Module 2
Random Variables

2.1. Random Variables and their distribution functions

Let $(\Omega, \mathcal{F}, P)$ be a given probability space. In some situations we may not be directly interested in the sample space Ω; rather we may be interested in some numerical aspect of Ω.

Example 2.1.1 A fair coin (head and tail are equally likely) is tossed three times independently. Then

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

and

$$P(\{\omega\}) = \frac{1}{8}, \quad \forall\, \omega \in \Omega.$$

Suppose that we are interested in number of heads in three tosses; i.e. we are interested in the function $X: \Omega \to \mathbb{R}$ defined by

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = TTT \\ 1, & \text{if } \omega \in \{HTT, THT, TTH\} \\ 2, & \text{if } \omega \in \{HHT, HTH, THH\} \\ 3, & \text{if } \omega = HHH \end{cases}$$

Clearly the values assumed by X are random with

$$Pr(X=0) = Pr(X=3) = \frac{1}{8}$$

and $$Pr(X=1) = Pr(X=2) = \frac{3}{8}.$$

Here

$$Pr(X \in \{0, 1, 2, 3\}) = 1.$$

Definition 2.1.1. Let $(\Omega, \mathcal{F}, P)$ be a given probability function. A real valued function $X: \Omega \to \mathbb{R}$ (defined on sample space Ω) is called a random variable (r.v.).

For a function $Y: \Omega \to \mathbb{R}$ and $A \subseteq \mathbb{R}$, define

$$Y^{-1}(A) = \{\omega \in \Omega : Y(\omega) \in A\}$$

Then it is straightforward to prove the following result.

Proposition 2.1.1. Let $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}$, and $A_\alpha \subseteq \mathbb{R}$, $\alpha \in \Lambda$, where Λ is an arbitrary index set. Let $Y: \Omega \to \mathbb{R}$ be a given function. Then

(a) If $A \cap B = \phi$, then $Y^{-1}(A) \cap Y^{-1}(B) = \phi$

(b) $Y^{-1}(A^c) = (Y^{-1}(A))^c$ (i.e. $Y^{-1}(\mathbb{R} - A) = Y^{-1}(\mathbb{R}) - Y^{-1}(A) = \Omega - Y^{-1}(A)$)

(c) $Y^{-1}\left(\bigcup_{\alpha \in \Lambda} A_\alpha\right) = \bigcup_{\alpha \in \Lambda} Y^{-1}(A_\alpha)$

(d) $Y^{-1}\left(\bigcap_{\alpha \in \Lambda} A_\alpha\right) = \bigcap_{\alpha \in \Lambda} Y^{-1}(A_\alpha)$

For a probability space $(\Omega, \mathcal{F}, P)$ and a r.v. $X: \Omega \to \mathbb{R}$, note that $\forall\ A \subseteq \mathbb{R}$

$$X^{-1}(A) = \{\omega \in \Omega : X(\omega) \in A\} \in \mathcal{F}.$$

Thus one can define a set function $P_X: \mathcal{B} \to [0, 1]$ by

$$P_X(B) = P(X^{-1}(B))$$
$$= P(\{\omega \in \Omega : X(\omega) \in B\}), \quad B \in \mathcal{B},$$

where $\mathcal{B}$ is some class of subsets of $\mathbb{R}$. As before, for all practical purposes we will take $\mathcal{B}$ to be power set of $\mathbb{R}$. We will simply write

$$P_X(B) = P(\{\omega \in \Omega : X(\omega) \in B\})$$
$$= Pr(X \in B), \quad B \in \mathcal{B}$$

We have the following scenarios.

$$(\Omega, \mathcal{F}, P)$$
$$\Big\downarrow X \quad \text{(random variable)}$$
$$(\mathbb{R}, \mathcal{B}, P_X)$$

Theorem 2.1.1. (Induced Probability Space/Measure) $(\mathbb{R}, \mathcal{B}, P_X)$ (as defined above) is a probability space, i.e. $P_X(\cdot)$ is a probability function defined on $\mathcal{B}$.

Proof.
$$P_X(\mathbb{R}) = \Pr(X \in \mathbb{R}) = P(X^{-1}(\mathbb{R})) = P(\Omega) = 1$$

Also, for any $B \in \mathcal{B}$,
$$P_X(B) = P(X^{-1}(B)) \geq 0.$$

Let $\{B_n\}_{n\geq 1}$ be a collection of mutually exclusive events in $\mathcal{B}$. Then

$$\begin{aligned} P_X\left(\bigcup_{n=1}^{\infty} B_n\right) &= P\left(X^{-1}\left(\bigcup_{n\geq 1}^{\infty} B_n\right)\right) \\ &= P\left(\bigcup_{n\geq 1}^{\infty} X^{-1}(B_n)\right) && \text{(Proposition 2.1.1.(c))} \\ &= \sum_{n\geq 1}^{\infty} P(X^{-1}(B_n)) && \text{(P is a prob. measure and using Proposition 2.1.1(a))} \\ &= \sum_{n\geq 1}^{\infty} P_X(B_n). \end{aligned}$$

Definition 2.1.2. The probability function P_X defined above is called the probability function/measure induced by r.v. X and $(\mathbb{R}, \mathcal{B}, P_X)$ is called the probability space induced by r.v. X.

