MTH 418a: Inference-I
Assignment No. 7: Neyman Pearson Lemma and Its Applications

1. Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. $f_\theta, \theta \in \Theta = \{\theta_0, \theta_1\}$, where θ_0 and θ_1 are known real constants. In each of the following cases, find an MP(α) test ($0 < \alpha < 1$)for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$, and determine if it is the unique MP(α) test. Wherever the MP(α) is not unique, find at at least two MP(α) tests.

   (i) $X_1 \sim \mathrm{N}(\theta, \sigma_0^2), \theta_0, \theta_1 \in \mathbb{R}, \alpha = 0.90, \sigma_0$ is a known positive constant;

   (ii) $X_1 \sim \mathrm{N}(\mu_0, \theta^2), \theta_0, \theta_1 \in (0, \infty), \alpha = 0.95, \mu_0$ is a known real constant;

   (iv) $X_1 \sim \mathrm{Exp}(\mu_0, \theta), \theta_0, \theta_1 \in \mathbb{R}, \alpha = 0.99, \mu_0$ is a known real constant;

   (v) $X_1 \sim \mathrm{Exp}(\theta, \sigma_0), \theta_0, \theta_1 \in \mathbb{R}, \alpha = 0.95, \sigma_0$ is a known positive constant;

   (vi) $X_1 \sim \mathrm{U}(0, \theta), \theta_0, \theta_1 \in (0, \infty), \alpha = 0.90$;

   (vii) $X_1 \sim \mathrm{U}(\theta, \theta + 1), \theta_0, \theta_1 \in \mathbb{R}, n \geq 2, \alpha = 0.99$;

   (viii) X_1 follows discrete uniform distribution on the set $\{1, 2, \ldots, \theta\}, \theta_0, \theta_1 \in \{2, 3, \ldots\}, \alpha = 0.95$,;

   (ix) $X_1 \sim \mathrm{Bin}(m, \theta), \theta_0, \theta_1 \in (0, 1), \alpha = 0.95, m$ is a known positive integer;

   (x) $X_1 \sim \mathrm{Poisson}(\theta), \theta_0, \theta_1 \in (0, \infty), \alpha = 0.90$.

2. Let $X_1, \ldots, X_5$ be a random sample from $\mathrm{Bin}(2, \theta)$ distribution, where $\theta \in \Theta = \{\frac{1}{3}, \frac{2}{3}\}$.

   (i) Find an MP(0.95) test for testings $H_0 : \theta = \frac{1}{3}$ vs. $H_1 : \theta = \frac{2}{3}$;

   (ii) Find an MP(0.9) test for testings $H_0 : \theta = \frac{2}{3}$ vs. $H_1 : \theta = \frac{1}{3}$.

3. Let $X_1, \ldots, X_5$ be a random sample from $\mathrm{Poisson}(\theta)$ distribution, where $\theta \in \Theta = \{1, \frac{3}{2}\}$.

   (i) Find an MP(0.95) test for testings $H_0 : \theta = 1$ vs. $H_1 : \theta = \frac{3}{2}$;

   (ii) Find an MP(0.9) test for testings $H_0 : \theta = \frac{3}{2}$ vs. $H_1 : \theta = 1$.

4. Let $X_1, \ldots, X_n$ be a random sample from p.m.f./p.d.f. $f_\theta, \theta \in \Theta$. In each of the following cases, find a test function based on a minimal sufficient statistic T and having the same power function as the test function ϕ.

