Module 5

Hypothesis Testing

Let $\underline{X}$ be a random observation having pdf/pmf $f \in \mathcal{P}$, where $\mathcal{P}$ is a given family of pdfs/pmfs.

Let $\mathcal{P}_0$ and $\mathcal{P}_1$ be subfamilies of $\mathcal{P}$ such that

$$\mathcal{P}_0 \cap \mathcal{P}_1 = \phi \quad \text{and} \quad \mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}.$$

Problem. Based on observed sample (say $\underline{X} = \underline{x}$), to decide which of the following two statements (hypotheses) is true?

$$H_0 : f \in \mathcal{P}_0$$
$$H_1 : f \in \mathcal{P}_1$$

H_0 is called the null hypothesis and H_1 is called the alternative hypothesis

$\mathcal{X}$: sample space of $\underline{X}$

Two possible actions after observing $\underline{X} = \underline{x}$:

a_0: do not reject H_0

a_1: reject H_0

Randomized Test Function: After observing $\underline{X} = \underline{x}$, another random experiment is conducted to decide in favor of a_0 or a_1. Thus a test function is a function

$$\phi : \mathcal{X} \to [0, 1]$$

such that

$\phi(\underline{x}) =$ Conditional prob. of rejecting H_0 (i.e. taking action a_1) given that $\underline{X} = \underline{x}$

$1 - \phi(\underline{x}) =$ Conditional prob. of not rejecting H_0 (i.e. taking action a_0) given that $\underline{X} = \underline{x}$

Remark If $\forall \underline{x} \in \mathcal{X}$, $\phi(\underline{x}) \in \{0, 1\}$, then after observing the sample $\underline{X} = \underline{x}$ no further randomization is necessary to choose from actions a_0 or a_1. Such test functions are called non-randomized test functions. Such tests are of the form

$$\phi(\underline{x}) = \begin{cases} 1, & \text{if } \underline{x} \in C \\ 0, & \text{otherwise} \end{cases};$$

where

$C = \{\underline{x} \in \mathcal{X} : \phi(\underline{x}) = 1\} =$ rejection region or critical region of ϕ for testing H_0

Goal: To find a reasonable test function ϕ.

Any test function ϕ may lead to two possible errors.

Type I error: rejecting H_0 when H_0 is true

Type II error: not rejecting H_0 when H_0 is wrong.

For $f \in \mathcal{P}$, define

$$\beta_\phi(f) = E_f(\phi(\underline{X})) \longrightarrow$$ Expected probability of rejecting H_0 when $f \in \mathcal{P}$ is the true distribution

For $f \in \mathcal{P}$, $\beta_\phi(f)$ is called the power function of test ϕ.

For $f \in \mathcal{P}_0$,

$\beta_\phi(f)$: Expected prob. of type-I error at $f \in \mathcal{P}_0$

For $f \in \mathcal{P}_1$

$1 - \beta_\phi(f) = E_f[1 - \phi(\underline{X})] \longrightarrow$ Expected prob. of type-II error at $f \in \mathcal{P}_1$

For $\theta \in \Theta_1$, $\beta_\phi(\theta)$ is called the power of the test at $\theta \in \Theta_1$

An ideal thing would be to find a test (function) ϕ_0 which uniformly minimizes $\beta_\phi(\theta)$, $\forall \theta \in \Theta_0$ and $1-\beta_\phi(\theta)$, $\forall \theta \in \Theta_1$. Unfortunately $\beta_\phi(\theta)$, $\forall \theta \in \Theta_0$, and $1-\beta_\phi(\theta)$ $\forall \theta \in \Theta_1$, can not be minimized simultaneously.

Example Consider test functions ϕ_0 and ϕ_1 such that

$\phi_0(\underline{x}) = 0$, $\forall \underline{x} \in \mathcal{X}$ and $\phi_1(\underline{x}) = 1$, $\forall \underline{x} \in \mathcal{X}$.

Note that ϕ_0 and ϕ_1 are no-data test functions where ϕ_0 never rejects H_0 and ϕ_1 always rejects H_0. Here,

for ϕ_0

prob. of type-I error $= \beta_{\phi_0}(\theta) = 0$, $\forall \theta \in \Theta_0$

prob. of type-II error $= 1-\beta_{\phi_0}(\theta) = 1$, $\forall \theta \in \Theta_1$

Whereas for the test ϕ_1

prob. of type-I error $= \beta_{\phi_1}(\theta) = 1$ $\forall \theta \in \Theta_0$

prob. of type-II error $= 1-\beta_{\phi_1}(\theta) = 0$, $\forall \theta \in \Theta_1$.

Clearly the test ϕ_0 is the best test as far as minimization of type-I error is concerned and the worst possible test for minimizing type-II error. Similarly the test ϕ_1 is the best test for minimizing type-II error and the worst test for minimizing type-I error.

Example Let $X \sim Bin(n, \theta)$ where $\theta \in (0, 1)$ is unknown and $n \geq 1$ is a known integer. Consider testing

$$H_0 : \theta \leq \theta_0$$

against $H_1 : \theta > \theta_0$,

where $\theta_0 \in (0, 1)$ is a fixed value.

Since larger values of θ (prob. of success) are captured in the data through larger values of $\hat{\frac{X}{n}}$ or X (proportion of successes or

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number of successes) a class of reasonable test functions is

$\mathcal{D}_0 = \{\phi_j : j = 0, 1, 2, \ldots, n-1\}$

Where

$$\phi_j(x) = \begin{cases} 1, & \text{if } x \in \{j+1, \ldots, n\} \\ 0, & \text{if } x \in \{0, 1, \ldots, j\} \end{cases}, \quad j = 0, 1, \ldots, n-1.$$

Consider minimizing the risk function (w.r.t. ϕ)

$$R_\phi(\theta) = E_\theta(\beta_\phi(\theta))\, I_{(0,\theta_0]}(\theta) + E_\theta((1-\beta_\phi(\theta))\, I_{(\theta_0, 1)}(\theta))$$

For $0 \le k < j \le n-1$

$$R_{\phi_j}(\theta) = \begin{cases} E_\theta(\beta_{\phi_j}(\theta)) = P_\theta(X \ge j+1), & \text{if } 0 < \theta \le \theta_0 \\ E_\theta(1-\beta_{\phi_j}(\theta)) = P_\theta(X \le j), & \text{if } \theta_0 < \theta < 1 \end{cases}.$$

For $0 \le k < j \le n-1$

$$R_{\phi_j}(\theta) - R_{\phi_k}(\theta) = \begin{cases} -P_\theta(k < X \le j) < 0, & \text{if } 0 < \theta \le \theta_0 \\ P_\theta(k < X \le j) > 0, & \text{if } \theta_0 < \theta < 1 \end{cases}.$$

Hence neither ϕ_j nor ϕ_k is better than the other. Thus controlling the two errors simultaneously does not seem feasible.

