MSO 201a : Probability and Statistics

Assignment No. 6

Model Solutions

Problem No. 1

$$F_n(x) = \begin{cases} 0, & x < -n \\ \frac{x+n}{2n}, & -n \le x < n \\ 1, & x \ge n \end{cases} \xrightarrow{n\to\infty} F(x) = \frac{1}{2}, \ \forall x \in \mathbb{R}$$

$\hookrightarrow$ not a d.f.

Problem No. 2

$$F_{Y_n}(t) = P(X_{1:n} \le \tfrac{t}{n}) = 1 - P(X_{1:n} > \tfrac{t}{n}) = 1 - P(X_i > \tfrac{t}{n}, i=1..n)$$

$$= 1 - \prod_{i=1}^{n} P(X_i > \tfrac{t}{n}) = 1 - [1 - F(\tfrac{t}{n})]^n, \quad t \in \mathbb{R}$$

$(\min\{X_1..X_n\} > y \Leftrightarrow X_i > y, \forall i=1..n)$

(a) $X_1 \sim U(0,\theta) \Rightarrow F(t) = \begin{cases} 0, & t<0 \\ t/\theta, & 0 \le t < \theta \\ 1, & t \ge \theta \end{cases}; \quad F_{Y_n}(t) = \begin{cases} 1-(1-0)^n, & t<0 \\ 1-(1-\frac{t}{n\theta})^n, & 0 \le t < n\theta \\ 1-(1-1)^n, & t \ge n\theta \end{cases}$

$$\xrightarrow{n\to\infty} F(t) = \begin{cases} 0, & t<0 \\ 1-e^{-\frac{t}{\theta}}, & t \ge 0 \end{cases}$$

$\downarrow$ d.f. of $Exp(\theta)$

Thus

$$Y_n \xrightarrow{d} Y \sim Exp(\theta)$$

$$X_{1:n} = \frac{1}{n} \times Y_n \xrightarrow{p} 0 \times Y = 0 \quad (\tfrac{1}{n} \to 0 \text{ as } n \to \infty)$$

(b) $X_1 \sim Exp(\theta) \Rightarrow F(t) = \begin{cases} 0, & t<0 \\ 1-e^{-t/\theta}, & t \ge 0 \end{cases}$

$$F_{Y_n}(t) = \begin{cases} 1-(1-0)^n, & t<0 \\ 1-(e^{-\frac{t}{n\theta}})^n, & t \ge 0 \end{cases} = \begin{cases} 0, & t<0 \\ 1-e^{-\frac{t}{\theta}}, & t \ge 0 \end{cases} = F(t)$$

$\downarrow$ does not depend on n

$\downarrow$ d.f. of $Exp(\theta)$

$\Rightarrow Y_n \xrightarrow{d} Y \sim Exp(\theta)$

$\Rightarrow X_{1:n} = \frac{1}{n} \times Y_n \xrightarrow{p} 0 \times Y = 0.$

Problem No. 3 $E(\bar{X}_n)=0$, $Var(\bar{X}_n)=\frac{1}{n^2}\sum_{i=1}^{n}Var(X_i)=\frac{1}{n^2}\sum_{i=1}^{n}i^{1/2}$

$Var(\bar{X}_n)=\frac{1}{\sqrt{n}}\times\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{1/2}\right)$. But $\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^{1/2}\rightarrow\int_0^1 t^{1/2}dt=\frac{2}{3}$

$\Rightarrow Var(\bar{X}_n)\rightarrow 0$. Thus $\bar{X}_n\xrightarrow{P}0$ $(E(\bar{X}_n)=0,\ Var(\bar{X}_n)\rightarrow 0)$.

Problem No. 4 (a) On contrary suppose that $a\neq b$. Let $|a-b|=\varepsilon$, where $\varepsilon>0$.
Then

$$P(|X_n-a|>\tfrac{\varepsilon}{2})=P(|b-a-(b-X_n)|>\tfrac{\varepsilon}{2})$$
$$\geq P(|b-a|-|b-X_n|>\tfrac{\varepsilon}{2})\quad\left(\text{since } |s|-|t|>\tfrac{\varepsilon}{2}\Rightarrow|s-t|>\tfrac{\varepsilon}{2}\right)$$
$$=P(|X_n-b|<\tfrac{\varepsilon}{2})\rightarrow 1 \text{ as } n\to\infty\quad(X_n\xrightarrow{P}b)$$

$\Rightarrow X_n \not\xrightarrow{P} a$.

(b) Fix $\varepsilon>0$. Then, by Markov's inequality

$$0\leq P(|X_n-a|>\varepsilon)\leq\frac{E(|X_n-a|^r)}{\varepsilon^r}\rightarrow 0 \text{ as } n\to\infty$$

$\Rightarrow \lim_{n\to\infty}P(|X_n-a|>\varepsilon)=0 \Rightarrow X_n\xrightarrow{P}a$

Problem No. 5 (a) (For the continuous case. For the discrete case proof is similar with integral replaced by summation)

$$E\left(\frac{|X|^r}{1+|X|^r}\right)=\int_{|x|\leq t}\frac{|x|^r}{1+|x|^r}f(x)dx+\int_{|x|\geq t}\frac{|x|^r}{1+|x|^r}f(x)dx\geq 0+\int_{|x|\geq t}\frac{t^r}{1+t^r}f(x)dx$$

