$$E(Y^2) = \frac{3}{16} E(X^4) = \frac{3}{80} a^4$$

$$Var(Y) = E(Y^2) - (E(Y))^2 = \frac{a^4}{60}.$$

(b) The required probability is

$$p = P(\max\{X, a-X\} > 2\min\{X, a-X\})$$

$$= P(a-X > 2X, X \le \tfrac{a}{2}) + P(X > 2(a-X), X > \tfrac{a}{2})$$

$$= P(X \le \tfrac{a}{3}) + P(X > \tfrac{2}{3}a)$$

$$= \frac{2}{3}.$$

$\boxed{\text{Problem 20}}$ (a) The d.f. of X is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} \frac{e^{-|t|}}{2}\,dt$$

$$= \begin{cases} \frac{e^{x}}{2}, & \text{if } x < 0 \\ 1 - \frac{e^{-x}}{2}, & \text{if } x \ge 0 \end{cases}$$

Quantile function of X is

$$Q(p) = F^{-1}(p) = \begin{cases} \ln(2p), & 0 < p < \frac{1}{2} \\ -\ln(2(1-p)), & \frac{1}{2} \le p < 1 \end{cases}$$

The desired random observation is

$$X = Q(U) = \begin{cases} \ln(2U), & \text{if } 0 < U < \frac{1}{2} \\ -\ln(2(1-U)), & \text{if } \frac{1}{2} \le U < 1. \end{cases}$$

(b) The d.f. of X is

$$G(x) = \begin{cases} 0, & \text{if } x < 0 \\ \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, & \text{if } k \le x < k+1,\ k = 0, 1, \ldots, n-1 \\ 1, & \text{if } x \ge n \end{cases}$$

The quantile function of X is

$$Q(p) = \inf\{x \in \mathbb{R} : G(x) \ge p\}$$

$$= \begin{cases} 0, & \text{if } 0 < p \le (1-\theta)^n \\ k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < p \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j},\ k = 1, 2, \ldots, n-1 \\ n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U < 1 \end{cases}$$

$\boxed{11/\underline{5}}$

The derived random observation is

$$X = \begin{cases} 0, & \text{if } 0 < U \le (1-\theta)^n \\ k, & \text{if } \sum_{j=0}^{k-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U \le \sum_{j=0}^{k} \binom{n}{j} \theta^j (1-\theta)^{n-j}, \\ & k = 1, 2, \ldots, n-1 \\ n, & \text{if } \sum_{j=0}^{n-1} \binom{n}{j} \theta^j (1-\theta)^{n-j} < U < 1. \end{cases}$$

$\boxed{\text{Problem 21}}$ $F_Y(y) = P(Y \le y) = \sum_{j=1}^{\theta} P(X - [X] \le y,\ j-1 \le X < j) = \sum_{j=1}^{\theta} P(X \le y+j-1,\ j-1 \le X < j)$

$$= \sum_{j=1}^{\theta} P(j-1 \le X \le \min(\theta, y+j-1)),\quad y \in \mathbb{R}.$$

Clearly, for $y<0$, $F_Y(y) = 0$ and for $y \ge 1$

$$F_Y(y) = \sum_{j=1}^{\theta} P(j-1 \le X \le j) = P(0 \le X \le \theta) = 1.$$

For $0 \le y < 1$, $F_Y(y) = \sum_{j=1}^{\theta} P(j-1 \le X \le y+j-1) = \sum_{j=1}^{\theta} \frac{y}{\theta} = y$

$$\Rightarrow F_Y(y) = \begin{cases} 0, & \text{if } y < 0 \\ y, & \text{if } 0 \le y < 1 \\ 1, & \text{if } y \ge 1 \end{cases} \Rightarrow Y \sim U(0,1)$$