The induced probability measure P_X describes the random behaviour of X.

Example 2.1.2. Consider the probability space defined in Example 2.1.1 where

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\},$$

$$P(\{\omega\}) = \frac{1}{8}, \quad \forall \omega \in \Omega$$

and $X: \Omega \to \mathbb{R}$ (number of heads in three tosses) is defined by

$$X(\omega) = \begin{cases} 0, & \text{if } \omega = TTT \\ 1, & \text{if } \omega \in \{HTT, THT, TTH\} \\ 2, & \text{if } \omega \in \{HHT, HTH, THH\} \\ 3, & \text{if } \omega \in HHH. \end{cases}$$

Obviously $X: \Omega \to \mathbb{R}$ is a r.v. with induced probability space given by $(\mathbb{R}, \mathcal{B}, P_X)$, where

$P_X(\{0\}) = P(\{TTT\}) = \frac{1}{8}$

$P_X(\{1\}) = P(\{HTT, THT, TTH\}) = \frac{3}{8}$

$P_X(\{2\}) = P(\{HHT, HTH, THH\}) = \frac{3}{8}$

$P_X(\{3\}) = P(\{HHH\}) = \frac{1}{8}$

and for any $B \in \mathcal{B}$

$$
\begin{aligned}
P_X(B) &= P(X^{-1}(B)) \\
&= P(\{\omega \in \Omega : X(\omega) \in B\}) \\
&= \sum_{\substack{i=0 \\ i \in B}}^{3} P_X(\{i\}).
\end{aligned}
$$

$\boxed{\text{Definition 2.1.3.}}$ Let X be a r.v. defined on probability space induced by X. and let $(\mathbb{R}, \mathcal{B}, P_X)$ denote the probability space induced by X. Define the function $F_X : \mathbb{R} \to \mathbb{R}$ by

$$
\begin{aligned}
F_X(x) &= P_r(X \le x) \\
&= P_X((-\infty, x]), \quad x \in \mathbb{R}.
\end{aligned}
$$

The function F_X is called the cumulative distribution function (c.d.f.) or simply the distribution function (d.f.) of r.v. X.

Whenever there is no ambiguity we will drop subscript X in F_X to represent d.f. of a r.v. by F (or G, H, ...).

It can be shown (in advanced courses) that the c.d.f. $F_X(\cdot)$ of a r.v. X determines the induced probability measure $P_X(\cdot)$ uniquely. Thus to study the random behavior of r.v. X it suffices to study its d.f. F

Example 2.1.3. Let us revisit Examples 2.1.1 & 2.1.2, so that

$$Pr(X=0) = P_X(\{0\}) = \frac{1}{8}$$

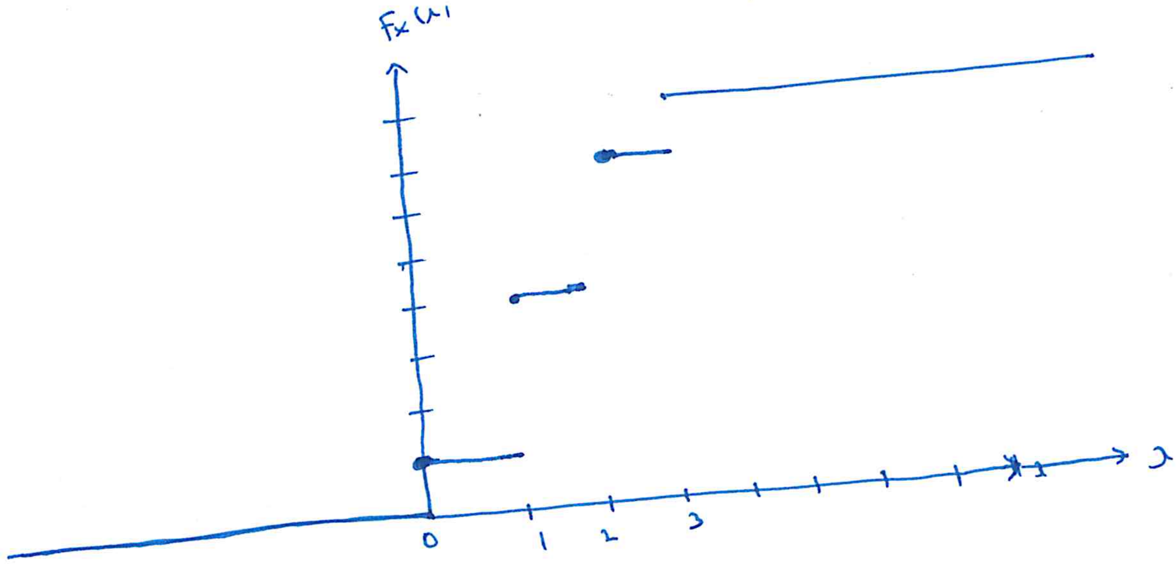
$$Pr(X=1) = P_X(\{1\}) = \frac{3}{8}$$

$$Pr(X=2) = P_X(\{2\}) = \frac{3}{8}$$

and $Pr(X=3) = P_X(\{3\}) = \frac{1}{8}$.