$\left(\text{for } x>0\ \ \psi(x)=\frac{x^r}{1+x^r}\uparrow \text{ in } x\right)$

$$=\frac{t^r}{1+t^r}P(|X|\geq t).\quad\ldots\quad(\text{I})$$

Also

$$E\left(\frac{|X|^r}{1+|X|^r}\right)=\int_{|x|\leq t}\frac{|x|^r}{1+|x|^r}f(x)dx+\int_{|x|\geq t}\frac{|x|^r}{1+|x|^r}f(x)dx$$
$$\leq\int_{|x|\leq t}\frac{t^r}{1+t^r}f(x)dx+\int_{|x|\geq t}f(x)dx\quad\left(\frac{|x|^r}{1+|x|^r}\leq 1\ \forall x,\ \psi(x)=\frac{x^r}{1+x^r}\uparrow\text{ in } x\right)$$
$$=\frac{t^r}{1+t^r}P(|X|\leq t)+P(|X|\geq t)$$
$$\leq\frac{t^r}{1+t^r}+P(|X|\geq t)\quad\ldots\quad(\text{II})$$

Combining (I) and (II) we get the result.

(b) First suppose that $\lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) = 0$. Using (a) for any $\varepsilon > 0$,

$$0 \le P(|X_n| \ge \varepsilon) \le \frac{1+\varepsilon^r}{\varepsilon^r} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \to 0$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n| \ge \varepsilon) = 0, \quad \forall \varepsilon > 0 \quad \Rightarrow X_n \xrightarrow{P} 0.$$

Conversely suppose that $X_n \xrightarrow{P} 0$. Fix $\varepsilon > 0$. Then by (a)

$$0 \le E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \le P(|X_n| \ge \varepsilon) + \frac{\varepsilon^r}{1+\varepsilon^r} \to \frac{\varepsilon^r}{1+\varepsilon^r}$$

$$\Rightarrow 0 \le \lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) \le \frac{\varepsilon^r}{1+\varepsilon^r}, \quad \forall \varepsilon > 0.$$

Since $\varepsilon > 0$ is arbitrary, we get

$$\lim_{n\to\infty} E\left(\frac{|X_n|^r}{1+|X_n|^r}\right) = 0.$$

Problem No. 6 (a) Fix $\varepsilon > 0$ Then

$$P(|a_n X_n| > \varepsilon) = P\left(|X_n| > \frac{\varepsilon}{|a_n|}\right) = P\left(|X_1| > \frac{\varepsilon}{|a_n|}\right) = 1 - \left[F\left(\frac{\varepsilon}{|a_n|}\right) - F\left(-\frac{\varepsilon}{|a_n|}\right)\right]$$

$$\le 1 - F\left(\frac{\varepsilon}{|a_n|}\right) - F\left(-\frac{\varepsilon}{|a_n|}\right) \to 1 - F(\infty) - F(-\infty) = 0$$

$\Rightarrow a_n X_n \xrightarrow{P} 0$ (Alt. $a_n X_n \overset{d}{=} a_n X_1 \xrightarrow{P} 0 \times X_1 = 0$)

(b) Fix $\varepsilon > 0$. Then

$$0 \le P(|X_n - a| \ge \varepsilon) = P(X_n \le a - \varepsilon) + P(X_n \ge a + \varepsilon)$$

$$\le P(Y_n \le a - \varepsilon) + P(Z_n \ge a + \varepsilon) \quad \left(\begin{array}{l} X_n \le a - \varepsilon \Rightarrow Y_n \le a - \varepsilon \\ X_n \ge a + \varepsilon \Rightarrow Z_n \ge a + \varepsilon \end{array} \right)$$

$$\le P(|Y_n - a| \ge \varepsilon) + P(|Z_n - a| \ge \varepsilon)$$

$$\to 0 + 0 = 0, \quad \text{as } n \to \infty$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n - a| \ge \varepsilon) = 0, \quad \forall \varepsilon > 0 \quad \Rightarrow X_n \xrightarrow{P} a.$$

(c) See Lecture notes. Alternatively: Fix $\varepsilon > 0$. Then $\exists\, n_0 \equiv n_0(\varepsilon)$ s.t. $|a_n - a| < \frac{\varepsilon}{2}$, $\forall n \ge n_0$

$0 \le P[|X_n + a_n - (c+a)| \geq \varepsilon) \le P[|X_n - c| + |a_n - c| \geq \varepsilon)$

$\le P(|X_n - c| \geq \frac{\varepsilon}{2}$ or $|a_n - c| \geq \varepsilon)$

$= P(|X_n - c| \geq \varepsilon/2), \quad \forall n \geq n_0$

$\longrightarrow 0, \quad$ as $n \to \infty \quad (X_n \xrightarrow{P} c)$

$\Rightarrow \lim_{n\to\infty} P(|X_n + a_n - (c+a)| \geq \varepsilon) = 0 \quad \Rightarrow \quad X_n + a_n \xrightarrow{P} c + a$

Also for any fixed $\varepsilon > 0$, $\exists\, n_0 = n_0(\varepsilon)$ s.t. $|c|\,|a_n - a| < \frac{\varepsilon}{2}, \; \forall n \geq n_0$ $\quad (c(a_n - a) \to 0)$

$\Rightarrow \quad 0 \le P(|a_n X_n - ac| \geq \varepsilon) = P(|a_n X_n - a_n c + a_n c - ac| \geq \varepsilon)$

$\le P(|a_n|\,|X_n - c| + |c|\,|a_n - a| \geq \varepsilon)$

$\le P(|a_n|\,|X_n - c| \geq \varepsilon/2$ or $|c|\,|a_n - a| \geq \frac{\varepsilon}{2})$

$= P(|a_n|\,|X_n - c| \geq \frac{\varepsilon}{2}), \quad \forall n \geq n_0$

For $\varepsilon > 0$ $\quad a_n \to a \Rightarrow \exists\, n_1 = n_1(\varepsilon)$ s.t. $|a_n - a| < \varepsilon, \; \forall n \geq n_1$

