MSO201a: Probability and Statistics

2019-2020: II Semester

Mid Semester Examination

Time Allowed: 120 Minutes

Maximum Marks: 50

Name:

Roll No.:

Problem No. 1:

(a) Three numbers are selected at random, without replacement, from the set $\{1, 2, \ldots, 50\}$. Find the probability that they form an arithmetic progression.

(b) In a probability space $(\Omega, \mathcal{F}, P)$, let A, B and C be pairwise independent events with $P(A \cap B) = 0.3$ and $P(B \cap C) = 0.2$. Show that $P(A \cup C) \geq \frac{11}{25}$. 4+4=8 Marks

Problem No. 2: Let X be a random variable having the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x < -1 \\ \frac{x+1}{4}, & \text{if } -1 \leq x < 0 \\ \frac{x+1}{3}, & \text{if } 0 \leq x < 1 \\ \frac{x+3}{6}, & \text{if } 1 \leq x < 2 \\ 1, & \text{if } x \geq 2 \end{cases}.$$

(a) Show that X is neither a discrete nor a continuous random variable;

(b) Decompose F as $F(x) = \alpha F_d(x) + (1-\alpha)F_c(x)$, $x \in \mathbb{R}$, where $\alpha \in [0,1]$, F_d is a distribution function of some discrete random variable and F_c is a distribution function of some continuous random variable. 3+5=8 Marks

Problem No. 3: Let X be a random variable having the probability density function

$$f(x) = \begin{cases} c|x|, & \text{if } -2 < x < 1 \\ 0, & \text{otherwise} \end{cases},$$

where c is a real constant. Find the value of c and the variance of X. Also derive the distribution function of X and hence verify that

$$E(X) = \int_0^\infty P(X > y)dy - \int_{-\infty}^0 P(X < y)dy.$$

1+3+2+2=8 Marks

Problem No. 4: Suppose that the random variable X has the distribution function

$$F(x) = \begin{cases} a + be^{x}, & \text{if } x < 0 \\ \frac{x^2}{4\pi^2}, & \text{if } 0 \le x < 2\pi \\ c + de^{-x}, & \text{if } x \ge 2\pi \end{cases},$$

where a, b, c and d are real constants. Find the values of a, b, c and d. Also derive the probability density/mass function X and the probability density/mass function of $Y = \cos X$.

2+3+5=10 Marks

Problem No. 5: Suppose that the random variable X has the moment generating function

$$M(t) = c \sum_{k=-2}^{2} \frac{e^{kt}}{k^2+1}, \quad -\infty < t < \infty,$$

where c is a real constant. Find the value of c. Derive the probability density/mass function of $Y = X^2 + |X|$ and hence find the distribution function of Y. 2+3+3=8 Marks

Problem No. 6:

(a) For any positive real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$, using Jensen's inequality, show that

$$\sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq (\sum_{i=1}^{n} a_i) \ln \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}.$$

(b) The marks scored by students of a college in a test are realizations of a random variable, having mean 120 and standard deviation 5 (or variance 25). According to the declared grading scheme, students securing between 112 and 128 will be awarded B grade. Using Chebyshev's inequality, find a lower bound on the proportion of students likely to receive B grade. **4+4=8 Marks**

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Model Solutions

Problem No. 1

(a) Total number of possible ways to select three numbers from the set $\{1, 2, \ldots, 50\} = \binom{50}{3}$

For AP, the selected numbers must be: a, $a+d$, $a+2d$, where $1 \le a < a+d \le a+2d \le 50$, a, $a+d$, $a+2d \in \{1, \ldots, 50\}$

$\Leftrightarrow$ $d \in \{1, 2, \ldots, 24\}$, $a \in \{1, 2, \ldots, 50-2d\}$

Total # of favorable cases $= \sum_{d=1}^{24} (50-2d) = 600$ ... **2 MARKS**

Required prob $= \dfrac{600}{\binom{50}{3}} = \dfrac{3}{98}$ ... **2 MARKS**

(b) $P(A \cap B) = 0.3$, $P(B \cap C) = 0.2$, A, B & C are pairwise independent

$\Rightarrow P(A)P(B) = 0.3$, $P(B)P(C) = 0.2$ $\Rightarrow \dfrac{P(A)}{P(C)} = \dfrac{3}{2}$.

Let $P(C) = x$. Then $x = P(C) \ge P(B \cap C) = 0.2 = \frac{1}{5}$; $P(A) = \frac{3}{2}x \in [0, 1]$

$\Rightarrow \frac{1}{5} \le x \le \frac{2}{3}$ ... **2 MARKS**

$P(A \cup C) = P(A) + P(C) - P(A \cap C)$

$= \frac{5}{2}x - \frac{3}{2}x^2 = h(x)$, $x \in [\frac{1}{5}, \frac{2}{3}]$, say.