A Common Approach

Assign a small upper bound $\alpha \in (0,1)$ to one of the error probabilities (say prob. of type-I error $\beta_\phi(\theta)$, $\theta \in \Theta_0$) and then attempt to minimize (maximize) the other error probability (power, say $\beta_\phi(\theta)$ $\forall \theta \in \Theta_1$), $1-\beta_\phi(\theta)$, $\theta \in \Theta_1$ (say prob. of type-II error). For a fixed $\alpha \in (0,1)$, let

$$\mathcal{Y} = \{\phi : \phi : \mathcal{X} \to [0,1],\ \beta_\phi(\theta) \le \alpha \ \forall \theta \in \Theta_0\}.$$

The goal is to find a $\phi^* \in \mathcal{T}$ such that for any other $\phi \in \mathcal{T}$

$$\beta_{\phi^*}(\theta) \geq \beta_{\phi}(\theta) \quad \forall\, \theta \in \Theta_1 \qquad \ldots \text{(I)}$$

or equivalently $1-\beta_{\phi^*}(\theta) \leq 1-\beta_{\phi}(\theta), \ \forall\, \theta \in \Theta_1$

The bound α is called the level of significance. The size of a test ϕ is the quantity

$$\sup_{\theta \in \Theta_0} \beta_{\phi}(\theta)$$

<u>Note:</u> (a) The level of significance α should lie in the interval $[0,1]$. If $\alpha = 0$, then

$$\beta_{\phi}(\theta) \leq 0, \quad \forall\, \theta \in \Theta_0$$

$$\Rightarrow E_{\theta}(\phi(\underline{X})) = 0, \quad \forall\, \theta \in \Theta_0$$

$$\Rightarrow P_{\theta}(\phi(\underline{X}) = 0) = 1, \quad \forall\, \theta \in \Theta_0$$

Such a test is generally undesirable.

(b) We frame our null and alternative hypothesis in such a way that the error of wrongly rejecting the null hypothesis H_0 is considered to be more serious than the error of wrongly not rejecting H_0. By doing so we guarantee that the probability of the error which is more serious is bounded above a desired level α. Note that the above formulation does not guarantee upper bound on the probability of Type II error, which may be large. Thus one should be very careful in the use of the test which is optimal under the above formulation, as it does not guarantee upper bound on Type-II error probability. Thus if such an optimal test rejects H_0 one may actually conclude the rejection of H_0. However if it does not reject H_0 one would not like to accept H_0. At the best one would conclude that there is not enough evidence to reject H_0. In short, under the above formulation, if the optimal test does not reject H_0 then one must make sure that the probability of Type II error

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(or the power) ... are not large (small) before actually accepting H_0.

<u>Example</u> Let $X_1, \ldots, X_n$ be i.i.d. $N(\theta, 1)$ where $\theta \in \mathbb{R}$ is unknown. Consider testing

$$H_0 : \theta \leq 0$$

against $H_1 : \theta > 0$.

Note that $\bar{X}$ is a complete and sufficient (and hence minimal sufficient) statistic for θ. It is reasonable to consider the class

$$\Theta_0 = \{ \phi_\alpha : \alpha \in (0, 1) \}$$

of tests where

$$\phi_\alpha(\underline{x}) = \begin{cases} 1 & \text{if } \bar{x} > c_\alpha \\ 0, & \text{if } \bar{x} \leq c_\alpha \end{cases}$$

where c_α is chosen so that the test has level of significance α.
We want a test with α- level of significance, i.e

$$\beta_{\phi_c}(\theta) = E_\theta [\phi_c(\underline{X})]$$

$$= P_\theta(\bar{X} > c)$$

$$\leq \alpha, \qquad \forall \; \theta \leq 0$$

$$\Rightarrow \quad \Phi(\sqrt{n}(c - \theta)) \geq 1 - \alpha \qquad \forall \; \theta \leq 0$$

$$\Rightarrow \quad \Phi(\sqrt{n}\, c) \geq 1 - \alpha \iff c_\alpha \geq \frac{1}{\sqrt{n}} \Phi^{-1}(1-\alpha).$$

One such choice is

$$c^* \equiv c^*_\alpha = \frac{1}{\sqrt{n}} \Phi^{-1}(1-\alpha).$$

For $n = 25$ and $\alpha = 0.05$

$$c^*_\alpha = c^*_{0.05} \equiv \frac{1}{5} \Phi^{-1}(0.95) = \frac{1.65}{5} = 0.33 = c^*, \text{ say}$$

Thus a UMP (0.05) test is

$$\phi^*_\alpha(\underline{x}) = \hat{\phi}_{c^*}(\underline{x}) = \begin{cases} 1, & \text{if } \bar{X} > 0.33 \\ 0, & \text{if } \bar{X} \le 0.33 \end{cases}$$

The power of the test is

$$\beta_{\hat{\phi}_{c^*}}(\theta) = P_\theta(\bar{X} > 0.33)$$
$$= 1 - \Phi(\sqrt{n}\,(0.33-\theta)), \quad \theta > 0$$
$$= 1 - \Phi(5\,(0.33-\theta)),$$
$$\in [.05, 1)$$

Near $\theta = 0$, power is very low but for θ far away from 0 power is large. For example the power at $\theta = 0.4$ is

$$\beta_{\hat{\phi}_{c^*}}(.4) = 1 - \Phi(-.35) = \Phi(.35) = 0.6368$$

<u>p-value of a test ϕ</u> It is the smallest possible level of significance $\hat{\alpha} \equiv \hat{\alpha}(\underline{x})$ at which H_0 would be rejected for computed $\phi_\alpha(\underline{x})$ (based on the observed sample $\underline{X} = \underline{x}$).

Thus the p-value of a test ϕ is

$$\hat{\alpha}(\underline{x}) \equiv \inf\{\alpha \in (0,1) : \phi_\alpha(\underline{x}) = 1\},$$

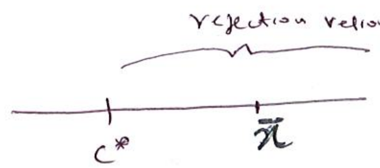
where $\underline{x}$ is the observed sample

<u>Example (Continued)</u> In the last example the p-value of the tests

$$\phi_\alpha(\underline{x}) = \begin{cases} 1, & \text{if } \bar{x} > c_\alpha \\ 0, & \text{otherwise} \end{cases} \qquad \left(c_\alpha \ge \frac{1}{\sqrt{n}}\Phi^{-1}(1-\alpha)\right)$$

is

$$\hat{\alpha}(\underline{x}) = \inf\{\alpha \in (0,1) : \phi_\alpha(\underline{x}) = 1\}$$



$$\alpha \ge \sup_{\theta \in \Theta_0} E_\theta(\phi_\alpha(\underline{X}))$$

$$\Rightarrow \hat{\alpha}(\underline{x}) = \sup_{H_0} P_\theta(\bar{X} > \bar{x}) = \sup_{\theta \le 0}[1 - \Phi(\sqrt{n}(\bar{x}-\theta))]$$
$$= 1 - \Phi(5\bar{x})$$

with the additional information provided by p-values, using p-values is typically more appropriate than using fixed level tests.