$\Rightarrow |a_n| < |a| + \varepsilon, \quad \forall n \geq n_1$

Thus

$0 \le P(|a_n X_n - ac| \geq \varepsilon) \le P(|a_n|\,|X_n - c| \geq \frac{\varepsilon}{2})$

$\le P((|a| + \varepsilon)|X_n - c| \geq \frac{\varepsilon}{2})$

$= P(|X_n - c| \geq \frac{\varepsilon}{2(|a| + \varepsilon)}) \to 0, \quad$ as $n \to \infty$

$\Rightarrow \lim_{n\to\infty} P(|a_n X_n - ac| \geq \varepsilon) = 0, \; \forall \varepsilon > 0 \quad \Rightarrow \quad a_n X_n \xrightarrow{P} ac$

(d) First suppose that $Y_n \xrightarrow{P} 0$. Then $|Y_n| \xrightarrow{P} 0$. And

$0 \le X_n \le |Y_n| \xrightarrow{P} 0.$

$\Rightarrow X_n \xrightarrow{P} 0 \quad$ (By (b)).

Conversely suppose $X_n \xrightarrow{P} 0$. Then

$$P(|Y_n| > \varepsilon) = \begin{cases} P(|X_n| > \varepsilon), & \text{if } a > \varepsilon \\ 0, & \text{if } a \le \varepsilon \end{cases} \longrightarrow 0 \quad \text{as } n \to \infty$$

$\Rightarrow Y_n \xrightarrow{P} 0$

Problem No. 7

(a) $$E\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i\right) = \frac{2}{n(n+1)} \times \frac{n(n+1)}{2}\times\mu = \mu$$

$$Var\left(\frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i\right) = \frac{4}{n^2(n+1)^2}\sum_{i=1}^{n} i^2\, Var(x_i) = \frac{4\sigma^2}{n^2(n+1)^2}\times\frac{n(n+1)(2n+1)}{6}$$

$$\to 0 \text{ as } n\to\infty.$$

$$\Rightarrow \frac{2}{n(n+1)}\sum_{i=1}^{n} i x_i \xrightarrow{P} \mu \qquad \left(E(T_n)\to\mu,\ Var(T_n)\to 0 \Rightarrow T_n \xrightarrow{P} \mu \right).$$

(b) $$E\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i\right) = \frac{6\mu}{n(n+1)(2n+1)} \times \frac{n(n+1)(2n+1)}{6} = \mu$$

$$Var\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i\right) = \frac{36\sigma^2}{n^2(n+1)^2(2n+1)^2}\sum_{i=1}^{n} i^4$$

$$= \frac{36 n^3\sigma^2}{(n+1)^2(2n+1)^2}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4\right)$$

$$\to 36\times 0\times\int_0^1 t^4\,dt = 0$$

$$\Rightarrow \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 x_i \xrightarrow{P} \mu.$$

Problem No. 8

We know that

$$m_{X_n}(t) = E(e^{tX_n}) = \sum_{x=0}^{\infty} e^{tx}\binom{n+x-1}{x} p_n^n (1-p_n)^x$$

$$= p_n^n \sum_{x=0}^{\infty}\binom{n+x-1}{x}\left((1-p_n)e^t\right)^x$$

$$= \frac{p_n^n}{\left(1-(1-p_n)e^t\right)^n}, \qquad t < -\ln(1-p_n)$$

$$= \left[\frac{1-\theta/n}{1-\frac{\theta}{n}e^t}\right]^n$$

$$= \left(1 + \frac{\theta(e^t-1)}{n\left(1-\frac{\theta}{n}e^t\right)}\right)^n$$

$$\to e^{\theta(e^t-1)} \quad \text{as } n\to\infty \quad \forall t\in\mathbb{R}$$

$\hookrightarrow$ mgf of Poisson(θ)

$X_n \sim \text{Gamma}(n, \frac{1}{n})$ (Note the typo in question)

Problem No. 9 (a) $E(X_n) = n \times \frac{1}{n} = 1, \ \forall n = 1, 2, \ldots$

$\text{Var}(X_n) = n \times \frac{1}{n^2} \longrightarrow 0$ as $n \to \infty$

$\Rightarrow X_n \xrightarrow{p} 1.$

(b) $M_{X_n}(t) = E(e^{tX_n}) = e^{\frac{t}{n} + \frac{1}{2}(1-\frac{1}{n})^2 t^2} \longrightarrow e^{t^2/2}, \ \forall t \in \mathbb{R}$

↳ mgf of $N(0,1)$ $= M_Z(t)$

$\Rightarrow X_n \xrightarrow{d} Z \sim N(0,1)$

(Alt: $P(X_n \le x) = \Phi\left(\frac{x - \frac{1}{n}}{1 - \frac{1}{n}}\right) \longrightarrow \Phi(x)$, as $n \to \infty$)

Problem No. 10 (a) Let $X_1, \ldots, X_{72}$ be a random sample with pdf $f(\cdot)$.