$\boxed{\text{Problem 22}}$ The d.f. corresponding to p.d.f. $f(x)$ is

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2(3-2x), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

and the Q.F. corresponding to F is $Q(p) = F^{-1}(p) =$ root of $3t^2 - 2t^3 - p = 0$. We know that $Q(U)$ has p.d.f. f. But

$$Q(U) = \text{root of } 3t^2 - 2t^3 - U = 0.$$

$\boxed{\text{Problem 23}}$ Let $I_n = \frac{1}{\Gamma n\, \theta^n} \int_t^{\infty} e^{-x/\theta} x^{n-1} dx = \frac{1}{\lfloor n-1} \int_{t/\theta}^{\infty} e^{-z} z^{n-1} dz,\ t>0$

$$= \frac{e^{-t/\theta}(t/\theta)^{n-1}}{\lfloor n-1} + \frac{1}{\lfloor n-2} \int_{t/\theta}^{\infty} e^{-z} z^{n-2} dz$$

$$= \frac{e^{-t/\theta}(t/\theta)^{n-1}}{\lfloor n-1} + I_{n-1}$$

$$= \frac{e^{-t/\theta}(t/\theta)^{n-1}}{\lfloor n-1} + \frac{e^{-t/\theta}(t/\theta)^{n-2}}{\lfloor n-2} + I_{n-2}$$

$$= \frac{e^{-t/\theta}(t/\theta)^{n-1}}{\lfloor n-1} + \frac{e^{-t/\theta}(t/\theta)^{n-2}}{\lfloor n-2} + \cdots + \frac{e^{-t/\theta}(t/\theta)}{\lfloor 1} + I_1$$

$$= \sum_{j=0}^{n-1} \frac{e^{-t/\theta}(t/\theta)^j}{\lfloor j},\quad t>0.$$

$\boxed{\text{Problem 24}}$ Suppose that $Y \sim Exp(\theta)$, for some $\theta>0$. Then for $s,t>0$

$$P(Y>s+t) = e^{-\frac{s+t}{\theta}} = e^{-\frac{s}{\theta}} e^{-\frac{t}{\theta}} = P(Y>s)\ P(Y>t).$$

$\Rightarrow$ Y has LOM property.

Conversely suppose that Y has LOM property, i.e.,

$$P(Y>s+t) = P(Y>s)\ P(Y>t) \quad \forall\ s,t>0$$

$$\Rightarrow \bar{F}(s+t) = \bar{F}(s)\bar{F}(t), \quad \forall\ s,t>0, \text{ where } \bar{F}(x)=P(Y>x),\ x>0.$$

$$\Rightarrow \bar{F}(s_1+s_2+\cdots+s_m) = \bar{F}(s_1)\bar{F}(s_2)\cdots\bar{F}(s_m), \quad \forall\ s_i>0,\ i=1,\dots,m$$

$$\Rightarrow \bar{F}\left(\frac{m}{n}\right) = \bar{F}(\underbrace{\tfrac{1}{n}+\cdots+\tfrac{1}{n}}_{m\text{-times}}) = \left[\bar{F}\left(\tfrac{1}{n}\right)\right]^m, \quad m,n\in\mathbb{N} \quad \dots\dots (A)$$

$$\Rightarrow \bar{F}(1) = \left[\bar{F}\left(\tfrac{1}{n}\right)\right]^n, \quad \forall\ n\in\mathbb{N} \quad \dots\dots (B)$$

Using (A) and (B) we get

$$\bar{F}\left(\tfrac{m}{n}\right) = \left[\bar{F}(1)\right]^{\frac{m}{n}}, \quad \forall\ m,n\in\mathbb{N} \quad \dots\dots (C)$$

Let $\lambda=\bar{F}(1)$, so that $0\le\lambda\le1$. Clearly, if $\lambda=0$ then using (B)

$$\bar{F}\left(\tfrac{1}{n}\right)=0, \quad \forall\ n\in\mathbb{N}$$

$$\Rightarrow F\left(\tfrac{1}{n}\right)=1 \quad \forall\ n\in\mathbb{N}$$

$$\Rightarrow \lim_{n\to\infty} F\left(\tfrac{1}{n}\right)=1 \quad \Rightarrow F(0)=1 \quad \left(\text{since } F \text{ is Continuous}\right),$$

which is not true as $F(0)=0$.

Similarly if $\lambda=1$ then

$$\bar{F}(n) = \bar{F}(\underbrace{1+\cdots+1}_{n\text{ times}}) = \left[\bar{F}(1)\right]^n = 1 \quad \forall\ n\in\mathbb{N}$$

$\Rightarrow F(n)=0, \quad \forall\ n\in\mathbb{N} \Rightarrow \lim_{n\to\infty} F(n)=0,$ which is not true. Thus $\lambda\in(0,1)$. Let $\lambda=e^{-\frac{1}{\theta}}$, for some $\theta>0$ ($\theta^{-1}=-\ln\lambda$). Then using (C)

$$\bar{F}(r) = e^{-r/\theta}, \quad \forall\ r\in\mathbb{Q}\cap(0,\infty) \text{ and some } \theta>0$$