In the example discussed above the level of the test $\hat{\phi}_{c^*}$ = size of the test. This may not be true always, especially in situations where the distributions of the statistic involved in the test function ϕ is discrete.

Example $X \sim Bin(2, \theta)$, $\theta \in \Theta = \{\frac{1}{2}, \frac{1}{4}\}$.

Consider testing

$$H_0: \theta = \frac{1}{2}$$

$$\text{against} \quad H_1: \theta = \frac{1}{4}$$

A reasonable class of test procedures is $\mathcal{D}_0 = \{\phi_j : j = 0, 1, 2\}$

$$\phi_j(x) = \begin{cases} 1 & \text{if } x \geq j \\ 0, & \text{otherwise.} \end{cases}$$

For test ϕ_0

$$\beta_{\phi_0}(\theta) = P(X \geq 0) = 1$$

For test ϕ_1

$$\beta_{\phi_1}(\theta) = P(X \geq 1) = 1-(1-\theta)^2$$

$$\sup_{H_0} \beta_{\phi_1}(\theta) = \frac{3}{4}$$

For test ϕ_2

$$\beta_{\phi_2}(\theta) = P(X \geq 2) = \theta^2$$

$$\sup_{H_0} \beta_{\phi_2}(\theta) = \frac{1}{4}$$

Thus

level = size

is possible only for levels $\alpha \in \{1, \frac{1}{4}, \frac{3}{4}\}$.

Example $X \sim Bin(2, \theta)$ where $\theta \in \Theta = [0,1)$ is unknown and $n>1$ is a known integer. Consider testing

$$H_0: \theta \le \frac{1}{2}$$

against $$H_1: \theta > \frac{1}{2}$$

Consider following class of randomized tests

$$\phi_{j,\gamma}(x) = \begin{cases} 1, & \text{if } x > j \\ \gamma & \text{if } x = j \\ 0, & \text{if } x < j \end{cases} \quad , j = -1, 0, 1, 2$$

Suppose we want

size = level α

Then

$$\beta_{\phi_{j,\gamma}}(\theta) = P_\theta(X > j) + \gamma P_\theta(X = j)$$

It can be shown that $\beta_{\phi_{j,\gamma}}(\theta)$ is an increasing function of θ. Thus

$$\sup_{H_0} \beta_{\phi_{j,\gamma}}(\theta) = \beta_{\phi_{j,\gamma}}\left(\tfrac{1}{2}\right) = P_{\theta=\frac{1}{2}}(X > j) + \gamma P_{\theta=\frac{1}{2}}(X = j)$$

Suppose level $\alpha = 0.90$. We have $P_{\theta=\frac{1}{2}}(X > -1) = 1$

$$P_{\theta=\frac{1}{2}}(X > 0) = 1 - \left(1-\tfrac{1}{2}\right)^2 = \frac{3}{4}$$

$$P_{\theta=\frac{1}{2}}(X > 1) = 1 - \left(1-\tfrac{1}{2}\right)^2 - 2\tfrac{1}{2}\left(1-\tfrac{1}{2}\right) = \frac{1}{4}$$

$$P_{\theta=\frac{1}{2}}(X > 2) = 0$$

Thus we take $j = 0$, and

$$\frac{3}{4} + \gamma P_{\theta=\frac{1}{2}}(X = 0) = 0.95$$

$$\gamma\left(1-\tfrac{1}{2}\right)^2 = 0.20$$

$$\Rightarrow \gamma = 0.80$$

Thus a test for which size $= \alpha$ is

$$\phi^*(x) = \begin{cases} 1 & \text{if } x = 1, 2 \\ 0.8 & \text{if } x = 0. \end{cases}$$

Uniformly Most Powerful Tests

Consider testing

$$H_0: f \in \mathcal{P}_0 \quad \text{vs} \quad H_1: f \in \mathcal{P}_1,$$

where $\mathcal{P}_0 \cap \mathcal{P}_1 = \phi$ and $\mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}$.

Let $\alpha \in (0, 1)$ be given level of significance.

Goal: To find a test funct ϕ^* for which

$$\sup_{f \in \mathcal{P}_0} \beta_{\phi^*}(f) \leq \alpha \quad \cdots\cdots \quad (1) \quad \left(\begin{array}{l}\text{level of} \\ \text{significance} = \alpha\end{array}\right)$$

and for any other test ϕ satisfying (1) (i.e. $\sup_{f \in \mathcal{P}_0} \beta_\phi(f) \leq \alpha$)

$$\beta_{\phi^*}(f) \geq \beta_\phi(f), \quad \forall f \in \mathcal{P}_1 \quad \left(\begin{array}{l}\text{maximum power amongst} \\ \text{tests having level} \\ \text{of significance } \alpha\end{array}\right)$$

Such a test ϕ^* is called uniformly most powerful test at level of significance α (UMP(α) test).

Remark: (1) Suppose that the statistic T is sufficient for $f \in \mathcal{P}$. Then, given any test function ϕ, consider the test

$$\phi^*(T) = E_f(\phi(X) \mid T).$$

Clearly ϕ^* depends on T alone and by virtue of sufficiency of T does not depend on f. Thus ϕ^* is a valid test function depending on sufficient statistic T alone.

Also

$$\beta_{\phi^*}(f) = E_f(\phi^*(T)) = E_f(E_f(\phi(X) \mid T))$$
$$= E_f(\phi(X)) = \beta_\phi(f),$$

i.e. ϕ^* and ϕ have the same power function.

Therefore, to find an UMP(α) test one may consider tests that are functions of minimal sufficient statistic T.

Simple and Composite Hypotheses:

A hypothesis H_0 (or H_1) is said to be simple iff $\mathcal{P}_0$ (or $\mathcal{P}_1$) contains exactly one pdf/pmf. A hypothesis which is not simple is called a composite hypothesis.

Example Let

$$\underline{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu \\ \mu \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right),$$

where $\underline{\theta} = (\mu, \rho) \in \Theta = \mathbb{R} \times (-1, 1)$. Then

$$H_0: \mu = 0$$

is a simple hypothesis iff ρ is known.

The Neyman Pearson Lemma

Consider testing simple null hypothesis $H_0: f = f_0$ against simple alternative hypothesis $H_1: f = f_1$; here f_0 and f_1 are known pdfs/pmfs; $\mathcal{P} = \{f_0, f_1\}$, $\mathcal{P}_0 = \{f_0\}$ and $\mathcal{P}_1 = \{f_1\}$.