Define

$$Y_i = \begin{cases} 1, & \text{if } X_i < 3 \\ 0, & \text{if } X_i \ge 3 \end{cases}, \quad i = 1, 2, \ldots, 72$$

and $Y = \sum_{i=1}^{72} X_i$

Required probability $= P(Y > 50)$

$= P(Y \geq 50.5)$ (Continuity Correction as Y takes only discrete values)

$Y_1, \ldots, Y_{72}$ are iid $\text{Bin}(1, p)$ where $p = P(X_1 < 3) = \int_1^3 \frac{1}{2x^2}\, dx = \frac{2}{3}$

By the CLT, for large n,

$$\frac{\sqrt{n}\left(\frac{Y}{n} - \frac{2}{3}\right)}{\sqrt{\frac{2}{3} \times (1 - \frac{1}{3})}} \approx Z \sim N(0,1),$$

$n = 72$ is large. Thus

$$3 \times \frac{\sqrt{72}\left(\frac{Y}{72} - \frac{2}{3}\right)}{\sqrt{2}} = \frac{Y}{4} - 12 \approx Z \sim N(0,1)$$

Required prob $= P(Y \geq 50.5) \approx P(4Z + 48 \geq 50.5)$

$= 1 - \Phi(.625) = 1 - .734 = 0.266$

(b) $E(X_1) = 3$, $Var(X_1) = 3$.

By the CLT ($n = 100$ is large)

$$\frac{\sqrt{100}\left(\frac{Y}{100} - 3\right)}{\sqrt{3}} \approx Z \sim N(0,1)$$

$$\frac{\frac{Y}{10} - 30}{\sqrt{3}} \approx Z \sim N(0,1)$$

(Continuity correction as Y takes only integer values)

$P(249.5 \le Y \le 300.5)$

$$P(250 \le Y \le 300) = P(249.5 \le 10\sqrt{3}Z + 300 \le 300.5)$$

(Note the typo in question)

$$= \Phi(.0289) - \Phi(-2.9152) = 0.512 - .002$$

$$= 0.51$$

(c) $X \stackrel{d}{=} \sum_{i=1}^{25} X_i$,

where $X_1, \ldots, X_{25}$ are iid $Bin(1, 0.6)$

$\mu = E(X_1) = 0.6$, $\sigma^2 = Var(X_1) = 0.6 \times 0.4 = 0.24$

By the CLT ($n = 25$ is reasonably large)

$$\frac{\sqrt{25}\left(\frac{X}{25} - 0.6\right)}{\sqrt{0.24}} \approx Z \sim N(0,1)$$

$$\Rightarrow \quad X \simeq 5\sqrt{0.24}\,Z + 15$$

Required prob $= P(10 \le X \le 16)$

$$= P(9.5 \le X \le 16.5)$$

$$\simeq P(9.5 \le 5\sqrt{0.24}\,Z + 15 \le 16.5)$$

$$= \Phi(0.6124) - \Phi(-2.2454)$$

$$= 0.730 - 0.012 = 0.718$$

Actual Prob: $P(X \le 16) - P(X \le 9) = 0.7265 - 0.0132$

$$= 0.7133$$

Problem No. 11 (a) Let $X_1, X_2, \ldots$ be iid Poisson(1) rvs $(E(X_1)=1,\ Var(X_1)=1)$. Let $Y=\sum_{i=1}^{n} X_i$. Then $Y \sim$ Poisson(n). By the CLT

$$\sqrt{n}\left(\frac{Y}{n}-1\right) \xrightarrow{d} Z \sim N(0,1) \text{ as } n\to\infty$$

$$\text{Required limit} = \lim_{n\to\infty}\left(e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}\right) = \lim_{n\to\infty} P(Y\le n)$$

$$= \lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{Y}{n}-1\right)\le 0\right) = P(Z\le 0) = \Phi(0) = \frac{1}{2}.$$

(b) Let $X_1, X_2, \ldots$ be iid Bin$(1,\frac{1}{2})$; $E(X_1)=\frac{1}{2}$, $Var(X_1)=\frac{1}{4}$. Let $Y_n=\sum_{i=1}^{n} X_i$. Then $Y \sim$ Bin$(n,\frac{1}{2})$. By CLT

$$\frac{\sqrt{n}\left(\frac{Y}{n}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z \sim N(0,1), \text{ as } n\to\infty.$$

Let $t_n = 2^{n}\sum_{k=0}^{r_n}\binom{n}{k} = P(Y_n \le r_n)$, $n=1,2,\ldots$

Then

$t_{2m} = P(Y_{2m}\le m)$ and $t_{2m+1} = P(Y_{2m+1}\le m)$, $m=1,2,\ldots$

By the CLT

$$\frac{\sqrt{2m}\left(\frac{Y_{2m}}{2m}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z\sim N(0,1), \quad \frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \xrightarrow{d} Z\sim N(0,1)$$

$$\lim_{m\to\infty} t_{2m} = \lim_{m\to\infty} P(Y_{2m}\le m) = \lim_{n\to\infty} P\left(\frac{\sqrt{2m}\left(\frac{Y_{2m}}{2m}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}\le 0\right) = \Phi(0)=\frac{1}{2}$$

$$\lim_{m\to\infty} t_{2m+1} = \lim_{m\to\infty} P(Y_{2m+1}\le m) = \lim_{m\to\infty} P\left(\frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}} \le \frac{\sqrt{2m+1}\left(\frac{m}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}\right)$$

$$= \lim_{m\to\infty} P\Bigg(\underbrace{\underset{\substack{\downarrow d \\ Z\sim N(0,1)}}{\frac{\sqrt{2m+1}\left(\frac{Y_{2m+1}}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}} - \underset{\substack{\downarrow \\ 0}}{\frac{\sqrt{2m+1}\left(\frac{m}{2m+1}-\frac{1}{2}\right)}{\sqrt{\frac{1}{4}}}}}_{\xrightarrow{d} Z\sim N(0,1)} \le 0\Bigg) = \Phi(0) = \frac{1}{2}.$$

$$\lim_{m\to\infty} t_{2m} = \lim_{m\to\infty} t_{2m+1} = \frac{1}{2} \Rightarrow \lim_{m\to\infty} t_m = \frac{1}{2}.$$