Then the d.f. of X is

$$F_X(x) = Pr(X \le x) = \begin{cases} 0, & x<0 \\ \frac{1}{8} & 0 \le x < 1 \\ \frac{4}{8} = \frac{1}{2} & 1 \le x < 2 \\ 7/8 & 2 \le x < 3 \\ 1 & x \geq 3 \end{cases}$$



Theorem 2.1.2. Let $F(\cdot)$ be the c.d.f. of a r.v. X defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $(\mathbb{R}, \mathcal{B}, P_X)$ be the probability space induced by X. Then

(a) F is non-decreasing

(b) F is right continuous

(c) $F(-\infty) = \lim_{n \uparrow \infty} F(-n) = 0$ and $F_X(\infty) = \lim_{n\uparrow\infty} F_X(n) = 1$.

Conversely any function $G(\cdot)$ satisfying properties (a)-(c) is a d.f. of some r.v. Y defined on a probability space $(\Omega^*, \mathcal{F}^*, P^*)$.

$\boxed{\text{Proof.}}$ (a) Let $-\infty < x < y < \infty$. Then

$$(-\infty, x] \subseteq (-\infty, y]$$
$$\Rightarrow P_X((-\infty, x]) \leq P_X((-\infty, y])$$
$$\Rightarrow F(x) \leq F(y).$$

(b) Since F is monotone and bounded below (by 0) $\lim_{h \downarrow 0} F(x+h)$ $\overset{= F(x+)}{\wedge}$ exists, $\forall x \in \mathbb{R}$

Therefore

$$\begin{aligned} F(x+) &= \lim_{h \downarrow 0} F(x+h) \\ &= \lim_{n \to \infty} F\left(x + \frac{1}{n}\right) \\ &= \lim_{n \to \infty} P_X\left(\left(-\infty, x + \frac{1}{n}\right]\right). \end{aligned}$$

Let $A_n = \left(-\infty, x + \frac{1}{n}\right]$, $n = 1, 2, \ldots$ Then $A_n \downarrow$ and $\bigcap_{n=1}^{\infty} \left(-\infty, x + \frac{1}{n}\right] = (-\infty, x]$.

Thus

$$\begin{aligned} F(x+) &= P_X\left(\bigcap_{n \geq 1}^{\infty} \left(-\infty, x + \frac{1}{n}\right]\right) \\ &= P_X((-\infty, x]) \\ &= F(x) \end{aligned}$$

(c) $$\begin{aligned} F(-\infty) &= \lim_{n \to \infty} F_X(-n) \\ &= \lim_{n \to \infty} P_X((-\infty, -n]) \\ &= P_X\left(\bigcap_{n \geq 1}^{\infty} (-\infty, -n]\right) \quad ((-\infty, -n] \downarrow) \\ &= P_X(\phi) \quad \left(\bigcap_{n=1}^{\infty} (-\infty, -n] = \phi\right) \\ &= 0 \end{aligned}$$

(d) $$\begin{aligned} F(+\infty) &= \lim_{n \to \infty} F_X(n) \\ &= \lim_{n \to \infty} P_X((-\infty, n]) \quad ((-\infty, n] \uparrow) \\ &= P_X\left(\bigcup_{n \geq 1}^{\infty} (-\infty, n]\right) \quad \left(\bigcup_{n \geq 1}^{\infty} (-\infty, n] = \mathbb{R}\right) \\ &= P_X(\mathbb{R}) \\ &= 1. \end{aligned}$$

$\boxed{\text{Remark 2.1.1.}}$ (a) Since any distribution function is monotone and bounded above (by 1) $\lim_{h \downarrow 0} F(x-h) = F(x-)$ exists $\forall x \in \mathbb{R}$.

Moreover

$$F(x-) = \lim_{h \downarrow 0} F(x-h)$$
$$= \lim_{n \to \infty} F(x-\tfrac{1}{n})$$
$$= \lim_{n \to \infty} P_X((x-\tfrac{1}{n}])$$
$$= P_X\left(\bigcup_{n=1}^{\infty} (-\infty, x-\tfrac{1}{n}]\right) \qquad ((-\infty, x-\tfrac{1}{n}] \uparrow)$$
$$= P_X((-\infty, x))$$
$$= Pr(X < x)$$

(b) Since, for any $x \in \mathbb{R}$, $F(x+)$ and $F(x-)$ exist F has only jump discontinuities ($F(x) = F(x+) > F(x-)$). Moreover since any d.f. F is monotone it has atmost countable number of discontinuities.

