MTH 418a: Inference-I
Assignment No. 5: Methods of Estimation and Rao Blackwell Theorem

1. (**Review Problem**) Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. $f_\theta, \theta \in \Theta$. In each of the following cases, find MME and MLE of θ. Also, determine if they are functions of a minimal sufficient statistic:

   (i) $X_1 \sim \mathrm{N}(\theta, \sigma_0^2), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant;

   (ii) $X_1 \sim \mathrm{N}(\mu_0, \theta^2), \Theta = (0, \infty), \mu_0$ is a known real constant;

   (iii) $X_1 \sim \mathrm{Exp}(\theta, \sigma_0), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant;

   (iv) $X_1 \sim \mathrm{Exp}(\mu_0, \theta), \Theta = (0, \infty), \mu_0$ is a known real constant;

   (v) $X_1 \sim U(\theta - \sigma_0, \theta + \sigma_0), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant;

   (vi) $X_1 \sim \mathrm{U}(\mu_0 - \theta, \mu_0 + \theta), \Theta = (0, \infty), \mu_0$ is a known real constant;

   (vii) $f_\theta(x) = \alpha_0 \frac{x^{\alpha_0 - 1}}{\theta^{\alpha_0}} I(0 < x < \theta), \Theta = (0, \infty), \alpha_0$ is a known positive constant;

   (viii) $f_\theta(x) = \alpha_0 \frac{\theta^{\alpha_0}}{x^{\alpha_0 + 1}} I(x > \theta), \Theta = (0, \infty), \alpha_0$ is a known positive constant;

   (ix) $X_1 \sim \mathrm{Bin}(m_0, \theta), \Theta = (0, 1), m_0$ is a known positive integer;

   (x) $X_1 \sim \mathrm{Bin}(\theta, p_0), \Theta = \{1, 2, \ldots\}, p_0 \in (0, 1)$ is a known constant;

   (xi) $X_1 \sim \mathrm{Poisson}(\theta), \Theta = (0, \infty)$;

   (xii) $X_1 \sim \mathrm{Gamma}(\theta, \alpha_0), \Theta = (0, \infty), \alpha_0$ is a known positive constant;

   (xiii) $X_1 \sim \mathrm{Gamma}(\sigma_0, \theta), \Theta = (0, \infty), \sigma_0$ is a known positive constant;

   (xiv) $X_1 \sim \mathrm{DEXP}(\theta, \sigma_0), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant;

   (xv) $X_1 \sim \mathrm{DEXP}(\mu_0, \theta), \Theta = (0, \infty), \mu_0$ is a known real constant;

2. Find two examples where the MLE is not unique.

3. Let $X \sim \mathrm{Gamma}(1, \alpha), \alpha > 0$. Show that $E(\ln X) = \psi(\alpha)$, where $\psi(\alpha) = \frac{d}{d\alpha} \ln \Gamma(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)}, \alpha > 0$, is called the digamma function. Show that:

   (i) $\psi(\alpha) < \ln \alpha, \forall \alpha > 0$;

   (ii) $\psi(\alpha)$ is an increasing function on $(0, \infty)$.

4. (i) Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\mu, \sigma^2)$, where $\underline{\theta} = (\mu, \sigma) \in \mathbb{R} \times (0, \infty) = \Theta$. Find the MLE and the MME of $\tau(\underline{\theta}) = P_{\underline{\theta}}(X_1 > 1)$;

   (ii) Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{Gamma}(\sigma, \alpha)$, where $\underline{\theta} = (\sigma, \alpha) \in (0, \infty) \times (0, \infty) = \Theta$. Find the MLEs and MMEs of $\underline{\theta}$ and $\tau(\underline{\theta}) = Var_{\underline{\theta}}(X_1)$;

   (iii) Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{DEXP}(\mu, \sigma), \underline{\theta} = (\mu, \sigma), \Theta = \mathbb{R} \times (0, \infty)$. Find the MLE and the MME of $\underline{\theta}$.

5. Let X_1, X_2 be a random sample from a p.d.f. $f_\theta(x) = \frac{2}{\theta^2}(\theta - x)I(0 < x < \theta), \theta \in (0, \infty) = \Theta$. Find the MME and the MLE of θ. Are they functions of a minimal sufficient statistic?

6. Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. $f_\theta, \theta \in \Theta$. In each of the following cases, find MME and MLE of θ. Also determine if they are functions of a minimal sufficient statistic:

   (i) $X_1 \sim \text{N}(\theta, \sigma_0^2), \Theta = (a, b), a, b\ (a < b)$ and σ_0 are known positive constants;

   (ii) $X_1 \sim \text{Exp}(\theta, \sigma_0), \Theta = (a, b), a, b\ (a < b)$ and σ_0 are known positive constants;

   (iii) $X_1 \sim \text{Bin}(m_0, \theta), \Theta = [\frac{1}{2}, 1)$, m_0 is a known positive integer;

   (iv) $X_1 \sim \text{U}(\theta_1, \theta_2), \underline{\theta} = (\theta_1, \theta_2), \Theta = \{(x, y) \in \mathbb{R}^2 : x < y\}$;

   (v) $X_1 \sim \text{U}(\theta_1 - \theta_2, \theta_1 + \theta_2), \underline{\theta} = (\theta_1, \theta_2), \Theta = \{(x, y) \in \mathbb{R}^2 : -\infty < x < \infty, y > 0\}$.

7. Let $X_1, \ldots, X_n$ be a random sample from $\text{Exp}(0, \theta), \theta \in \Theta = (0, \infty)$. Let c be a positive constant. What is being observed is only those X_is whose values are less than c. Suppose that the observed values are $Y_1, \ldots, Y_m$ and remaining $(n - m)$ observations that exceed c are not observed. Find the MLE of θ.

8. Prove the following inequalities for any non-degenerate random variable X: (i) $E(e^X) > e^{E(X)}$; (ii) $E(X)E(\frac{1}{X}) > 1$, provided $P(X > 0) = 1$; (iii) $E(\ln X) < \ln E(X)$, provided $P(X > 0) = 1$. Hence prove the AM-GM-HM inequality

   $$\sum_{i=1}^{n} a_i w_i > \prod_{i=1}^{n} a_i^{w_i} > \frac{1}{\sum_{i=1}^{n} \frac{w_i}{a_i}},$$

   for any positive constants $w_1, \ldots, w_n, a_1, \ldots, a_n$, such that $\sum_{i=1}^{n} w_i = 1$ and not all a_is are the same.

9. Let $X_1, \ldots, X_n$ be a random sample from $\text{N}(\theta, 1), \theta \in \mathbb{R} = \Theta$. For estimating θ under a loss function, consider the randomized estimator $\delta(\underline{x}) \sim \text{N}(\frac{x_1 + x_2}{2}, 1)$.

   (i) Find a randomized estimator that is based on a minimal sufficient statistic and has the same risk function as $\delta(\underline{x})$;

   (ii) Under the squared error loss function find a non-randomized estimator better than δ.

10. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from $\text{Exp}(0, \theta), \theta \in (0, \infty) = \Theta$. For estimating θ under a loss function, consider the randomized estimator $\delta(\underline{x}) \sim \text{U}(\frac{n-1}{n}\overline{x}_{n-1}, \overline{x}_{n-1})$, where $\overline{x}_{n-1} = \frac{1}{n-1}\sum_{i=1}^{n-1} x_i$.

    (i) Find a randomized estimator based on a minimal sufficient statistic having the same risk function as $\delta(\underline{X})$;

    (ii) Under the squared error loss function find a non-randomized estimator better than δ.

11. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from $\mathrm{N}(\theta, 1)$, where $\theta \in \mathbb{R} = \Theta$ is the unknown parameter. For estimating θ, suppose that the loss function is either the absolute error loss function or the squared error loss function.

   (i) Find an estimator dominating the estimator $\delta_1(\underline{X}) = \sum_{i=1}^{n} \alpha_i X_i$, where $\alpha_1, \ldots, \alpha_n$ are positive constants such that $\sum_{i=1}^{n} \alpha_i = 1$ and not all α_is are the same;

   (ii) Let $n = 2m + 1$. Find a randomized estimator, based on $\overline{X}$, which has the same risk function as the sample median $\delta_2(\underline{X}) = X_{(m+1)}$;

   (iii) Find an estimator better than $\delta_2(\underline{X}) = X_{(m+1)}$.

12. Let $X \sim \mathrm{Bin}(n, \theta), \theta \in (0, 1) = \Theta$. For estimating θ under the squared error loss function, consider the randomized estimator δ, such that $P(\delta(x) = \frac{x}{n}) = P(\delta(x) = \frac{1}{2}) = \frac{1}{2}$. Find a nonrandomized estimator better than δ. Also calculate the risk functions of the two estimators.

13. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{U}(0, \theta)$, where $\theta \in (0, \infty) = \Theta$ is the unknown estimator. Consider estimation of θ under the absolute error loss function.

   (i) Find an estimator, based on a minimal sufficient statistics, that is better than $\delta_0(\underline{X}) = \overline{X}$;

   (ii) Find an estimator, based on a minimal sufficient statistics, that is better than $\delta_1(\underline{X}) = X_{(n-1)}$.