Now let $x\in\mathbb{Q}^c\cap(0,\infty)$. Then $\exists$ a sequence $\{r_n\}_{n\ge1}$ in $\mathbb{Q}\cap(0,\infty)$ such that $\lim_{n\to\infty} r_n=x$. Therefore

$$\begin{aligned}\bar{F}(x) &= \bar{F}\left(\lim_{n\to\infty} r_n\right)\\ &= \lim_{n\to\infty}\bar{F}(r_n)\\ &= \lim_{n\to\infty} e^{-\frac{r_n}{\theta}}\\ &= e^{-\frac{x}{\theta}}.\end{aligned}$$

$$\Rightarrow \bar{F}(x)=P(Y>x)=e^{-x/\theta}, \quad \forall\ x>0 \Rightarrow Y\sim Exp(\theta).$$

$\boxed{\text{Problem 25}}$ We have

$$I_{k,n} = P(Y \le p) = \frac{1}{B(k, n-k+1)} \int_0^p t^{k-1}(1-t)^{n-k}\,dt = \frac{\lfloor n}{\lfloor k-1 \lfloor n-k} \int_0^p t^{k-1}(1-t)^{n-k}\,dt$$

$$= \frac{\lfloor n}{\lfloor k-1 \lfloor n-k} \left\{ \frac{p^k(1-p)^{n-k}}{k} + \frac{n-k}{k} \int_0^p t^k (1-t)^{n-k-1}\,dt \right\}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + I_{k+1,n}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + \binom{n}{k+1} p^{k+1}(1-p)^{n-k-1} + I_{k+2,n}$$

$$= \binom{n}{k} p^k (1-p)^{n-k} + \binom{n}{k+1} p^{k+1}(1-p)^{n-k-1} + \cdots + \binom{n}{n-1} p^{n-1}(1-p) + I_{n,n}$$

$$= \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j} = P(X \ge k).$$

$\boxed{\text{Problem 26}}$ (a) $-1 \le \alpha \le 1$, $-1 \le [2\Phi(x)-1] \le 1$

$$\Rightarrow \alpha\,[2\Phi(x_1)-1]\,[2\Phi(x_2)-1] \ge -1$$

$$\Rightarrow f_{\underset{\sim}{X}}(\underset{\sim}{x}) \ge 0, \quad \forall\, \underset{\sim}{x} \in \mathbb{R}^2$$

Also

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\underset{\sim}{X}}(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi(x_1)\phi(x_2)\,dx_1\,dx_2 + \alpha \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [2\Phi(x_1)-1][2\Phi(x_2)-1]\,\phi(x_1)\phi(x_2)\,d\underset{\sim}{x}$$

$$= \left(\int_{-\infty}^{\infty}\phi(x_1)\,dx_1\right)\left(\int_{-\infty}^{\infty}\phi(x_2)\,dx_2\right) + \alpha \left(\underbrace{\int_{-\infty}^{\infty}(2\Phi(x_1)-1)\,\phi(x_1)\,dx_1}_{2\Phi(x_1)-1=t \;\Rightarrow\; 2\phi(x_1)dx_1 = dt}\right) \times \left(\int_{-\infty}^{\infty}(2\Phi(x_2)-1)\,\phi(x_2)\,dx_2\right)$$

$2\Phi(x_2)-1 = t$

$2\phi(x_2)\,dx_2 = dt$

$$= 1 + \frac{\alpha}{4}\left(\int_{-1}^{1} t\,dt\right)\left(\int_{-1}^{1} t\,dt\right)$$

$$= 1$$

$\Rightarrow f_{\underset{\sim}{X}}$ is a p.d.f.