   (i) $X_1 \sim \mathrm{N}(\theta, \sigma_0^2), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant, $\phi(\underline{x}) = 1$, if $\frac{x_1 + x_2}{2} < 1$; $= 0$, otherwise;

   (ii) $X_1 \sim \mathrm{N}(\mu_0, \theta), \Theta = (0, \infty), \mu_0$ is a known real constant, $\phi(\underline{x}) = 1$, if $|x_1 - \mu_0| < 1$; $= 0$, otherwise;

   (iii) $X_1 \sim \mathrm{Exp}(\theta, \sigma_0), \Theta = \mathbb{R}, \sigma_0$ is a known positive constant, $\phi(\underline{x}) = 1$, if $x_{(2)} > 1$; $= 0$, otherwise;

(iv) $X_1 \sim \text{U}(0,\theta), \Theta = (0,\infty)$, $\phi(\underline{x}) = 1$, if $x_1 < 1$; $= 0$, otherwise;

(v) $X_1 \sim \text{Bin}(5,\theta), \Theta = (0,1)$, $\phi(\underline{x}) = 1$, if $x_1 + x_2 < 2$; $= 0$, otherwise;

(x) $X_1 \sim \text{Poisson}(\theta), \Theta = (0,\infty)$, $\phi(\underline{x}) = 1$, if $\frac{x_1+x_2}{2} < 3$; $= 0$, otherwise.

5. Let X be a single observation from a p.d.f. $f \in \mathcal{P} = \{f_0, f_1\}$, where $f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, -\infty < x < \infty$ and $f_1(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}, -\infty < x < \infty$. For $\alpha \in (0,1)$, find an MP(α) test for testing $H_0 : f \equiv f_0$ against $H_1 : f \equiv f_1$. Is MP(α) test unique?

6. Let $X_1, \ldots, X_n$ be a random sample from a p.d.f. $f \in \mathcal{P} = \{f_0, f_1\}$, where $f_0(x) = 1, 0 < x < 1$ and $f_1(x) = 1, 1 < x < 2$. For $\alpha \in (0,1)$, find an MP(α) test for testing $H_0 : f \equiv f_0$ against $H_1 : f \equiv f_1$. Find the power of MP(α) test and show that it is not unique. Find at least three different MP(α) tests.

7. Let f_0 be a given p.d.f./p.m.f. and $\mathcal{P}_1$ be a family of p.d.f.s/p.m.f.s such that $f_0 \notin \mathcal{P}_1$. Suppose that there is a test ϕ^* of level α $(0 < \alpha < 1)$ such that, for every $f_1 \in \mathcal{P}_1$, ϕ^* is MP(α) test for testing $H_0 : f \equiv f_0$ vs. $H_1 : f \equiv f_1$. Then show that ϕ^* is an UMP(α) test for testing $H_0 : f \equiv f_0$ against $H_1 : f \in \mathcal{P}_1$.

8. Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be disjoint family of p.d.f.s/p.m.f.s. Let f_0 be a given p.d.f./p.m.f. in $\mathcal{P}_0$. Suppose that there is a test ϕ^* of level α $(0 < \alpha < 1)$ such that, for every $f_1 \in \mathcal{P}_1$, ϕ^* is an MP(α) test for testing $H_0 : f \equiv f_0$ vs. $H_1 : f \equiv f_1$. Then show that ϕ^* is an UMP(α) test for testing $H_0 : f \in \mathcal{P}_0$ against $H_1 : f \in \mathcal{P}_1$, provided $\sup_{f \in \mathcal{P}_0} \beta_f(\phi^*) \leq \alpha$.

9. Use Problems 4 and 5 to generalize the findings in Problem 1.

10. Let f_0 and f_1 be distinct and known p.d.f.s/p.m.f.s and let $\beta_{\phi^*}(f), f \in \mathcal{P} = \{f_0, f_1\}$, be the power function of an MP(α) $(0 < \alpha < 1)$ test ϕ^* for testing $H_0 : f \equiv f_0$ against $H_1 : f \equiv f_1$. Show that $\beta_{\phi^*}(f_1) > \alpha$.

11. Let ϕ^* be an MP(α) $(0 < \alpha < 1)$ test for testing a simple hypothesis H_0 against a simple alternative hypothesis H_1. Let $\beta < 1$ be the power of MP(α) test ϕ^* under H_1. Show that $1 - \phi^*$ is MP($1 - \beta$) test for testing H_1 vs. H_0.