$\boxed{\text{Problem No. 12}}$ (a)

$$0 \le |\bar{X}_n| \le \frac{|X_1| + \dots + |X_n|}{n} \le \max\{|X_1|, \dots, |X_n|\} \xrightarrow{P} 0, \text{ as } n\to\infty$$

$$\Rightarrow |\bar{X}_n| \xrightarrow{P} 0, \text{ i.e. } \bar{X}_n \xrightarrow{P} 0 \qquad (\text{Problem } 6(5)),$$

Converse may not be true. To see a counter example, let $X_1, X_2, \dots$ be iid $U(-1, 0)$. Then $E(X_1) = -\frac{1}{2}$, $\bar{X}_n \xrightarrow{P} -\frac{1}{2}$ (WLLN)

Let $S_n = \max\{X_1, \dots, X_n\}$. Then, for any $\varepsilon > 0$

$$P(|S_n| > \varepsilon) = P(-S_n > \varepsilon) = P(S_n < -\varepsilon) = P(X_i < -\varepsilon, i = 1 \dots n)$$

$$= \prod_{i=1}^{n} P(X_i < -\varepsilon) = (P(X_1 < -\varepsilon))^n = \begin{cases} 0, & \text{if } \varepsilon > 1 \\ (1-\varepsilon)^n & \text{if } 0 < \varepsilon \le 1 \end{cases}$$

$\to 0$, as $n \to \infty$

$$\Rightarrow S_n \xrightarrow{P} 0.$$

(b) Let $Y_i = -\ln X_i$, $i = 1, 2, \dots$ Then $Y_1, Y_2, \dots$ are iid Exp(1) r.v.s ($E(Y_1) = 1$, $Var(Y_1) = 1$). By WLLN

$$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{P} E(Y_1) = 1$$

$$\Rightarrow -\ln Z_n = -\frac{1}{n}\sum_{i=1}^{n} \ln X_i = \bar{Y} \xrightarrow{P} 1$$

$$\Rightarrow \ln Z_n \xrightarrow{P} -1 \Rightarrow e^{\ln Z_n} \xrightarrow{P} e^{-1}$$ ($g(x) = e^x$, $x \in \mathbb{R}$ is a continuous function)

$$\Rightarrow Z_n \xrightarrow{P} e^{-1}.$$

$\boxed{\text{Problem No. 13}}$ (a) $$M_{T_n}(t) = E(e^{tT_n}) = \prod_{i=1}^{n} E(e^{tE_i}) = \prod_{i=1}^{n}(1-t)^{-1}$$

$$= (1-t)^{-n}, \quad t < 1$$

mgf of Gamma(n, 1)

$$\Rightarrow T_n \sim \text{Gamma}(n, 1)$$

(b) $E(E_1) = 1$ and $Var(E_1) = 1$. By CLT

$$\sqrt{n}\left(\frac{T_n}{n} - 1\right) \xrightarrow{d} Z \sim N(0,1)$$

$$\Rightarrow \lim_{n\to\infty} P\left(\sqrt{n}\left(\frac{T_n}{n}-1\right) \le \lambda\right) = \Phi(\lambda), \quad \forall \lambda \in \mathbb{R}.$$

$$\Rightarrow \lim_{n\to\infty} P(T_n \le n + \lambda\sqrt{n}) = \int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\Rightarrow \lim_{n\to\infty} \int_0^{n+\lambda\sqrt{n}} \frac{e^{-t} t^{n-1}}{\Gamma n} dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-t^2/2} dt, \quad \forall \lambda \in \mathbb{R} \quad (T_n \sim Gamma(n, 1) \text{ by (a)})$$

(c) For large n, (using (b)) we have

$$\int_0^{n+\lambda\sqrt{n}} \frac{e^{-t} t^{n-1}}{\Gamma n} dt \simeq \underbrace{\int_{-\infty}^{\lambda} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt}_{\text{smooth fun of } \lambda} \quad \forall \lambda \in \mathbb{R}$$

Taking derivatives on both side we get

$$\frac{e^{-(n+\lambda\sqrt{n})} (n+\lambda\sqrt{n})^{n-1}}{\Gamma n} \times \sqrt{n} = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{2}}, \quad \forall \lambda \in \mathbb{R}$$

For $\lambda = 0$, we get

$$\frac{e^{-n} n^{n-\frac{1}{2}}}{\Gamma n} \simeq \frac{1}{\sqrt{2\pi}} \Rightarrow \Gamma n \simeq \sqrt{2\pi}\, e^{-n} n^{n-\frac{1}{2}}.$$

$\boxed{\text{Problem No. 14}}$ (a) The jt pdf of (X_1, X_2) is

$$f_{X_1 X_2}(x_1, x_2) = \frac{1}{\pi^2} \cdot \frac{1}{(1+x_1^2)(1+x_2^2)}, \quad -\infty < x_i < \infty, \ i = 1, 2.$$

For $\alpha \in (0,1)$, define $Y = \alpha X_1 + (1-\alpha) X_2$, $Z = X_1$.

$$\begin{aligned} y &= \alpha x_1 + (1-\alpha) x_2 \\ z &= x_1 \end{aligned} \Rightarrow \begin{aligned} x_1 &= z \\ x_2 &= \frac{y - \alpha z}{1-\alpha} \end{aligned} \qquad J = \begin{vmatrix} 0 & 1 \\ \frac{1}{1-\alpha} & -\frac{\alpha}{1-\alpha} \end{vmatrix} = -\frac{1}{1-\alpha}$$