Theorem (The Neyman-Pearson Lemma) Consider testing the simple null hypothesis

$$H_0: f = f_0$$

against simple alternative hypothesis

$$H_1: f = f_1$$

(I) (Existence of UMP(α) test). For any $\alpha \in [0, 1]$, there exists a UMP(α) test given by

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } f_1(\underline{x}) > c f_0(\underline{x}) \\ \gamma, & \text{if } f_1(\underline{x}) = c f_0(\underline{x}) \\ 0, & \text{if } f_1(\underline{x}) < c f_0(\underline{x}) \end{cases} \quad \cdots (A),$$

where $\gamma \in [0, 1]$ and $c \geq 0$ are constants satisfying

$$E_{f_0}(\phi^*(\underline{x})) = \alpha \qquad (c = \infty \text{ are allowed})$$

(II) (Uniqueness). Let ϕ^* be the MP(α) test defined in (I) above. If ϕ^{**} is UMP(α) test then

$$\phi^{**}(x) = \phi^*(x) = \begin{cases} 1, & \text{if } f_1(x) > c f_0(x) \\ 0 & \text{if } f_1(x) < c f_0(x), \end{cases} \quad \text{a.s. } P$$

i.e. UMP(α) tests can essentially differ only on the set $\{x : f_1(x) = c f_0(x)\}$, $c \geq 0$.

<u>Proof.</u> (i) We provide the proof for the case $\alpha \in (0, 1)$ as the proof for $\alpha \in \{0, 1\}$ follows trivially.

<u>Claim I:</u> There exist $\gamma \in [0, 1]$ and $c \geq 0$ such that

$$E_{f_0}(\phi^*(\underline{x})) = \alpha.$$

Let

$$\beta(t) = P_{f_0}(f_1(x) > t f_0(x)) = 1 - P_{f_0}(f_1(x) \leq t f_0(x))$$

Then $\beta(t) \downarrow$ in $t \in [0, \infty]$, $\beta(t)$ is right continuous, $\beta(+\infty) = \lim_{t \uparrow \infty} \beta(t) = 0$ and $\beta(0-) = \lim_{t \uparrow 0} \beta(t) = 1$

Thus there exists a $c \in [0, \infty]$ such that

$$\beta(c) \leq \alpha \leq \beta(c-).$$

Set

$$\gamma = \begin{cases} \dfrac{\alpha - \beta(c)}{\beta(c-) - \beta(c)}, & \text{if } \beta(c-) \neq \beta(c) \\ 0, & \text{if } \beta(c-) = \beta(c) \end{cases}$$

Then

$$\beta(c-) - \beta(c) = P_{f_0}\left(f_1(x) = c f_0(x)\right)$$

and

$$E_{f_0}(\phi^*(\underline{x})) = P_{f_0}\left(f_1(x) > c f_0(x)\right) + \gamma P_{f_0}\left(f_1(x) = c f_0(x)\right)$$
$$= \alpha.$$

Claim II: ϕ^* in (A) is UMP(α) test.

Let $\phi(\cdot)$ be any other test at level of significance α, i.e. let

$$E_{f_0}(\phi(\underline{x})) \leq \alpha.$$

Consider

$$\Delta(x) := [\phi^*(x) - \phi(x)][f_1(x) - c f_0(x)]$$

Then $\Delta(x) \geq 0$, a.s. P and thus

$$\int [(\phi^*(x) - \phi(x))(f_1(x) - c f_0(x))]\,dx \geq 0$$

$$\Rightarrow E_{f_1}[(\phi^*(x) - \phi(\underline{x}))] \geq c\, E_{f_0}[(\phi^*(x) - \phi(x))]$$

$$\Rightarrow \beta_{\phi^*}(f_1) - \beta_\phi(f_1) \geq c\,[\underbrace{\beta_{\phi^*}(f_0)}_{=\alpha} - \underbrace{\beta_\phi(f_0)}_{\leq \alpha}]$$

$$\geq 0.$$

$$\Rightarrow \beta_{\phi^*}(f_1) \geq \beta_\phi(f_1)$$

$\Rightarrow$ ϕ^*, is UMP(α).

This proves (I)

(II) Let ϕ^{**} be a UMP(α) test and let ϕ^* be as in (I). Consider

$$\Delta(x) = [\phi^*(x) - \phi^{**}(x)][f_1(x) - c f_0(x)]$$

Then

$\Delta(x) > 0$, on $A = \{x : \phi^*(x) - \phi^{**}(x) \neq 0,\ f_1(x) \neq c f_0(x)\}$

$\Delta(x) = 0$ on A^c

Also

$$\int [\phi^*(x) - \phi^{**}(x)]\,[f_1(x) - c f_0(x)]\,dx$$

$$= -c \int [\phi^*(x) - \phi^{**}(x)]\, f_0(x)\, dx \qquad \left(\begin{array}{c}\text{both } \phi^* \text{ and } \phi^{**} \text{ have} \\ \text{the same power}\end{array}\right)$$

$$\leq 0$$

$$\Rightarrow \int_A \underbrace{[\phi^*(x) - \phi^{**}(x)]\,[f_1(x) - c f_0(x)]}_{\geq 0 \text{ on } A}\, dx \leq 0$$

$\Rightarrow$ Set A is negligible

$\Rightarrow$ For almost all values of x, $\phi^*(x) = \phi^{**}(x)$ or $f_1(x) = c f_0(x)$.

Remark: (i) If the set $B = \{x : f_1(x) = c f_0(x)\}$ is negligible then we have a unique UMP(α) test; otherwise UMP(α) tests are randomized on the set B.

(ii) There always exists a UMP(α) test for which size $= \alpha$. (the level of significance).

Example Let $f_0 \equiv U(0,1)$ and $f_1 \equiv U(0,2)$. For testing

$$H_0: f \equiv f_0$$
$$\text{vs} \quad H_1: f \equiv f_1$$

show that the test

$$\phi(x) = \begin{cases} \alpha, & \text{if } 0 < x < 1 \\ 1, & \text{if } 1 < x < 2 \end{cases};$$

is UMP(α).

Example. Let $f_0 \equiv U(0,1)$ and $f_1 \equiv U(1,2)$. For testing

$$H_0: f \equiv f_0$$
$$\text{vs} \quad H_1: f \equiv f_1$$

Show that any test of the following form is UMP(α)

$$\phi(x) = \begin{cases} \upsilon, & \text{if } 0 < x < 1 \\ 1, & \text{if } 1 < x < 2 \end{cases},$$

where $\upsilon \in [0, \alpha]$. Show that power of each of these tests is 1. Among these UMP(α) test find the test with smallest Type-I error.