(c) A distribution function F is continuous at $a \in \mathbb{R}$ iff

$$F(a) = F(a-).$$

(d) For any $a \in \mathbb{R}$

$$Pr(X=a) = Pr(X \le a) - Pr(X<a)$$
$$= F(a) - F(a-)$$

Thus a d.f. F is continuous at $a \in \mathbb{R}$ iff

$$Pr(X=a) = F(a) - F(a-) = 0.$$

(e) We have, for $-\infty < a < b < \infty$

$$Pr(a < X \le b) = Pr(X \le b) - Pr(X \le a) = F(b) - F(a)$$
$$Pr(a < X < b) = Pr(X < b) - Pr(X \le a) = F(b-) - F(a)$$
$$Pr(a \le X < b) = Pr(X < b) - Pr(X < a) = F(b-) - F(a-)$$
$$Pr(a \le X \le b) = Pr(X \le b) - Pr(X < a) = F(b) - F(a-)$$
$$\therefore \; Pr(X < a) = F(a-)$$
and $Pr(X \ge a) = 1 - F(a-)$

(b) It can be shown (in advanced courses) that the d.f. F determines the induced probability function $P_X(\cdot)$ uniquely. Thus it suffices to study the d.f. of a r.v.

$\boxed{\text{Example 2.1.4.}}$ Consider the function $G: \mathbb{R} \to \mathbb{R}$ defined by

$$G(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{3}, & \text{if } 0 \le x < 1 \\ \frac{1}{2}, & \text{if } 1 \le x < 2 \\ \frac{2}{3}, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \ge 3 \end{cases}$$

(a) Show that G is d.f. of some r.v. Y;

(b) Find $\Pr(X = a)$ for various values of $a \in \mathbb{R}$;

(c) Find $\Pr(X < 3)$, $\Pr(X \ge \frac{1}{2})$, $\Pr(2 < X \le 4)$, $\Pr(1 \le X < 2)$, $\Pr(2 \le X \le 3)$ and $\Pr(\frac{1}{2} < X < 3)$.

$\boxed{\text{Solution}}$ (a) Clearly G is non-decreasing in $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$ and $(3, \infty)$. Moreover

$$G(0) - G(0-) = 0 \ge 0$$

$$G(1) - G(1-) = \frac{1}{2} - \frac{1}{3} > 0$$

$$G(2) - G(2-) = \frac{2}{3} - \frac{1}{2} > 0$$

$$G(3) - G(3-) = 1 - \frac{2}{3} > 0$$

It follows that G is non-decreasing.

Clearly G is continuous (and hence right continuous) on $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$ and $(3, \infty)$. Moreover

$$G(0+) - G(0) = 0 - 0 = 0$$

$$G(1+) - G(1) = \frac{1}{2} - \frac{1}{2} = 0$$

$$G(2+) - G(2) = \frac{2}{3} - \frac{2}{3} = 0$$

$$G(3) - G(3+) = 1 - 1 = 0$$

$\Rightarrow$ G is right continuous on $\mathbb{R}$

Also

$$G(-\infty) = \lim_{n\to\infty} G(-n) = 0$$
$$G(+\infty) = \lim_{n\to\infty} G(n) = 1$$

Thus G is a d.f. of some r.v. Y.

(b) The set of discontinuity points of F is
$$D = \{1, 2, 3\}$$

Thus
$$Pr(X=a) = G(a) - G(a-) = 0, \quad \forall\, a \neq 1, 2, 3$$
$$Pr(X=1) = G(1) - G(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$
$$Pr(X=2) = G(2) - G(2-) = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$
$$Pr(X=3) = G(3) - G(3-) = 1 - \frac{2}{3} = \frac{1}{3}$$

(c)
$$Pr(X<3) = G(3-) = \frac{2}{3}$$
$$Pr(X \geq \tfrac{1}{2}) = 1 - G(\tfrac{1}{2}-) = 1 - \frac{1}{6} = \frac{5}{6}$$
$$Pr(2 < X \leq 4) = G(4) - G(2) = 1 - \frac{2}{3} = \frac{1}{3}$$
$$Pr(1 \leq X < 2) = G(2-) - G(1-) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$
$$Pr(2 \leq X \leq 3) = G(3) - G(2-) = 1 - \frac{1}{2} = \frac{1}{2}$$
$$Pr(\tfrac{1}{2} < X < 3) = G(3-) - G(\tfrac{1}{2}) = \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$$

2.2. Discrete Random Variables

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X: \Omega \to \mathbb{R}$ be a r.v. with induced probability space $(\mathbb{R}, \mathcal{B}, P_X)$ and d.f. F.