Jt pdf of (Y, Z) is

$$f_{YZ}(y, z) = \frac{1-\alpha}{\pi^2} \cdot \frac{1}{(1+z^2)((1-\alpha)^2 + (\alpha z - y)^2)}, \quad -\infty < y, z < \infty.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{Y,Z}(y, z)\, dz = \frac{1-\alpha}{\pi^2} \int_{-\infty}^{\infty} \frac{1}{(1+z^2)((1-\alpha)^2+(\alpha z-y)^2)}\, dz, \quad -\infty<y<\infty.$$

Let $\frac{1}{(1+z^2)((1-\alpha)^2+(\alpha z-y)^2)} = \frac{2Az}{1+z^2} + \frac{B}{1+z^2} + \frac{2\alpha(\alpha z-y)C}{(1-\alpha)^2+(\alpha z-y)^2} + \frac{D}{(1-\alpha)^2+(\alpha z-y)^2}$

$$\Leftrightarrow\ 2Az\left[\alpha^2 z^2 - 2\alpha y z + (1-\alpha)^2 + y^2\right] + B\left(\alpha^2 z^2 - 2\alpha y z + (1-\alpha)^2 + y^2\right) + 2\alpha(\alpha z^3 - yz^2 + \alpha z - y)C + D(z^2+1) = 1$$

$$\Rightarrow\ 2\alpha^2(A+C)z^3 + \left[-4\alpha y A + B\alpha^2 - 2\alpha y C + D\right]z^2 + \left[2((1-\alpha)^2+y^2)A - 2\alpha y B + 2\alpha^2 C\right]z + \left[(1-\alpha)^2+y^2\right]B - 2\alpha y C + D = 1$$

$\Rightarrow\ 2\alpha^2(A+C) = 0 \quad \Rightarrow \quad \boxed{C = -A} \ \ldots (I)$

$-4\alpha y A + B\alpha^2 - 2\alpha y C + D = 0 \quad \Rightarrow \quad \boxed{D = 2\alpha y A - B\alpha^2} \ldots (II)$

$2((1-\alpha)^2+y^2)A - 2\alpha y B + 2\alpha^2 C = 0 \quad \Rightarrow \quad \boxed{\alpha y B = (1-2\alpha+y^2)A} \ldots (III)$

$((1-\alpha)^2+y^2)B - 2\alpha y C + D = 1 \quad \Rightarrow \quad \boxed{D = 1 - 2\alpha y A - ((1-\alpha)^2+y^2)B} \ldots (IV)$

(II) and (IV) give

$2\alpha y A - B\alpha^2 = 1 - 2\alpha y A - B\left[(1-\alpha)^2+y^2\right]$

$\Rightarrow \quad \boxed{B(1-2\alpha+y^2) = 1 - 4\alpha y A} \ \ldots (V)$

(III) and (V) give

$\frac{1-2\alpha+y^2}{\alpha y} A = \frac{1-4\alpha y A}{1-2\alpha+y^2}$

$\Rightarrow \quad \boxed{A = \frac{\alpha y}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2}}$

Putting the value of A in (III) we get

$\boxed{B = \frac{1-2\alpha+y^2}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2}}$; $\boxed{C = \frac{-\alpha y}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2} = -A}$

Using II

$D = \frac{2\alpha^2 y^2}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2} - \frac{(1-2\alpha+y^2)\alpha^2}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2}$

$\boxed{D = \frac{\alpha^2(y^2+2\alpha-1)}{(1-2\alpha+y^2)^2 + 4\alpha^2 y^2}}$

Thus

$$f_Y(y) = \frac{1-\alpha}{\pi^2}\Big[\left\{ A\ln(1+z^2) + C\ln\{(1-\alpha)^2 + (\alpha z - y)^2\}\right\}_{z=-\infty}^{\infty}$$
$$+ B\pi + D\frac{1-\alpha}{\alpha}\int_{-\infty}^{\infty} \frac{dt}{(1-\alpha)^2 + (1-\alpha)^2 t^2}\Big]$$

$$= \frac{1-\alpha}{\pi^2}\Big[A\left\{\ln \frac{1+z^2}{(1-\alpha)^2+(\alpha z-y)^2}\right\}_{z=-\infty}^{\infty} + B\pi + \frac{D\pi}{\alpha(1-\alpha)}\Big]$$

$$= \frac{1}{\pi}\Big[(1-\alpha)B + \frac{D}{\alpha}\Big]$$

$$(1-\alpha)D + \frac{D}{\alpha} = \frac{(1-\alpha)(1-2\alpha+y^2)}{(1-2\alpha+y^2)^2+4\alpha^2y^2} + \frac{\alpha(y^2+2\alpha-1)}{(1-2\alpha+y^2)^2+4\alpha^2y^2}$$

$$= \frac{y^2+4\alpha^2-4\alpha+1}{y^4+(4\alpha^2-4\alpha+2)y^2+(1-2\alpha)^2}$$

$$= \frac{y^2+(2\alpha-1)^2}{[y^2+(2\alpha-1)^2][y^2+1]} = \frac{1}{y^2+1}.$$

Thus $f_Y(y) = \frac{1}{\pi}\cdot\frac{1}{1+y^2}$, $-\infty < y < \infty$.