(b) For fixed $x_1 \in \mathbb{R}$,

$$f_{X_1}(x_1) = \int_{-\infty}^{\infty} f_{\underline{X}}(x_1, x_2)\,dx_2 = \phi(x_1)\int_{-\infty}^{\infty}\phi(x_2)\,dx_2 + \alpha\,\phi(x_1)[2\Phi(x_1)-1]\int_{-\infty}^{\infty}[2\Phi(x_2)-1]\phi(x_2)\,dx_2$$

$$= \phi(x_1) \quad \left[\int_{-\infty}^{\infty}\phi(x_2)\,dx_2 = 1,\ \int_{-\infty}^{\infty}[2\Phi(x_2)-1]\,\phi(x_2)\,dx_2 = 0 \text{ as in (a)}\right]$$

By symmetry

$$f_{X_2}(x_2) = \phi(x_2), \quad \forall\, x_2 \in \mathbb{R}$$

(c) For $\alpha=0$, clearly $\underset{\sim}{X}=(X_1, X_2) \sim N_2(0, 0, 1, 1, 0)$. For $\alpha \neq 0$, obviously $\underset{\sim}{X}=(X_1, X_2) \not\sim N_2$.

$\boxed{\text{Problem 27}}$ We know that $(Y_1, Y_2) \sim N_2(\ ) \iff$ every linear combination of Y_1 and Y_2 has univariate normal distribution

(a) $\underbrace{t_1 Y + t_2 Z}_{\text{linear combination of } Y \text{ and } Z} = (t_1 a_1 + t_2 a_3) X_1 + (t_1 a_2 + t_2 a_4) X_2 \sim N_1$ (since $(X_1, X_2) \sim N_2$)

$\Rightarrow (Y, Z) \sim N_2$

$E(Y) = a_1\mu_1 + a_2\mu_2 = \theta_1$ (say); $E(Z) = a_3\mu_1 + a_4\mu_2 = \theta_2$ (say)

$Var(Y) = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + 2a_1a_2\sigma_1\sigma_2\rho = \tau_1^2$ (say)

$Var(Z) = a_3^2\sigma_1^2 + a_4^2\sigma_2^2 + 2a_3a_4\sigma_1\sigma_2\rho = \tau_2^2$ (say)

$Cov(Y, Z) = a_1a_3\sigma_1^2 + (a_1a_4 + a_2a_3)\sigma_1\sigma_2\rho + a_2a_4\sigma_2^2 = 0$ (say)

Then $(Y, Z) \sim N_2(\theta_1, \theta_2, \tau_1^2, \tau_2^2, \rho)$

(b) $Y \sim N_1(\theta_1, \tau_1^2)$ and $Z \sim N_1(\theta_2, \tau_2^2)$.

$\boxed{\text{Problem 28}}$ (a) $Y|X=2 \sim N_1\left(8 + \frac{0.6\times 3}{4}(2-5),\ 9(1-(0.6)^2)\right)$

$= N_1(6.65, 5.76)$

$P(5<Y<11|X=2) = \Phi\left(\frac{11-6.65}{\sqrt{5.76}}\right) - \Phi\left(\frac{5-6.65}{\sqrt{5.76}}\right) = \Phi(1.8125) - \Phi(-0.6875)$

$\simeq \Phi(1.8) + \Phi(.7) - 1 = .722$.

$X \sim N_1(5, 16) \Rightarrow P(4<X<6) = \Phi\left(\frac{6-5}{4}\right) - \Phi\left(\frac{4-5}{4}\right) = \Phi(.25) - \Phi(-.25)$

$= 2\Phi(.25) - 1 = 2\times .5987 - 1$

$Y \sim N_1(8, 9) \Rightarrow P(7<Y<9) = \Phi\left(\frac{9-8}{3}\right) - \Phi\left(\frac{7-8}{3}\right) = 2\Phi(\tfrac{1}{3}) - 1$

$\simeq 2\times 0.6293 - 1$.

(b) $Y|X=5 \sim N_1\left(10 + \frac{\rho\times 5}{1}(5-5),\ 25(1-\rho^2)\right) = N_1(10, 25(1-\rho^2))$

$P(4<Y<16|X=5) = 0.954 \Rightarrow \Phi\left(\frac{16-10}{5\sqrt{1-\rho^2}}\right) - \Phi\left(\frac{4-10}{5\sqrt{1-\rho^2}}\right) = 0.954$

$\Rightarrow \Phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) = .977 \Rightarrow \frac{6}{5\sqrt{1-\rho^2}} = 2 \Rightarrow \rho = \frac{4}{5} = 0.8$ (as $\rho>0$).