Definition 2.2.1. The r.v. X is said to be a discrete r.v. if there exists a countable set S (finite or infinite) such that
$$Pr(X=x) = F(x) - F(x-) > 0 \quad \forall\, x \in S$$
and
$$Pr(X \in S) = 1.$$
The set S is called the support of r.v. X.

Remark 2.2.1. (a) If S is the support of a discrete r.v. X, then clearly

$$S = \{x \in \mathbb{R} : F(x) - F(x-) > 0\} \quad \left(\text{since } P(X \in S) = 1 \text{ and } F(x) - F(x-) > 0 \ \forall x \in S\right)$$

$= $ set of discontinuity points of F

(b) Note that if x is a discontinuity point of d.f. F then

$F(x) - F(x-) = $ size of jump of F at x.

Thus a r.v. X is of discrete type

$\Leftrightarrow$ sum of jump points of F equals 1 $\left(P(X \in S) = \sum_{x \in S} P(X = x) = \sum_{x \in S} [F(x) - F(x-)] = 1 \right)$

Example 2.2.1 In Example 2.1.4 the set of discontinuity points of F is $D = \{1, 2, 3\}$ and

$$\sum_{x \in D} [F(x) - F(x-)] = \frac{1}{6} + \frac{1}{6} + \frac{1}{3} = \frac{2}{3} < 1$$

$\Rightarrow$ Y is not a discrete r.v.

Example 2.2.2 Consider the d.f. (see Example 2.1.3)

$$G(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{1}{8}, & \text{if } 0 \le x < 1 \\ \frac{1}{2}, & \text{if } 1 \le x < 2 \\ 7/8, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \ge 3 \end{cases}$$

The set of discontinuity points of G is $D = \{0, 1, 2, 3\}$ with

$$\sum_{x \in D} [F(x) - F(x-)] = \frac{1}{8} + \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{7}{8} - \frac{1}{2}\right) + \left(1 - \frac{7}{8}\right) = 1$$

$\Rightarrow$ Y is a discrete r.v. with support $S = D = \{0, 1, 2, 3\}$.

Definition 2.2.2. Let X be a r.v. with c.d.f. F_X and support S_X.
Define the function $f_X : \mathbb{R} \to \mathbb{R}$ by

$$f_X(x) = \begin{cases} \Pr(X=x) = F_X(x) - F_X(x-), & \text{if } x \in S_X \\ 0, & \text{otherwise} \end{cases}$$

The function f_X is called the probability mass function (p.m.f.) of r.v. X.

Whenever their is no ambiguity we will drop subscript X in F_X, S_X and f_X to represent the d.f. of X by F, the support of X by S and the p.m.f. of X by f.

Remark 2.2.2. (a) Let X be a discrete r.v. with p.m.f. and d.f. F.
Let S be the support of X.
Then, for any $A \subseteq \mathbb{R}$, $\Pr(X \in A \cap S) =$

$$\Pr(X \in A) = \sum_{x \in A \cap S} f(x), \quad (A \cap S \subseteq S \text{ and thus } A \cap S \text{ is a countable set}),$$

where S is support of X. Moreover

$$F(x) = \sum_{y \in S_X \cap (-\infty, x]} f(y)$$

Moreover, for any $x \in S$, $f(x) = F(x) - F(x-)$.

(b) Clearly a d.f. determines the p.m.f. uniquely and vice-versa.
Thus it suffices to study the p.m.f. of a discrete r.v.

(c) Let X be a discrete r.v. with p.m.f. f and support S.
Then $f : \mathbb{R} \to \mathbb{R}$ satisfies

(i) $f(x) > 0 \quad \forall x \in S$

(ii) $\sum_{x \in S} f(x) = 1$.

Conversely suppose that $g : \mathbb{R} \to \mathbb{R}$ is a function such that, for some countable set T,

(i) $g(x) > 0 \quad \forall x \in T$

and (ii) $\sum_{x \in T} g(x) = 1$.

Then $g(\cdot)$ is the p.m.f. of some discrete r.v. having support T.