$h'(x) = \frac{5}{2} - 3x > 0$, $\forall x \in [\frac{1}{5}, \frac{2}{3}]$

$\Rightarrow h(x) \in [h(\frac{1}{5}), h(\frac{2}{3})] = [\frac{11}{25}, 1]$

$\Rightarrow P(A \cup C) \ge \frac{11}{25}$. ... **2 MARKS**

Problem No. 2

(a) Let D be the set of discontinuity points of F. Then

$$D = \{0, 2\} \neq \phi \Rightarrow X \text{ is not continuous} \quad \ldots \text{1 MARK}$$

Also

$$\sum_{x \in D} [F(x) - F(x-)] = \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \left(1 - \tfrac{5}{6}\right) = \tfrac{1}{4} \neq 1 \quad \ldots \text{2 MARKS}$$

$\Rightarrow X$ is not discrete

(b) $$\sum_{x \in D} [F(x) - F(x-)] = \frac{1}{4} \Rightarrow \alpha = \frac{1}{4}$$

$$\alpha F_d(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{12}, & 0 \le x < 2 \\ \frac{1}{4}, & x \ge 2 \end{cases} \Rightarrow F_d(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{3}, & 0 \le x < 2 \\ 1, & x \ge 2 \end{cases} \quad \ldots \text{2 MARKS}$$

$$\frac{3}{4} F_c(x) = F(x) - \alpha F_d(x) = \begin{cases} 0, & x < -1 \\ \frac{x+1}{4}, & -1 \le x < 0 \\ \frac{4x+3}{12}, & 0 \le x < 1 \\ \frac{2x+5}{12}, & 1 \le x < 2 \\ \frac{3}{4}, & x \ge 2 \end{cases}$$

$\Rightarrow$

$$F_c(x) = \begin{cases} 0, & x < -1 \\ \frac{x+1}{3}, & -1 \le x < 0 \\ \frac{4x+3}{9}, & 0 \le x < 1 \\ \frac{2x+5}{9}, & 1 \le x < 2 \\ 1, & x \ge 2 \end{cases} \quad \ldots \text{2 MARKS}$$

Problem No. 3

(a) $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow c\int_{-2}^{1} |x|\,dx = 1 \Rightarrow c = \frac{2}{5}$ .. **1 MARK**

$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \frac{2}{5}\int_{-2}^{1} x|x|\,dx = -\frac{14}{15}$ **1 MARK**

$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{2}{5}\int_{-2}^{1} x^2|x|\,dx = \frac{17}{10}$

$Var(X) = E(X^2) - (E(X))^2 = \frac{373}{450}$ **2 MARKS**

$F(x) = \int_{-\infty}^{x} f(t)\,dt$

$$= \begin{cases} 0, & x < -2 \\ \frac{2}{5}\int_{-2}^{x} |t|\,dt & -2 \le x < 1 \\ 1, & x \geq 1 \end{cases} = \begin{cases} 0, & x < -2 \\ \frac{1}{5}(4 - x^2), & -2 \le x < 0 \\ \frac{1}{5}(4 + x^2), & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}$$

2 MARKS

$$\int_{0}^{\infty} P(X > y)\,dy - \int_{-\infty}^{0} P(X < y)\,dy = \int_{0}^{1}\left[1 - \frac{1}{5}(4 + x^2)\right]dx - \int_{-2}^{0}\frac{1}{5}(4 - x^2)\,dx$$

$$= -\frac{14}{15} = E(X)$$ **2 MARKS**

Problem No. 4

$F(0)=0 \Rightarrow F(x) = a + be^{x} = 0, \quad \forall\, x<0$

$\Rightarrow a = b = 0$

$F(2\pi)=1 \Rightarrow F(x) = c + de^{-x} = 1, \quad \forall\, x > 2\pi \Rightarrow c=1$ and $d=0$

2 MARKS

$\Rightarrow$ $\boxed{a=b=d=0 \text{ and } c=1}$

$$F(x) = \begin{cases} 0, & x<0 \\ \frac{x^2}{4\pi^2}, & 0 \le x < 2\pi \\ 1, & x \ge 2\pi \end{cases}$$

F is continuous and differentiable almost everywhere with

$$F'(x) = \begin{cases} \frac{x}{2\pi^2}, & 0<x<2\pi \\ 0, & \text{otherwise (wherever } F' \text{ exists)} \end{cases}$$

and $\int_{-\infty}^{\infty} F'(x)\,dx = \frac{1}{2\pi^2}\int_0^{2\pi} x\,dx = 1$

$\Rightarrow$ X is continuous with a pdf $f(x) = \begin{cases} \frac{x}{2\pi^2}, & 0<x<2\pi \\ 0, & \text{otherwise} \end{cases}$

3 MARKS

We have $S_X^0 = (0,\pi) \cup (\pi, 2\pi) = S_{1,X}^{(0)} \cup S_{2,X}^{(0)}$.