(b) By (a) the result is true for $n=2$ (taking $\alpha = \frac{1}{2}$). Now suppose that the result is true for some $n = m \in \{2, 3, \dots\}$

Then

$$\bar{X}_{m+1} = \frac{1}{m+1}\sum_{i=1}^{m+1} X_i = \frac{m}{m+1}\bar{X}_m + \frac{1}{m+1}X_{m+1} = \alpha\underbrace{\bar{X}_m}_{C(0,1)} + (1-\alpha)\underbrace{X_{m+1}}_{C(0,1)}$$

Independent

Thus, by (a)

$$\bar{X}_{m+1} \overset{d}{=} X_1.$$

(c) $\bar{X}_n \overset{d}{=} X_1 \xrightarrow{d} X_1$ as $n\to\infty$

$\Rightarrow$ $\bar{X}_n$ does not converge in probability to any constant, it converges in distribution to $X_1 \sim C(0,1)$. In fact the distribution of $\bar{X}_n$ does not depend on n.

$\boxed{\text{Problem No. 15}}$ Let $T_1, T_2, \ldots$ be iid Poisson(4) rvs; $E(T_1) = Var(T_1) = 4$. Also $\sum_{i=1}^{n} T_i \sim$ Poisson$(4n)$, i.e

$X_n \overset{d}{=} \sum_{i=1}^{n} T_i$ and $Y_n \overset{d}{=} \bar{T}_n$, where $\bar{T}_n = \frac{1}{n}\sum_{i=1}^{n} T_i$.

(a) By WLLN $\bar{T}_n \xrightarrow{p} 4$, i.e. $Y_n \xrightarrow{p} 4$

(b) $Y_n \xrightarrow{p} 4 \Rightarrow Y_n^2 \xrightarrow{p} 4^2 = 16$ & $\sqrt{Y_n} \xrightarrow{p} \sqrt{4} = 2$ ($g_1(x) = x^2$, $x \in \mathbb{R}$ and $g_2(x) = \sqrt{x}$, $x > 0$ are continuous fns)

$\Rightarrow Y_n^2 + \sqrt{Y_n} \xrightarrow{p} 16 + 2 = 18$

(c) $$\frac{n^2 Y_n^2 + n Y_n}{n Y_n + n^2} = \frac{Y_n^2 + \frac{1}{n} Y_n}{\frac{Y_n}{n} + 1} \xrightarrow{p} \frac{16 + 0 \times 4}{0 \times 4 + 1} = 16$$

$\boxed{\text{Problem No. 16}}$ By the CLT

$Z_n \xrightarrow{d} Z \sim N(0,1)$

(a) $Y_n \xrightarrow{p} 4 \Rightarrow \frac{Y_n}{4} \xrightarrow{p} 1 \Rightarrow \frac{4 Z_n}{Y_n} = \frac{Z_n}{Y_n/4} \xrightarrow{d} \frac{Z}{1} = Z \sim N(0,1)$

$$\frac{16 Z_n^2}{Y_n^2} = \frac{Z_n^2}{\left(\frac{Y_n}{4}\right)^2} \xrightarrow{d} \frac{Z^2}{1} = \chi^2_1.$$

$\left\{ T_n \xrightarrow{d} T,\ S_n \xrightarrow{p} a \Rightarrow S_n T_n \xrightarrow{d} aT;\quad T_n \xrightarrow{d} T,\ g(\cdot) \text{ is cont} \Rightarrow g(T_n) \xrightarrow{d} g(T) \right\}$

$$\frac{4n + Y_n}{n Y_n + Y_n^2} = \frac{4 + \frac{Y_n}{n}}{Y_n + \frac{Y_n^2}{n}} \xrightarrow{p} \frac{4 + 0 \times 4}{4 + 0 \times 4^2} = 1$$

$$\Rightarrow \frac{4n + Y_n}{n Y_n + Y_n^2} Z_n \xrightarrow{d} 1 \times Z = Z \sim N(0,1).$$

(b) Here we will require the assumption that $\exists n_0$ s.t. $P_\mu(\bar{X}_n > 0) = 1$, $\forall n \geq n_0$.

$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Z \sim N(0,1)$

Using Delta-method with $g(x) = \ln x$, $x > 0$, we get

$\sqrt{n}(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} g'(\mu) Z \sim N(0, (g'(\mu))^2)$

$\sqrt{n}(\ln \bar{X}_n - \ln \mu) \xrightarrow{d} N(0, \frac{1}{\mu^2})$

(c) $$\frac{n^\delta (\bar{X}_n - \mu)}{\sigma} = n^{\delta - \frac{1}{2}} \times \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{p} 0 \times Z = 0, \text{ for } \delta < 0.5.$$

Thus for $\delta < \frac{1}{2}$, $\frac{n^\delta (\bar{X}_n - \mu)}{\sigma} \xrightarrow{p} 0$ and

$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0,1)$.

(d) (i) Let $g(x) = x^2$, $x \in \mathbb{R}$, $g'(x) = 2x$

Case I $\mu \neq 0$, so that $g'(\mu) = 2\mu \neq 0$.

Using Delta-method with $g(x) = x^2$, $x \in \mathbb{R}$,

$\sqrt{n}\,(g(\bar{X}_n) - g(\mu)) \xrightarrow{d} \sigma g'(\mu) Z \sim N(0, 4\mu^2\sigma^2)$

$\sqrt{n}\,(\bar{X}_n^2 - \mu^2) \xrightarrow{d} N(0, 4\mu^2\sigma^2)$

Case II $\mu = 0$, so that $g'(\mu) = 2\mu = 0$.