$\boxed{\text{Example 2.2.3.}}$ Let Y be a r.v. having d.f.

$$G(x) = \begin{cases} 0, & \text{if } x<0 \\ \frac{1}{8}, & \text{if } 0 \le x < 1 \\ \frac{1}{2}, & \text{if } 1 \le x < 2 \\ \frac{7}{8}, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \geq 3 \end{cases}$$

We have seen in Example 2.2.2 that Y is a discrete r.v. with support $S = \{0, 1, 2, 3\}$. The p.m.f. of Y is $g: \mathbb{R} \to \mathbb{R}$, where

$$g(0) = G(0) - G(0-) = \frac{1}{8} - 0 = \frac{1}{8}$$

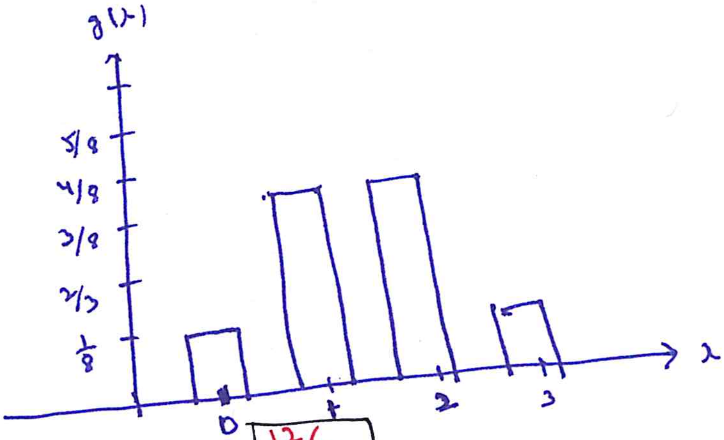
$$g(1) = G(1) - G(1-) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$$

$$g(2) = G(2) - G(2-) = \frac{7}{8} - \frac{1}{2} = \frac{3}{8}$$

and $g(3) = G(3) - G(3-) = 1 - \frac{7}{8} = \frac{1}{8}$

Thus the p.m.f. of Y is

$$g(x) = \begin{cases} \frac{1}{8}, & \text{if } x = 0, 3 \\ \frac{3}{8}, & \text{if } x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$



$\boxed{\text{Example 2.2.4.}}$ A fair die (all outcomes are equally likely) is tossed repeatedly and independently until a 6 is observed. Then X is a discrete r.v. with support $S = \{1, 2, 3, \ldots\}$,

p.m.f.

$$f(x) = Pr(X=x) = \begin{cases} \left(\frac{5}{6}\right)^{x-1} \frac{1}{6}, & \text{if } x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$$

and p.m.f.

$$\begin{cases} 0, & \text{if } x < 1 \\ \frac{1}{6}, & \text{if } 1 \le x < 2 \\ \frac{5}{36}, & \text{if } 2 \le x < 3 \\ \vdots & \\ \sum_{j=1}^{i} \left(\frac{5}{6}\right)^{j-1} \frac{1}{6} = 1 - \left(\frac{5}{6}\right)^{i}, & \text{if } i \le x < i+1 \\ \vdots & \end{cases}$$

2.3. Continuous Random Variables

Let X be a r.v. with d.f. F.

$\boxed{\text{Definition 2.3.1.}}$ The r.v. X is said to be a continuous r.v. if there exists a non-negative integral function $f: \mathbb{R} \to [0, \infty)$ such that, for any $x \in \mathbb{R}$,

$$F(x) = Pr(X \le x) = \int_{-\infty}^{x} f(t)\,dt$$

The function $f(\cdot)$ is called the probability density function (p.d.f.) of X. The support of the continuous r.v. X is the set $S = \{x \in \mathbb{R} : F(x+h) - F(x-h) > 0 \ \forall h > 0\}$.

Remark 2.3.1. (i) From the fundamental theorem of Calculus we know that the definite integral

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

is a continuous function of $\mathbb{R}$. Thus the d.f. F of any continuous r.v. X is continuous everywhere on $\mathbb{R}$. In particular

$$\Pr(X = x) = F(x) - F(x-) = 0, \quad \forall x \in \mathbb{R}$$

Generally if A is any countable subset of $\mathbb{R}$ then for any continuous r.v. X

$$\Pr(X \in S) = \sum_{x \in S} \Pr(X = x) = 0.$$

(ii) There are random variables that are neither discrete nor continuous (see Example 2.1.4). Such random variables will not be studied here.

(iii) If X is a continuous r.v. then

$$\Pr(X < x) = \Pr(X \le x) = F(x), \quad \forall x \in \mathbb{R}$$

$$\Pr(X \ge x) = 1 - \Pr(X < x) = 1 - F(x), \quad \forall x \in \mathbb{R}$$

and, for $-\infty < a < b < \infty$,

$$\Pr(a < X < b) = \Pr(a \le X < b) = \Pr(a < X \le b) = \Pr(a \le X \le b)$$
$$= F(b) - F(a)$$
$$= \int_{-\infty}^{b} f(t)\,dt - \int_{-\infty}^{a} f(t)\,dt$$
$$= \int_{a}^{b} f(t)\,dt.$$

(iv) Let $f(\cdot)$ be the p.d.f. of a continuous r.v. X and let $E \subseteq \mathbb{R}$ be any countable subset of $\mathbb{R}$. Define $g: \mathbb{R} \to [0, \infty)$ by

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{R} \cap E^c \\ c_x, & \text{if } x \in E \end{cases},$$

where $c_x \ge 0$ are arbitrary. Then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} g(t)\,dt, \quad \forall x \in \mathbb{R}$$

and, thus, g is also a p.d.f. of X. Thus the p.d.f. of a continuous r.v. is not unique.