Example X = sample of size one from $\mathcal{P} = \{f_0, f_1\}$,

where
$$f_0(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < \infty \quad (N(0, 1))$$
$$f_1(x) = \frac{1}{4} e^{-\frac{|x|}{2}}, \quad -\infty < x < \infty \quad (DE(0, 2))$$

Here
$$P_f(f_1(X) = c f_0(X)) = 0, \quad \forall f \in \mathcal{P}.$$

Thus the unique UMP(α) test is

$$\phi^*(x) = \begin{cases} 1, & \text{if } f_1(x) > c f_0(x) \\ 0, & \text{if } f_1(x) < c f_0(x) \end{cases}$$

Note that

$$\phi^*(x) = 1 \iff \frac{f_1(x)}{f_0(x)} > c$$
$$\iff |x|^2 - |x| > d, \quad \text{for some } d \in (-\infty, \infty)$$
$$\iff |x| < \frac{1 - \sqrt{1+4d}}{2} \text{ or } |x| > \frac{1 + \sqrt{1+4d}}{2} \quad (1 + 4d > 0)$$
$$\iff |x| < 1 - t \text{ or } |x| > t, \quad \text{for some } t > \frac{1}{2}.$$

Case I. $\frac{1}{2} < t \le 1$

$$E_{f_0}(\phi^*(X)) = P_{f_0}(|X| < 1 - t) + P_{f_0}(|X| > t) \downarrow t$$
$$\in [P_{f_0}(|X| > 1), P_{f_0}(|X| < \tfrac{1}{2}) + P_{f_0}(|X| > \tfrac{1}{2}))$$
$$= [0.3374, 1)$$

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Thus choice $t \in (\frac{1}{2}, 1]$ can be used only when $\alpha \in [0.3374, 1)$

$\phi^*(x) = \begin{cases} 1 & |x| < 1-t \text{ or } |x| > t \\ 0 & \text{o.w.} \end{cases}$

Case I $t \geq$ ½ (i.e. $\alpha \in (0, 0.3374)$)

$$\phi^*(x) = \begin{cases} 1, & |x| > t \\ 0, & \text{o.w.} \end{cases}$$

$$P_{\theta_0}(|X| > t) = \alpha$$

$$2(1 - \Phi(t)) = \alpha$$

$$t = \Phi^{-1}(1 - \alpha/2).$$

Example Let $X_1, \dots, X_n$ be iid $Bin(1, \theta)$, $\theta \in \Theta = (0, 1)$

Consider testing

$$H_0: \theta = \theta_0$$

$$\text{vs} \quad H_1: \theta = \theta_1,$$

where $0 < \theta_0 < \theta_1 < 1$ are fixed constants.

A UMP(α) test is

$$\phi^*(Y) = \begin{cases} 1, & \lambda(Y) > c \\ \gamma, & \lambda(Y) = c \\ 0 & \lambda(Y) < c \end{cases},$$

where $Y = \sum_{i=1}^{n} X_i \sim Bin(n, \theta)$ and

$$\lambda(Y) = \frac{f_{\theta_1}(\underline{x})}{f_{\theta_0}(\underline{x})} = \frac{\prod_{i=1}^{n} \theta_1^{x_i}(1-\theta_1)^{1-x_i}}{\prod_{i=1}^{n} \theta_0^{x_i}(1-\theta_0)^{1-x_i}}$$

$$= \left(\frac{\theta_1(1-\theta_0)}{\theta_0(1-\theta_1)}\right)^Y \left(\frac{1-\theta_1}{1-\theta_0}\right)^n$$

Clearly $\lambda(Y) \uparrow$ in Y. Thus, for some $m \geq 0$

$$\phi^*(Y) = \begin{cases} 1, & \text{if } Y > m \\ \gamma, & Y = m \\ 0 & Y < m \end{cases}$$

where m and γ satisfy

$$E_{\theta_0}(\phi^*(Y)) = \alpha$$

$$P_{\theta_0}(Y > m) + \gamma P_{\theta_0}(Y = m) = \alpha$$

$$\Rightarrow \sum_{j=m+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} + \gamma \binom{n}{m} \theta_0^m (1-\theta_0)^{n-m} = \alpha$$

If for some $m \geq 0$,

$$\sum_{j=m+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} = \alpha,$$

then $\gamma = 0$ and ϕ^* is non-randomized.

In general m is such that

$$\sum_{j=m+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} \leq \alpha \leq \sum_{j=m}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j}$$

and

$$\gamma = \frac{\alpha - \sum_{j=m+1}^{n} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j}}{\binom{n}{m} \theta_0^m (1-\theta_0)^{n-m}},$$

provided $P_{\theta_0}(X = m) > 0$.

Remark: The above test ϕ^* does not depend on θ_1 as long as $\theta_1 > \theta_0$. Therefore by definition of UMP(α) test ϕ^* is in fact UMP(α) test even for testing

$$H_0: \theta = \theta_0$$
$$\text{vs} \quad H_1: \theta > \theta_0$$

Lemma Suppose that for every $\theta_1 \in \Theta_1$, a test ϕ^* is UMP(α) for testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$. Then ϕ^* is UMP(α) for testing $H_0: \theta = \theta_0$ vs $H_1: \theta \in \Theta_1$.

Proof Let ϕ be some test of α level of significance for testing $H_0: \theta = \theta_0$ vs $H_1: \theta \in \Theta_1$. Then

$$E_{\theta_0}(\phi^*(X)) = E_{\theta_0}(\phi(\underline{X})) \leq \alpha.$$

Let $f_1 \in \mathcal{P}_1$. Then, since ϕ^* is UMP(α) test for testing $\breve{H}_0: f \equiv f_0$ vs $H_1^*: f \equiv f_1$, it follows that

$$\beta_{\phi^*}(f_1) \geq \beta_\phi(f_1)$$

Since $f_1 \in \mathcal{P}_1$ was arbitrary, we have

$$\beta_{\phi^*}(f_1) \geq \beta_\phi(f_1), \quad \forall\ f_1 \in \mathcal{P}_1.$$

$\Rightarrow$ ϕ^* is UMP(α) test for testing

$$H_0: f \equiv f_0$$
$$\text{vs} \quad H_1: f \in \mathcal{P}_1.$$

<u>Lemma</u> (<u>Generalized Neyman Pearson Lemma</u>). Let $f_1, \ldots, f_{m+1}$ be real-valued functions on $\mathbb{R}^p$. For given constants $\alpha_1, \ldots, \alpha_m$, let

$$\mathcal{D} = \{\phi : \phi: \mathbb{R}^p \to [0,1],\ \int \phi(\underline{x}) f_i(\underline{x})\, d\underline{x} \leq \alpha_i,\ i=1,\ldots,m\}$$

and

$$\mathcal{D}_0 = \{\phi : \phi: \mathbb{R}^p \to [0,1],\ \int \phi(\underline{x}) f_i(\underline{x})\, d\underline{x} = \alpha_i,\ i=1,\ldots,m\}.$$

If there are constants $c_1, \ldots, c_m$ such that

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } f_{m+1}(\underline{x}) > \sum_{i=1}^m c_i f_i(\underline{x}) \\ 0, & \text{if } f_{m+1}(\underline{x}) < \sum_{i=1}^m c_i f_i(\underline{x}) \end{cases}$$

is a member of $\mathcal{D}_0$, then ϕ^* maximizes $\int \phi(\underline{x}) f_{m+1}(\underline{x})\, d\underline{x}$ over $\phi \in \mathcal{D}_0$. If $c_i \geq 0\ \forall\ i=1,\ldots,m$, then ϕ^* maximizes $\int \phi f_{m+1}\, d\underline{x}$ over $\phi \in \mathcal{D}$.