$h(x) = \cos x$ is monotone in each $S_{i,X}^{(0)}$, $i=1,2$.

$S_{1,X}^{(0)} = (0,\pi)$

$h_1^{-1}(y) = \cos^{-1} y$

$\frac{d}{dy} h_1^{-1}(y) = -\frac{1}{\sqrt{1-y^2}}$

$h(S_{1,X}^{(0)}) = (-1,1)$

$S_{2,X}^{(0)} = (\pi, 2\pi)$

$h_2^{-1}(y) = 2\pi - \cos^{-1} y$

$\frac{d}{dy} h_2^{-1}(y) = \frac{1}{\sqrt{1-y^2}}$

$h(S_{2,X}^{(0)}) = (-1,1)$

2 MARKS

Thus the pdf of $Y=\cos X$ is

$$f_Y(y) = f_X(h_1^{-1}(y))\left|\frac{d}{dy}h_1^{-1}(y)\right| I_{h(S_{1,X})}(y) + f_X(h_2^{-1}(y))\left|\frac{d}{dy}h_2^{-1}(y)\right| I_{h(S_{2,X}^{(0)})}(y)$$

$$= \frac{\cos^{-1} y}{2\pi^2\sqrt{1-y^2}} I_{(-1,1)}(y) + \frac{(2\pi - \cos^{-1} y)}{2\pi^2\sqrt{1-y^2}} I_{(-1,1)}(y)$$

$$= \begin{cases} \frac{1}{\pi}\cdot\frac{1}{\sqrt{1-y^2}}, & -1<y<1 \\ 0, & \text{otherwise} \end{cases}$$

3 MARKS

4/6

Problem No 5

$M_X(0) = 1 \Rightarrow c \sum_{k=-2}^{2} \frac{1}{k^2+1} = 1 \Rightarrow c = \frac{5}{12}$. **2 MARKS**

The p.m.f. of X is

$$f_X(x) = P(X = x) = \begin{cases} \frac{5}{12(x^2+1)}, & x = -2, -1, 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Clearly $S_Y = \{0, 2, 6\}$ and the p.m.f. of Y is

$$f_Y(y) = P(Y = y) = P(X^2 + |X| = y) = \begin{cases} P(X=0), & y = 0 \\ P(|X|=1), & y = 2 \\ P(|X|=2), & y = 6 \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} \frac{5}{12}, & y = 0, 2 \\ \frac{1}{6}, & y = 6 \\ 0, & \text{otherwise} \end{cases}$$

3 MARKS

The d.f. of Y is

$$F_Y(y) = P(Y \le y) = \begin{cases} 0, & y < 0 \\ \frac{5}{12}, & 0 \le y < 2 \\ 5/6, & 2 \le y < 6 \\ 1, & y \ge 6 \end{cases}$$

3 MARKS

Problem No. 6

(a) Consider a rv X having the p.m.f.

$$f_X(x) = \begin{cases} \dfrac{a_i}{\sum_{j=1}^{n} a_j}, & \text{if } x = \dfrac{b_i}{a_i}, \quad i = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$$

Clearly $f_X(\cdot)$ is a proper p.m.f. Let

$$h(x) = -\ln x, \quad x > 0,$$

so that h is a convex function. On applying the Jensen inequality we get

$$E(h(X)) \geq h(E(X)) \quad \ldots \quad \text{2 MARKS}$$

$$\Rightarrow \sum_{i=1}^{n} \left(-\ln \frac{b_i}{a_i}\right) \times \frac{a_i}{\sum_{j=1}^{n} a_j} \geq -\ln\left(\sum_{i=1}^{n} \frac{b_i}{a_i} \times \frac{a_i}{\sum_{j=1}^{n} a_j}\right)$$

$$\Rightarrow \sum_{i=1}^{n} a_i \ln \frac{a_i}{b_i} \geq \left(\sum_{j=1}^{n} a_j\right) \ln\left(\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}\right) \quad \ldots \quad \text{2 MARKS}$$

(b) X: mark obtained by a typical student

$\mu = E(X) = 120, \quad \sigma^2 = Var(X) = 25$

By Chebyshev's inequality

$$P(-k\sigma < X - \mu < k\sigma) \geq 1 - \frac{1}{k^2}, \quad \forall\, k > 0$$

2 MARKS

Thus

$$P(112 < X < 128) = P(-8 < X - \mu < 8)$$

$$= P(-1.6\sigma < X - \mu < 1.6\sigma)$$

$$\geq 1 - \frac{1}{(1.6)^2} = \frac{39}{64}$$

Proportion of students likely to get B grade $= \frac{39}{64} \times 100 = 60.93\%$.

2 MARKS