We state the following theorem without proof.

$\boxed{\text{Theorem 2.3.1.}}$ Let X be a r.v. with d.f. F. Suppose that F is differentiable everywhere except (possibly) on a countable set E. Further suppose that

$$\int_{-\infty}^{\infty} F'(t)\,dt = 1.$$

Then X is a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} F'(x), & \text{if } x \in E^c \\ 0, & \text{if } x \in E \end{cases}.$$

$\boxed{\text{Remark 2.3.1.}}$ (a) The p.d.f. determines the d.f. uniquely. Converse is not true. However the d.f. determines the p.d.f. almost uniquely (they may vary in sets that have no length (or have zero content)). Thus it is enough to study the p.d.f. of a continuous r.v.

(b) Let X be a continuous r.v. with p.d.f. $f(\cdot)$. Then

(i) $f(x) \geq 0, \quad \forall x \in \mathbb{R}$

and (ii) $\int_{-\infty}^{\infty} f(t)\,dt = 1$.

Conversely suppose that $g: \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $g(x) \geq 0, \quad \forall x \in \mathbb{R}$

(ii) $\int_{-\infty}^{\infty} g(t)\,dt = 1$.

Then $g(\cdot)$ is the p.d.f. of some continuous r.v. having

support $T = \{x \in \mathbb{R} : \int_{x-h}^{x+h} g(t)\,dt > 0, \ \forall h > 0\}$.

$\boxed{\text{Example 2.3.1.}}$ Let X be a r.v. with d.f.

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x}{4}, & \text{if } 0 \leq x < 1 \\ \frac{x}{3}, & \text{if } 1 \leq x < 2 \\ \frac{3x}{8}, & \text{if } 2 \leq x < \frac{5}{2} \\ 1, & \text{if } x \geq 5/2 \end{cases}$$

Examine whether X is continuous r.v. or a discrete r.v. or none?

Solution Let D be the set of discontinuity points of F. Then

$$D = \{1, 2, \tfrac{5}{2}\}$$

$D \neq \phi \Rightarrow X$ is not a Continuous r.v.

$$\therefore \sum_{x \in D} [F(x) - F(x-)] = \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(\tfrac{3}{4} - \tfrac{2}{3}\right) + \left(1 - \tfrac{15}{16}\right)$$

$$= \tfrac{11}{48} < 1$$

$\Rightarrow X$ is not a discrete r.v.

Thus X is neither a discrete r.v. nor a Continuous r.v.

Example 2.3.2 Let X be a r.v. with d.f.

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{x^2}{2}, & \text{if } 0 \le x < 1 \\ \frac{x}{2}, & \text{if } 1 \le x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}$$

Show that X is a Continuous r.v. Find the p.d.f. of X. and support of X.

Solution Clearly F is Continuous everywhere. Moreover F is differentiable every where except at three (Countable) points $0, 1$ and 2 and

$$F'(x) = \begin{cases} 0, & \text{if } x < 0 \\ x, & \text{if } 0 < x < 1 \\ \frac{1}{2}, & \text{if } 1 < x < 2 \\ 0, & \text{if } x \geq 2 \end{cases}$$

Moreover

$$\int_{-\infty}^{\infty} F'(x)\,dx = \int_0^1 x\,dx + \int_1^2 \tfrac{1}{2}\,dx = 1.$$

$\Rightarrow X$ is a Continuous r.v. with p.d.f.

$$f(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ \frac{1}{2}, & \text{if } 1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

The support of X is

$$S = \{x \in \mathbb{R} : F(x+h) - F(x-h) > 0 \;\; \forall h > 0\} = [0, 2]$$

$$= \{x \in \mathbb{R} : \int_{x-h}^{x+h} f(t)\,dt > 0 \;\; \forall h > 0\}$$

Example 2.3.3. Let X be a continuous r.v. with p.d.f.

$$f(x) = \begin{cases} x^2, & \text{if } 0<x<1 \\ c e^{-x}, & \text{if } x>1 \\ 0, & \text{otherwise}, \end{cases}$$

where $c \geq 0$ is a constant.

(a) Find the value of c.

(b) Find $P(\frac{1}{2} \leq X \leq 2)$

(c) Find the support of X

(d) Find the d.f. of X.