<u>Proof.</u> <u>Case I</u> <u>General c_is</u>

Let $\phi \in \mathcal{D}_0$. Consider

$$\Delta(\underline{x}) = [\phi^*(\underline{x}) - \phi(\underline{x})]\,[f_{m+1}(\underline{x}) - \sum_{i=1}^m c_i f_i(\underline{x})]$$

Then

$$\Delta(\underline{x}) \geq 0, \quad \forall \underline{x}$$

$$\Rightarrow \int [\phi^*(\underline{x}) - \phi(\underline{x})] [f_{m+1}(\underline{x}) - \sum_{i=1}^{m} c_i f_i(\underline{x})] d\underline{x} \geq 0$$

$$\Rightarrow \int [\phi^*(\underline{x}) - \phi(\underline{x})] f_{m+1}(\underline{x}) \geq \sum_{i=1}^{m} c_i [\int \phi^*(\underline{x}) f_i(\underline{x}) d\underline{x} - \int \phi(\underline{x}) f_i(\underline{x}) d\underline{x}] \quad \ldots (I)$$

$$\Rightarrow \int \phi^*(\underline{x}) f_{m+1}(\underline{x}) d\underline{x} - \int \phi(\underline{x}) f_{m+1}(\underline{x}) d\underline{x}$$

$$\geq \sum_{i=1}^{m} c_i [\alpha - \alpha] \quad [\text{since } \phi, \phi^* \in \tilde{\mathcal{D}}_0]$$

$$= 0$$

Case II $\quad c_i \geq 0, \ \forall i = 1, \ldots, m$

Let $\phi \in \mathcal{D}_0$. Then again (I) yields

$$\int \phi^*(\underline{x}) f_{m+1}(\underline{x}) d\underline{x} - \int \phi(\underline{x}) f_{m+1}(\underline{x}) d\underline{x} \geq \sum_{i=1}^{m} c_i [\alpha - \alpha] = 0.$$

Lemma (Existence of c_is): Under the notation of above lemma the set $C = \{ (\int \phi f_1 dx, \ldots, \int \phi f_m dx) : \phi : \mathbb{R}^n \to [0,1] \}$ is a closed and convex set. If $(\alpha_1, \ldots, \alpha_m)$ is an interior point of C, then there exist constants $c_1, \ldots, c_m$ such that ϕ^* defined in above lemma is in $\mathcal{D}_0$.

Monotone Likelihood Ratio Property: A family $\mathcal{P} = \{ f_\theta : \theta \in \Theta \}$ ($\Theta \subseteq \mathbb{R}$) of pdfs/pmfs is said to have the monotone likelihood ratio (MLR) property in statistic $T(\underline{X})$ if for any $\theta_1 < \theta_2$ ($\theta_1, \theta_2 \in \Theta$), $\dfrac{f_{\theta_2}(\underline{x})}{f_{\theta_1}(\underline{x})}$ is a non-decreasing function of $T(\underline{x})$ for all $\underline{x} \in \{ t : f_{\theta_1}(t) > 0 \} \cup \{ t : f_{\theta_2}(t) > 0 \}$.

Lemma Suppose that the family $\mathcal{P} = \{ f_\theta : \theta \in \Theta \}$ $(\Theta \subseteq \mathbb{R})$ of pdfs has MLR in $T(\underline{x})$. If $\psi(\cdot)$ is a non-decreasing function of T, then $g(\theta) = E_\theta(\psi(T))$ is a non-decreasing function of $\theta \in \Theta$.

Proof. Let $\theta_1, \theta_2 \in \Theta$ be such that $\theta_1 < \theta_2$. Define

$$A = \{x : f_{\theta_1}(x) > f_{\theta_2}(x)\}, \quad B = \{x : f_{\theta_1}(x) < f_{\theta_2}(x)\}$$
$$C = \{x : f_{\theta_1}(x) = f_{\theta_2}(x)\}$$
$$a = \sup_{x \in A} \psi(T(x)), \quad b = \inf_{x \in B} \psi(T(x)).$$

Since $f_{\theta_2}(\underline{x}) / f_{\theta_1}(\underline{x}) \uparrow$ and $\psi(\cdot) \uparrow$, it follows that $b \geq a$. Thus

$$g(\theta_2) - g(\theta_1) = \int \psi(T(\underline{x})) [f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})] d\underline{x}$$

$$= \int_A \underbrace{\psi(T(\underline{x}))}_{\leq a} \underbrace{[f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})]}_{<0} d\underline{x} + \int_B \underbrace{\psi(T(\underline{x}))}_{\geq b} \underbrace{[f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})]}_{\geq 0}$$

$$\geq a \int_A [f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})] d\underline{x} + b \int_B [f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})] d\underline{x}$$

$$= (b-a) \int_B [f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})] d\underline{x} \qquad \left[\int [f_{\theta_2}(\underline{x}) - f_{\theta_1}(\underline{x})] d\underline{x} = 0 \right]$$

$$\geq 0.$$

Example Let $\Theta \subseteq \mathbb{R}$ and let $\eta(\theta) \uparrow$ in $\theta \in \Theta$. Then the one-parameter exponential family with pmf/pdf

$$f_\theta(\underline{x}) = \exp\{\eta(\theta) T(\underline{x}) - \xi(\theta)\} h(\underline{x}) \quad \cdots \quad (*)$$

has MLR in $T(\underline{x})$.

Solution For $\theta_2 > \theta_1$

$$\frac{f_{\theta_2}(\underline{x})}{f_{\theta_1}(\underline{x})} = e^{-(\xi(\theta_2) - \xi(\theta_1))} \, e^{\overbrace{(\eta(\theta_2) - \eta(\theta_1))}^{} T(\underline{x})} \quad \uparrow \text{ in } T(\underline{x})$$

with $\eta(\theta_2) - \eta(\theta_1) > 0$.