Problem 29 (a)-(b) $t_1U + t_2V = \underbrace{(at_1+bt_2)X + (bt_1-at_2)Y}_{\text{linear combination of } (X,Y)} \sim N_1$ (as $(X,Y)\sim N_2$)

$\Rightarrow (U,V) \sim N_2$

$E(U) = E(V) = 0$, $Var(U) = Var(V) = (a^2+b^2)\sigma^2 = \tau^2$ (say); $Cov(U,V) = 0$

$\Rightarrow (U,V) \sim N_2(0, 0, \tau^2, \tau^2, 0) \Rightarrow$ U and V are i.i.d. $N(0,\tau^2)$ r.v.s.

(c) Taking $a = b = \frac{1}{\sqrt{2}}$ in (b) the result follows.

Problem 30 (a) $M_Y(t) = E(e^{tX_1X_2}) = E(E(e^{tX_1X_2}|X_1))$

$X_2 | X_1 = x_1 \sim N(\rho x_1, 1-\rho^2) \Rightarrow E(e^{tX_1X_2}|X_1) = e^{(tX_1)(\rho X_1) + \frac{t^2X_1^2(1-\rho^2)}{2}}$

$= e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}$

$X_1^2 \sim \chi_1^2 \Rightarrow M_Y(t) = E\left[e^{X_1^2\left(\rho t + \frac{(1-\rho^2)t^2}{2}\right)}\right]$

$= [1 - 2\rho t - (1-\rho^2)t^2]^{-1/2}, \quad -\frac{1}{1-\rho} < t < \frac{1}{1+\rho}.$

(b) $M_Y^{(1)}(t) = \frac{1}{2}[1-2\rho t - (1-\rho^2)t^2]^{-\frac{3}{2}}[2\rho + 2(1-\rho^2)t]$

$M_Y^{(2)}(t) = \frac{3}{4}[1-2\rho t-(1-\rho^2)t^2]^{-5/2}[2\rho + 2(1-\rho^2)t]^2 +$

$[1-2\rho t - (1-\rho^2)t^2]^{-3/2}(1-\rho^2)$

$E(Y^2) = M_Y^{(2)}(0) = 1 + 2\rho^2.$

(c) $E(X_1^2X_2^2) = E(X_1^2E(X_2^2|X_1)) = E[X_1^2(1-\rho^2+\rho^2X_1^2)]$

$= 1-\rho^2 + \rho^2E(X^4) = 1-\rho^2+3\rho^2 \quad (X_1 \sim N(0,1))$

$= 1+2\sigma^2.$

Problem 31 $f(x,y) = \begin{cases} \frac{1}{\pi}e^{-\frac{1}{2}(x^2+y^2)}, & \text{if } \{x<0 \text{ and } y<0\} \text{ or } \{x>0 \text{ and } y>0\} \\ 0 & \text{otherwise} \end{cases}$

$f_{X_1}(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_{-\infty}^{0}\frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy, & \text{if } x<0 \\ \int_{0}^{\infty}\frac{1}{\pi}e^{-\frac{x^2+y^2}{2}}dy, & \text{if } x\geq 0 \end{cases}$

$= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \quad \Rightarrow X_1 \sim N(0,1)$

By symmetry $X_2 \sim N(0,1)$

Obviously $f(x,y)$ is not a p.d.f. of N_2

$\boxed{\text{Problem 32}}$

$$f_X(x) = \int_{-\infty}^{\infty} g_\rho(x,y)\,dy = \frac{1}{2}\Big[\underbrace{\int_{-\infty}^{\infty} f_\rho(x,y)\,dy}_{\substack{\text{Marginal of } X_1 \sim N(0,1) \\ \text{in } (X_1,X_2)\sim N_2(0,0,1,1,\rho)}} + \underbrace{\int_{-\infty}^{\infty} f_{-\rho}(x,y)\,dy}_{\substack{\text{Marginal of } Y_1 \sim N(0,1) \\ \text{in } (Y_1,Y_2)\sim N_2(0,0,1,1,-\rho)}}\Big]$$

$$= \frac{1}{2}[\phi(x)+\phi(x)] = \phi(x), \quad -\infty<x<\infty$$

$$\Rightarrow X \sim N(0,1).$$

By symmetry $Y \sim N(0,1)$

Obviously $(X,Y) \not\sim N_2$ unless $\rho = 0$.