$\sqrt{n}\,\bar{X}_n \xrightarrow{d} \sigma Z \sim N(0, \sigma^2)$, $\bar{X}_n \xrightarrow{p} 0$

$\sqrt{n}\,(\bar{X}_n^2 - \mu^2) = \sqrt{n}\,\bar{X}_n^2 \quad \bar{X}_n \times \sqrt{n}\,\bar{X}_n \xrightarrow{p} 0 \times Z = 0.$

(ii) $\sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{d} \sigma Z \sim N(0, \sigma^2)$

$(\sqrt{n}\,(\bar{X}_n - \mu))^2 \xrightarrow{d} \sigma^2 Z^2 \sim \sigma^2 \chi_1^2$

$n(\bar{X}_n - \mu)^2 \xrightarrow{d} \sigma^2 Z^2 \sim \sigma^2 \chi_1^2.$

(iii) $\sqrt{n}\,(\bar{X}_n - \mu)^2 = \frac{1}{\sqrt{n}} \times n(\bar{X}_n - \mu)^2 \xrightarrow{p} 0 \times \sigma^2 \chi_1^2 = 0.$

$\boxed{\text{Problem No. 17}}$

$E(X_i) = \theta$, $Var(X_i) = \theta^2$. By the CLT

$\frac{\sqrt{n}\,(\bar{X}_n - \theta)}{\theta} \xrightarrow{d} Z \sim N(0, 1)$

$\sqrt{n}\,(\bar{X}_n - \theta) \xrightarrow{d} \theta Z \sim N(0, \theta^2)$

Using Delta-method with $g(x) = \frac{1}{x}$, $x > 0$ (so that $g'(x) = -\frac{1}{x^2}$, $x > 0$).

$\sqrt{n}\,(g(\bar{X}_n) - g(\theta)) \xrightarrow{d} g'(\theta)\theta Z = -\frac{Z}{\theta} \sim N(0, \frac{1}{\theta^2})$

$\sqrt{n}\,(\frac{1}{\bar{X}_n} - \frac{1}{\theta}) \xrightarrow{d} N(0, \frac{1}{\theta^2}).$

$\boxed{\text{Problem No. 18}}$

Let $Y_i = -\ln X_i$, $i = 1, 2, \ldots$. Then $Y_1, Y_2, \ldots$ are iid Exp(1) rvs ($E(Y_1) = 1$, $Var(Y_1) = 1$). By WLLN & CLT

$\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i \xrightarrow{p} 1$ and $\sqrt{n}\,(\bar{Y}_n - 1) \xrightarrow{d} Z \sim N(0, 1)$

$\sqrt{n}\,(-\ln G_n - 1) \xrightarrow{d} Z \sim N(0, 1)$

Apply Delta-method with $g(x) = e^{-x}$, $x \in \mathbb{R}$, we get ($g'(x) = -e^{-x}$)

$\sqrt{n}\,(g(-\ln G_n) - g(1)) \xrightarrow{d} e^{-1} Z \sim N(0, \frac{1}{e^2})$

$\sqrt{n}\,(G_n - \frac{1}{e}) \xrightarrow{d} N(0, \frac{1}{e^2})$, $\sigma^2 = \frac{1}{e^2}$.

Problem No. 19

$$Q_n = \frac{\sum_{i=1}^{n}(X_i-\bar{X})(Y_i-\bar{Y})}{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)(Y_i-\upsilon) - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu)$$

$$= \frac{n}{n-1}\bar{Z} - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu),$$

where $Z_i = (X_i-\mu)(Y_i-\upsilon)$, $i=1,2,\ldots$, $\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i$, $Z_1, Z_2, \ldots$ are iid with $E(Z_1) = \sigma\tau\rho$ and $Var(Z_1) = E((X_1-\mu)^2(Y_1-\upsilon)^2) - E((X_1-\mu)(Y_1-\upsilon))^2$

$$= \sigma^2\tau^2\delta - \sigma^2\tau^2\rho^2$$

$$= \sigma^2\tau^2(\delta-\rho^2).$$

(a) By WLLN

$\bar{Z} \xrightarrow{p} \sigma\tau\rho$, $\bar{X}-\mu \xrightarrow{p} 0$, $\bar{Y}-\mu \xrightarrow{p} 0$.

Thus

$Q_n \xrightarrow{p} 1\times\sigma\tau\rho - 1\times 0\times 0 = \sigma\tau\rho$

Also $S_n^2 \xrightarrow{p} \sigma^2$ and $T_n^2 \xrightarrow{p} \tau^2$ (see lecture notes)

$\Rightarrow S_n \xrightarrow{p} \sigma$ and $T_n \xrightarrow{p} \tau$

$\Rightarrow R_n = \frac{Q_n}{S_n T_n} \xrightarrow{p} \frac{\sigma\tau\rho}{\sigma\tau} = \rho.$

(b) $\sqrt{n}(Q_n - \rho\sigma\tau) = \sqrt{n}\left(\frac{n}{n-1}\bar{Z} - \frac{n}{n-1}(\bar{X}-\mu)(\bar{Y}-\mu) - \rho\sigma\tau\right)$

$$= \frac{n}{n-1}\sqrt{n}(\bar{Z}-\rho\sigma\tau) + \frac{\sqrt{n}}{n-1}\rho\sigma\tau - \frac{n}{n-1}(\sqrt{n}(\bar{X}-\mu))(\bar{Y}-\mu)$$

By CLT and WLLN, $\sqrt{n}(\bar{Z}-\rho\sigma\tau) \xrightarrow{d} U \sim N(0, \sigma^2\tau^2(\delta-\rho^2))$, $\sqrt{n}(\bar{X}-\mu) \xrightarrow{d} Z \sim N(0,1)$, $\bar{Y}-\mu \xrightarrow{p} 0$. Thus

$\sqrt{n}(Q_n-\rho\sigma\tau) \xrightarrow{d} 1\times U + 0\times\rho\sigma\tau - 1\times Z\times 0 = U \sim N(0, \sigma^2\tau^2(\delta-\rho^2)).$