Examples of (*) are; when $X_1, \ldots, X_n$ is a random sample from

(I) $Bin(m, \theta)$, $\theta \in \Theta \subseteq (0, 1)$, $m \in \{1, 2, \ldots\}$ is known/fixed;

(II) $Poisson(\theta)$, $\theta \in \Theta \subseteq (0, \infty)$;

(III) $NB(r, \theta)$, $\theta \in \Theta \subseteq (0, 1)$, $r \in \{1, 2, \ldots\}$ is known/fixed

(IV) $N(\theta, \sigma_0^2)$, $\theta \in \Theta \subseteq \mathbb{R}$, $\sigma_0 > 0$ is known/fixed

(V) $N(\mu_0, \theta)$, $\theta \in \Theta \subseteq (0, \infty)$, $\mu_0 \in \mathbb{R}$ is known/fixed

(VI) $Exp(\mu_0, \theta)$, $\theta \in \Theta \subseteq (0, \infty)$, $\mu_0 \in \mathbb{R}$ is known/fixed

(VII) $Exp(\theta, \sigma_0)$, $\theta \in \Theta \subseteq \mathbb{R}$, $\sigma_0 > 0$ is known/fixed

(VIII) $Gamma(\alpha_0, \theta)$, with scale parameter $\theta \in \Theta \subseteq \mathbb{R}$, $\alpha_0 > 0$ is known/fixed

(IX) $DE(\mu_0, \theta)$, $\theta \in \Theta \subseteq (0, \infty)$, $\mu_0 \in \mathbb{R}$ is known/fixed;

__Example__ Let $X_1, \ldots, X_n$ be iid $U(0, \theta)$, $\theta > 0$.

Then

$$f_\theta(\underline{x}) = \frac{1}{\theta^n} I_{(0, \theta)}(x_{(n)})$$

For $\theta_2 > \theta_1$,

$$\frac{f_{\theta_2}(\underline{x})}{f_{\theta_1}(\underline{x})} = \begin{cases} \frac{\theta_1^n}{\theta_2^n}, & 0 < x_{(n)} < \theta_1 \\ \infty & \theta_1 < x_{(n)} < \theta_2 \end{cases} \quad \uparrow \text{ in } x_{(n)}$$

$\Rightarrow$ $\{f_\theta : \theta > 0\}$ has MLR in $X_{(n)}$.

__Example__ Let $X_1, \ldots, X_n$ be iid $Exp(\theta, \sigma_0)$ $\theta \in \mathbb{R}$, $\sigma_0 > 0$ is fixed. Show that the family $\{f_\theta : \theta \in \mathbb{R}\}$ has MLR in $T(\underline{X}) = X_{(1)}$.

Example Consider a random sample X of size 1 from the Cauchy pdf

$$f_\theta(x) = \frac{1}{\pi} \cdot \frac{1}{1+(x-\theta)^2}, \quad -\infty < x < \infty, \quad \theta \in \Theta = \mathbb{R}$$

For $\theta_2 > \theta_1$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \frac{1+(x-\theta_1)^2}{1+(x-\theta_2)^2} \to 1, \quad \text{as } x \to \pm\infty$$

$$\to \frac{1+\theta_1^2}{1+\theta_2^2} \quad \text{at } x = 0$$

$\Rightarrow$ the family $\{f_\theta : \theta \in \Theta\}$ does not have MLR in X.

UMP(α) Tests for One Sided Hypotheses

One - Sided Hypotheses : Hypotheses of the form

$$H_0: \theta \le \theta_0 \quad (\text{or } H_0: \theta \ge \theta_0)$$

$$\text{vs} \quad H_1: \theta > \theta_0 \quad (\text{or } H_1: \theta < \theta_0)$$

are called one - sided hypotheses for any fixed constant θ_0.

Theorem. Suppose that the distribution of $\underline{X}$ has the p.d.f. / p.m.f. in the ~~(i.e. $f_{\theta_2}(x)/f_{\theta_1}(x)$ is strictly increasing in $T(\underline{x})$)~~ parametric family $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$ that has strict MLR in $T(\underline{X})$, where $\Theta \subseteq \mathbb{R}$. Consider testing $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$, where θ_0 is a given constant.

(a) There exist a UMP(α) test, given by

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } T(\underline{x}) > c \\ \gamma, & \text{if } T(\underline{x}) = c \\ 0, & \text{if } T(\underline{x}) < c, \end{cases}$$

where c and $\gamma \in [0, 1]$ are determined by $\beta_{\phi^*}(\theta_0) = \alpha$

(b) $\beta_{\phi^*}(\theta)$ is strictly increasing for all θ's for which $0 < \beta_{\phi^*}(\theta) < 1$.

(c) For any $\theta < \theta_0$ ϕ^* minimizes $\beta_\phi(\theta)$ (type I error prob. of ϕ) among all tests ϕ satisfying $\beta_\phi(\theta_0) = \alpha$.

(d) For any $\theta_1 \in \Theta$ ϕ^* is UMP (α^*) for testing $H_0: \theta \le \theta_1$ vs $H_1: \theta > \theta_1$ with $\alpha^* = \beta_{\phi^*}(\theta_1)$.

Proof. (a) First consider testing

$$H_0^*: \theta = \theta_0$$
$$\text{vs} \quad H_1^*: \theta = \theta_1,$$

where $\theta_1 > \theta_0$. Then a UMP(α) test is

$$\phi^* = \begin{cases} 1, & \text{if } f_{\theta_1}(\underline{x}) > k f_{\theta_0}(\underline{x}) \\ \gamma, & \text{if } f_{\theta_1}(\underline{x}) = k f_{\theta_0}(\underline{x}) \\ 0, & \text{if } f_{\theta_1}(\underline{x}) < k f_{\theta_0}(\underline{x}) \end{cases}$$

Since $\dfrac{f_{\theta_1}(\underline{x})}{f_{\theta_0}(\underline{x})}$ is ~~strictly~~ $\uparrow$ in $T(\underline{x})$, the UMP(α) test is

$$\phi^* = \begin{cases} 1 & \text{if } T(\underline{x}) > c \\ \gamma & \text{if } T(\underline{x}) = c \\ 0 & \text{if } T(\underline{x}) < c \end{cases}$$

as long as $\gamma \in [0,1]$ and c satisfy

$$\beta_{\phi^*}(\theta_0) = \alpha$$

$$\Rightarrow \quad P_{\theta_0}(T(\underline{x}) > c) + \gamma P_{\theta_0}(T(\underline{x}) = c) = \alpha.$$

Since ϕ^* does not depend on θ_1 (as long as $\theta_1 > \theta_0$) it is UMP(α) for testing

$$H_0^*: \theta = \theta_0$$
$$\text{vs} \quad H_1^*: \theta > \theta_0$$

To show that ϕ^* is UMP(α) for testing $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$, it suffices to show that $\sup_{\theta \le \theta_0} \beta_{\phi^*}(\theta) \le \alpha$. We have

$$\beta_{\phi^*}(\theta) = E_\theta(\phi^*(\underline{x})), \quad \theta \in \Theta$$

But $\phi^*(\underline{x})$ is an increasing function of $T(\underline{x})$ and the pdf of $\underline{x}$ has MLR in $T(\underline{x})$. Therefore

$$\beta_{\phi^*}(\theta) = E_\theta(\phi^*(\underline{x})) \uparrow \theta$$

$$\Rightarrow \quad \beta_{\phi^*}(\theta) \le \beta_{\phi^*}(\theta_0) = \alpha, \quad \forall\, \theta \le \theta_0 \quad \Rightarrow \sup_{\theta \le \theta_0} \beta_{\phi^*}(\theta) \le \alpha.$$

(b) Let $\theta_1 < \theta_2$ be such that $\beta_{\phi^*}(\theta_i) \in (0,1)$, $i=1,2$.