$\boxed{\text{Problem 33}}$ (a) Since $X, Y \sim N(0,1)$, $E(X)=E(Y)=0$, $Var(X)=Var(Y)=1$

$$E(XY) = \frac{1}{2}\Big[\underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_\rho(x,y)\,dx\,dy}_{\substack{E(X_1X_2) \\ \text{where } (X_1,X_2)\sim N_2(0,0,1,1,\rho)}} + \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{-\rho}(x,y)\,dx\,dy}_{\substack{E(Y_1Y_2) \\ \text{where } (Y_1,Y_2)\sim N_2(0,0,1,1,-\rho)}}\Big]$$

$$= \frac{1}{2}[\rho + (-\rho)] = 0$$

$$\Rightarrow Cov(X,Y) = E(XY) - E(X)E(Y) = 0 \Rightarrow Corr(X,Y) = 0$$

(b) Obviously X and Y are not independent unless $\rho = 0$.

$\boxed{\text{Problem 34}}$ (a) For $\sum_{i=1}^{4} x_i = 28$, $0 \le x_i \le 28$, $x_i \in \mathbb{N}$, $i = 1(1)4$

$$P(X_1=x_1,\dots,X_4=x_4 \mid X_5=2) = \frac{P(X_1=x_1,\dots,X_4=x_4, X_5=2)}{P(X_5=2)}$$

$$= \frac{\dfrac{30!}{x_1!\,x_2!\,x_3!\,x_4!\,2!}\,\theta_1^{x_1}\cdots\theta_4^{x_4}\,\theta_5^{2}}{\binom{30}{2}\theta_5^2(1-\theta_5)^{28}} \qquad (X_5 \sim Bin(30,\theta_5))$$

$$= \frac{28!}{x_1!\cdots x_4!}\left(\frac{\theta_1}{1-\theta_5}\right)^{x_1}\cdots\left(\frac{\theta_4}{1-\theta_5}\right)^{x_4}$$

$$\Rightarrow (X_1, X_2, X_3) \sim Mult(30, \theta_1, \theta_2, \theta_3) \text{ and } X_4 = 30 - \sum_{i=1}^{3} X_i$$

(b) Let X_1 = # of times we get a sum of 12 in 7 independent casts of a pair of dice

X_2 = " " . . . . . . sum of 8 . . . . . .

θ_1 = P(getting a sum of 12 in a cast) $= \frac{1}{36}$

$\theta_2 =$ P(getting a sum of 8 in a cast) $= \frac{5}{36}$

Required prob $= P(X_1 = 1, X_2 = 2)$

$$= \frac{\lfloor 7}{\lfloor 1 \lfloor 2 \lfloor 4} \left(\frac{1}{36}\right)\left(\frac{5}{36}\right)^2 \left(1-\frac{6}{36}\right)^4.$$

$\boxed{\text{Problem 35}}$ (a) Let $Z_1 \sim \chi^2_{n_1}$ and $Z_2 \sim \chi^2_{n_2}$ be independent. Then $\frac{Z_1}{2} \sim GAM(\frac{n_1}{2}, 1)$ and $\frac{Z_2}{2} \sim GAM(\frac{n_2}{2}, 1)$ are independent and

$$\Rightarrow X \stackrel{d}{=} \frac{Z_1/n_1}{Z_2/n_2} = \frac{n_2}{n_1}\frac{Z_1}{Z_2}$$

$$\Rightarrow Y_1 \stackrel{d}{=} \frac{n_2}{n_2 + n_2 \frac{Z_1}{Z_2}} = \frac{Z_2}{Z_1 + Z_2} = \frac{Z_2/2}{\frac{Z_1}{2}+\frac{Z_2}{2}} \sim Be\left(\frac{n_1}{2}, \frac{n_2}{2}\right)$$

(b) $Z_1 \sim N(0,1)$ and $Z_2^2 \sim \chi^2_1$ are independent

$$\Rightarrow \frac{Z_1}{\sqrt{Z_2^2/1}} \sim \chi^2_1, \quad \text{i.e.} \quad \frac{Z_1}{|Z_2|} \sim \chi^2_1.$$

Let $Y = \frac{Z_1}{Z_2}$. Then, for $y \in \mathbb{R}$,

$$F_Y(y) = P\left(\frac{Z_1}{Z_2} \le y\right)$$

$$= P\left(\frac{Z_1}{Z_2} \le y, Z_2 > 0\right) + P\left(\frac{Z_1}{Z_2} \le y, Z_2 < 0\right)$$

$$= P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 > 0\right) + P\left(-\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right).$$