Solution (a) We have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 x^2\,dx + \int_1^{\infty} c e^{-x}\,dx = 1$$

$$\Rightarrow \frac{1}{3} + c e^{-1} = 1$$

$$\Rightarrow c = \frac{2e}{3}$$

(b) $$P(\tfrac{1}{2} \leq X \leq 2) = \int_{1/2}^{2} f(x)\,dx$$

$$= \int_{1/2}^{1} x^2\,dx + c\int_1^2 e^{-x}\,dx$$

$$= \frac{1}{3}\left(1-\frac{1}{8}\right) + c\left(e^{-1}-e^{-2}\right)$$

$$= \frac{7}{24} + \frac{2}{3}\left(1-e^{-1}\right).$$

(c) The support of X is

$$S = \left\{x \in \mathbb{R} : \int_{x-h}^{x+h} f(t)\,dt > 0 \ \forall h>0\right\}$$

$$= [0, \infty)$$

(d) The d.f. of X is

$$F(x) = \int_{-\infty}^{x} f(t)\,dt.$$

For $x<0$, clearly $F(x)=0$. For $0 \le x < 1$,

$$F(x) = \int_0^x t^2\,dt = \frac{x^3}{3}$$

For $x \ge 1$

$$F(x) = \int_0^1 t^2\,dt + c\int_1^x e^{-t}\,dt$$
$$= \frac{1}{3} + c\,(e^{-1} - e^{-x})$$
$$= \frac{1}{3} + \frac{2}{3}\left(1 - e^{-(x-1)}\right).$$

$\boxed{\text{Remark 2.3.2.}}$ Let X be a continuous r.v. with p.d.f. $f(\cdot)$. If f is continuous at $x_0 \in \mathbb{R}$ then

$$f(x) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{x-\frac{\delta}{2}}^{x+\delta/2} f(t)\,dt$$

$\Rightarrow \Pr(x - \frac{\delta}{2} \le X \le x + \frac{\delta}{2}) \approx \delta f(x)$, for small $\delta > 0$

i.e. $\Pr(x \le X \le x + dx) \simeq f(x)\,dx$

~~Definition 2.3.2. Let X be a continuous r.v. with support S and suppose that S is an interval (may be unbounded). Further suppose that F is strictly increasing on S so that F^{-1} is well defined. Then $x = F^{-1}(y)$ iff $F(x) = y$, $y \in (0,1)$, $x \in \mathbb{R}$. For any $p \in (0,1)$ let $\xi_p \in \mathbb{R}$ be such that $F(\xi_p) = p$ (or $\Pr(X \le \xi_p) = p$) then ξ_p is called the p-th quantile of F.~~

$\boxed{\text{Definition 2.3.2.}}$ Let X be a continuous r.v. with d.f. F and support S, where S is an interval (may be unbounded). Further suppose that F is strictly increasing on S. For any $p \in (0,1)$ the p-th quantile of the d.f. F is defined to be the value $\xi_p \in S$ such that $F(\xi_p) = p$ ($\Pr(X \le \xi_p) = p$), i.e. the p-th quantile is defined by $\xi_p = F^{-1}(p)$. 0.5th-quantile is called median of F, 0.25-th quantile is called the lower quartile of F and 0.75-th quantile is called the upper quartile of F.

Example 2.3.4 Let X be a r.v. with p.d.f.

$$f(x) = \begin{cases} \frac{x}{2}, & \text{if } 0<x<2 \\ 0 & \text{otherwise} \end{cases}$$

Find the lower quartile, the median and the upper quartile of f

Solution For $0 \le x \le 2$, the d.f. of X is

$$F(x) = \int_0^x \frac{t}{2}\,dt = \frac{x^2}{4}$$

and the support of X is $[0, 2]$.

$$F(\xi_p) = p$$

$$\Rightarrow \frac{\xi_p^2}{4} = p$$

$$\Rightarrow \xi_p = 2\sqrt{p}$$

Median $\xi_{\frac{1}{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

Lower quartile $\xi_{\frac{1}{4}} = \frac{2}{2} = 1$

Upper quartile $\xi_{\frac{3}{4}} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$

Example 2.3.5. Value at risk (VaR) is a measure that quantifies the level of financial risk in investments over a specific time frame. VaR is used by firms to have the idea on amount of assets needed to cover losses. Calculation of VaR requires three parameters:

- given portfolio
- time horizon
- level of confidence

If the VaR of a portfolio of stocks is Rs. 100 million with one-day horizon and a confidence level 90%. the interpretation is that there is a 10%. chance of losses exceeding Rs. 100 million Such a loss should be anticipated about once in 10 days (because of 10%. of chances of loss).

Let

$V_0 =$ Current value of the investment

$V_1 =$ future value

Then return on investment is

$$R = \frac{V_1 - V_0}{V_0}$$

The probability distribution of R is modeled by c.d.f. F or p.d.f. f. Let the confidence level desired is $1-p$. Then VaR is the value v_0 such that

$$p = \Pr(V_0 - V_1 \geq v_0)$$

$$= F\left(-\frac{v_0}{V_0}\right)$$

i.e. $v_0 = -V_0 \xi_p$, where ξ_p is the p-th quantile of F.