Since $(Z_1, Z_2) \stackrel{d}{=} (-Z_1, Z_2)$, we have

$$P\left(-\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right) = P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right)$$

$$\Rightarrow F_Y(y) = P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 > 0\right) + P\left(\frac{Z_1}{|Z_2|} \le y, Z_2 < 0\right)$$

$$= P\left(\frac{Z_1}{|Z_2|} \le y\right), \quad \forall y \in \mathbb{R}$$

$$\Rightarrow Y \stackrel{d}{=} \frac{Z_1}{|Z_2|} \sim t_1$$

(c) $\frac{2X_1}{\theta}$ and $\frac{2X_2}{\theta}$ are i.i.d. χ^2_2

$$\Rightarrow Z = \frac{X_1}{X_2} = \frac{X_1/\theta}{X_2/\theta} = \frac{\chi^2_2/2}{\chi^2_2/2} \Big\rangle \text{ independent}$$

$$\sim F_{2,2}$$

(d) $\dfrac{X_1/n_1}{X_2/n_2} = \dfrac{\chi^2_{n_1}/n_1}{\chi^2_{n_2}/n_2} \Big\rangle$ independent

$$\sim F_{n_1, n_2}$$

Also $X_3 \sim \chi^2_{n_3}$ and $X_1 + X_2 \sim \chi^2_{n_1+n_2}$ are independent

$$\Rightarrow \frac{X_3/n_3}{(X_1+X_2)/(n_1+n_2)} = \frac{\chi^2_{n_3}/n_3}{\chi^2_{n_1+n_2}/n_1+n_2} \Big\rangle \text{ independent}$$

$$\sim F_{n_3, n_1+n_2}$$

It suffices to show that $Y_1 = \frac{X_1}{X_2}$ and $Y_2 = \frac{X_3}{X_1+X_2}$ are independent. The joint p.d.f of (X_1, X_2, X_3) is

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1) f_{X_2}(x_2) f_{X_3}(x_3)$$

$$= \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2} \Gamma\frac{n_3}{2}} e^{-\frac{x_1+x_2+x_3}{2}} x_1^{\frac{n_1}{2}-1} x_2^{\frac{n_2}{2}-1} x_3^{\frac{n_3}{2}-1}, \quad x_i > 0,\ i = 1,2,3.$$

Let $Y_1 = \frac{X_1}{X_2}$, $Y_2 = \frac{X_3}{X_1+X_2}$ and $Y_3 = X_2$, i.e., $X_1 = Y_1 Y_3$

$X_2 = Y_3$ and $X_3 = Y_2 Y_3 (1+Y_1)$

$$J = \begin{vmatrix} y_3 & 0 & y_1 \\ 0 & 0 & 1 \\ y_2 y_3 & y_3(1+y_1) & y_2(1+y_1) \end{vmatrix} = -y_3^2(1+y_1)$$

$x_i > 0,\ i = 1,2,3 \Rightarrow y_1 y_3 > 0,\ y_3 > 0,\ y_2 y_3 (1+y_1) > 0 \Rightarrow y_i > 0,\ i = 1,2,3$

The joint pdf of (Y_1, Y_2, Y_3) is given by

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2} \Gamma\frac{n_3}{2}} e^{-\frac{(y_1y_3 + y_3 + y_2 y_3(1+y_1))}{2}} (y_1 y_3)^{\frac{n_1}{2}-1} \times y_3^{\frac{n_2}{2}-1} (y_2 y_3(1+y_1))^{\frac{n_3}{2}-1} y_3^2 (1+y_1)$$

$$= \frac{1}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2} \Gamma\frac{n_3}{2}} e^{-\frac{y_3(1+y_1)(1+y_2)}{2}} \times y_1^{\frac{n_1}{2}-1} (1+y_1)^{\frac{n_3}{2}} \times y_2^{\frac{n_3}{2}-1} y_3^{\frac{n_1+n_2+n_3}{2}-1},$$

$y_i > 0,\ i = 1,2,3$

The joint pdf of (Y_1, Y_2) is

$$f_{Y_1, Y_2}(y_1, y_2) = \int_0^\infty f_{Y_1, Y_2, Y_3}(y_1, y_2, y_3)\, dy_3 , \qquad y_1 > 0,\ y_2 > 0$$

$$= \frac{\Gamma\frac{n_1+n_2+n_3}{2}}{2^{\frac{n_1+n_2+n_3}{2}} \Gamma\frac{n_1}{2} \Gamma\frac{n_2}{2} \Gamma\frac{n_3}{2}} \left(\frac{2}{(1+y_1)(1+y_2)}\right)^{\frac{n_1+n_2+n_3}{2}} y_1^{\frac{n_1}{2}-1} (1+y_1)^{\frac{n_3}{2}} \times y_2^{\frac{n_3}{2}} \cdots$$

$$= h_1(y_1) h_2(y_2) \qquad y_i > 0,\ i = 1, 2$$

$\Rightarrow$ Y_1 and Y_2 are independent.

Problem 36 The joint p.d.f. of $(X_{1:n}, \ldots, X_{n:n})$ is

$$g(x_1, \ldots, x_n) = \underline{|n} \prod_{i=1}^{n} \frac{1}{\theta} e^{-\frac{x_i}{\theta}} , \qquad 0 < x_1 < \cdots < x_n < \infty$$

$$= \frac{\underline{|n}}{\theta^n} e^{-\frac{\sum_{i=1}^n x_i}{\theta}} , \qquad 0 < x_1 < \cdots < x_n < \infty$$

Let $Z_1 = n X_{1:n}$, $Z_2 = (n-1)(X_{2:n} - X_{1:n})$, ..., $Z_n = X_{n:n} - X_{n-1:n}$

$\Rightarrow$ $X_{1:n} = \frac{Z_1}{n}$, $X_{2:n} = \frac{Z_1}{n} + \frac{Z_2}{n-1}$, ..., $X_{n:n} = \frac{Z_1}{n} + \frac{Z_2}{n-1} + \cdots + \frac{Z_{n-1}}{2} + Z_n$

$$J = \begin{vmatrix} 1/n & 0 & 0 \cdots 0 \\ 1/n & \frac{1}{n-1} & 0 \cdots 0 \\ \vdots & & \\ \frac{1}{n} & \frac{1}{n-1} & \cdots \quad 1 \end{vmatrix} = \frac{1}{\underline{|n}}$$

$X_{1:n} + X_{2:n} + \cdots + X_{n:n} = Z_1 + Z_2 + \cdots + Z_n$.

$0 < x_1 < \cdots < x_n < \infty \Rightarrow 0 < \frac{z_1}{n} < \frac{z_1}{n} + \frac{z_2}{n-1} < \cdots < \frac{z_1}{n} + \frac{z_2}{n-1} + \cdots + z_n < \infty$

$\Rightarrow z_1 > 0, z_2 > 0, \ldots, z_n > 0$

Thus the joint pdf of $(Z_1, \ldots, Z_n)$ is

$$h(z_1, \ldots, z_n) = \frac{\underline{|n}}{\theta^n} e^{-\frac{\sum_{i=1}^n z_i}{\theta}} \times \frac{1}{\underline{|n}} , \quad z_i > 0, \quad i = 1, \ldots, n$$

$$= \prod_{i=1}^{n} \left(\frac{1}{\theta} e^{-\frac{z_i}{\theta}}\right) , \quad z_i > 0, \quad i = 1, \ldots, n$$

$\Rightarrow$ $Z_1, Z_2, \ldots, Z_n$ are i.i.d. $Exp(\theta)$

$\boxed{20/5}$

We have

$$X_{r:n} = \sum_{i=1}^{r} \frac{Z_i}{n-i+1}$$

$$\Rightarrow E(X_{r:n}) = \sum_{i=1}^{r} \frac{E(Z_i)}{n-i+1} = \theta \sum_{i=1}^{r} \frac{1}{n-i+1}$$

$$Var(X_{r:n}) = \sum_{i=1}^{r} \frac{Var(Z_i)}{(n-i+1)^2} \qquad (Z_i\text{'s are Independent})$$

$$= \theta^2 \sum_{i=1}^{r} \frac{1}{(n-i+1)^2}$$

$$Cov(X_{r:n}, X_{s:n}) = Cov\left(\sum_{i=1}^{r} \frac{Z_i}{n-i+1}, \sum_{i=1}^{s} \frac{Z_i}{n-i+1} \right)$$

$$= Var\left(\sum_{i=1}^{r} \frac{Z_i}{n-i+1} \right) \qquad (Z_i\text{'s are Independent})$$

$$= \theta^2 \sum_{i=1}^{r} \frac{1}{(n-i+1)^2} = Var(X_{r:n}),$$

$1 \le r < s \le n.$

—— o ——