Consider testing

$$H_0 : \theta = \theta_1 \quad \cdots\cdots \quad (A)$$
$$\text{vs } H_1 : \theta = \theta_2$$

Then ϕ^* is UMP(α^*) test of level $\alpha^* = E_{\theta_1}(\phi^*) \in (0,1)$.

Consider the no-data test

$$\phi_0(\underline{x}) = \alpha^*, \quad \forall \underline{x}$$

Then, $E_{\theta_1}(\phi_0) = \alpha^*$ and thus we have (since ϕ^* is UMP(α) for (A))

$$E_{\theta_2}(\phi_0(\underline{x})) \le E_{\theta_2}(\phi^*) \iff \alpha^* \le E_{\theta_2}(\phi^*)$$

$$\Rightarrow E_{\theta_1}(\phi^*) = \alpha^* \le E_{\theta_2}(\phi^*)$$

Moreover,

$$E_{\theta_1}(\phi^*) = E_{\theta_2}(\phi^*) \iff E_{\theta_2}(\phi_0) = E_{\theta_2}(\phi^*)$$

$\Rightarrow$ ϕ_0 is also UMP(α^*).

By uniqueness of UMP(α^*) test

$$\phi_0(\underline{x}) = \begin{cases} 1, & f_{\theta_1}(\underline{x}) > k f_{\theta_2}(\underline{x}) \\ 0 & f_{\theta_1}(\underline{x}) < k f_{\theta_2}(\underline{x}) \end{cases}$$

$$\parallel$$
$$\alpha^* \in (0,1)$$

$$\Rightarrow P_\theta(f_{\theta_1}(\underline{x}) = k f_{\theta_2}(\underline{x})) = 1 \quad \forall \theta \in \{\theta_0, \theta_1\}$$

$$\Rightarrow k = 1 \text{ and } f_{\theta_1}(\underline{x}) = f_{\theta_2}(\underline{x}) \longrightarrow \text{Contradiction}$$

(c) Let $\theta_{00} < \theta_0$. Consider testing

$$H_0^* : \theta = \theta_0$$
$$\text{vs } H_1^* : \theta = \theta_{00}$$

UMP($1-\alpha$) test

$$\phi^{**} = \begin{cases} 1 & T(\underline{x}) < c^* \\ \gamma^* & T(\underline{x}) = c^* \\ 0, & T(x) > c^* \end{cases}$$

Where $E_{\theta_0}(\phi^{**}) = 1-\alpha$

$$1-\phi^{**} = \begin{cases} 1 & T(x) > c^* \\ 1-\gamma^* & T(\underline{x}) = c \\ 0 & T(x) < c^* \end{cases}$$

Clearly $E_{\theta_0}(1-\phi_{\alpha\alpha}) = \alpha$

Comparing with ϕ^* we observe that $\phi^{**} = 1-\phi^*$ is UMP $(1-\alpha)$ test for testing the above hypothesis.

Let ϕ_0 be any test of level α for testing

$$H_0: \theta \le \theta_0$$
$$\text{vs} \quad H_1: \theta > \theta_0.$$

Then $1-\phi_0$ is a valid test for testing H_0^* vs H_1^*.

Also $E_{\theta_0}(1-\phi_0) \ge 1-\alpha$. Since $\phi^{**} = 1-\phi^*$ is UMP$(1-\alpha)$ test for H_0^* vs H_1^* we have

$$E_{\theta_{00}}(1-\phi^*) \ge E_{\theta_{00}}(1-\phi_0)$$
$$\Rightarrow \quad E_{\theta_{00}}(\phi_0) \ge E_{\theta_{00}}(\phi^*)$$

(d) Similarly to (a)

Remark: Under the set-up of last theorem consider testing $H_0: \theta \ge \theta_0$ vs $H_1: \theta < \theta_0$. As in last theorem, it can be shown that the test

$$\phi^{**}(\underline{T}) = \begin{cases} 1 & \text{if } T < c \\ \gamma, & \text{if } T = c \\ 0 & \text{if } T > c, \end{cases}$$

where γ and c are chosen so that $E_{\theta_0}(\phi^{**}(\underline{x})) = \alpha$, is UMP$(\alpha)$ for testing the above hypotheses.

Corollary: Suppose that $\underline{X}$ has p.d.f/p.m.f.

$$f_\theta(x) = e^{\eta(\theta) T(x) - S(\theta)} h(x), \quad x \in \mathcal{X}, \quad \theta \in \Theta \subseteq \mathbb{R} \text{ and } \alpha \in (0,1)$$

where $\eta(\theta)$ is an increasing function of θ. Let $\theta_0 \in \Theta$ be fixed. Then a UMP(α) test for testing $H_0: \theta \le \theta_0$ $(\theta \ge \theta_0)$ vs $H_1: \theta > \theta_0$ $(\theta < \theta_0)$ is

$$\phi^*(T) = \begin{cases} 1 & \text{if } T(\underline{x}) > (<)\, c \\ \gamma & \text{if } T(\underline{x}) = c \\ 0 & \text{if } T(\underline{x}) < (>)\, c, \end{cases}$$

where c and γ are such that $\beta_{\phi^*}(\theta_0) = \alpha$.

If $\eta(\theta)$ is a decreasing or $\eta(\theta)$ is strictly increasing but $H_0: \theta \geq \theta_0$ (vs $H_1: \theta < \theta_0$) the result is still valid by reversing inequalities in ϕ^*.

<u>Example</u> Let $X_1, \ldots, X_n$ be iid $N(\theta, \sigma_0^2)$ where $\theta \in \Theta = \mathbb{R}$ is unknown and $\sigma_0 > 0$ is known. Consider testing

$$H_0: \theta \leq \theta_0$$
$$\text{vs } H_1: \theta > \theta_0,$$

at α-level of significance.

where $\theta_0 \in \mathbb{R}$ is a fixed constant

We have

$$f_\theta(\underline{x}) = \exp\Big[\underbrace{\frac{n\theta}{\sigma_0^2}}_{\eta(\theta)}\bar{x} - \overbrace{\Big\{\frac{n\theta^2}{2\sigma_0^2} + \frac{n}{2}\ln(2\pi\sigma_0^2)\Big\}}^{\xi(\theta)}\Big] \underbrace{e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^n x_i^2}}_{h(\underline{x})}$$

$$= \exp\left[\eta(\theta)T(\underline{x}) - \xi(\theta)\right] h(\underline{x}), \quad \underline{x} \in \mathcal{X} = \mathbb{R}^n,$$

where $\eta(\theta) = \frac{n\theta}{\sigma_0^2}$ is strictly increasing in θ. Thus UMP(α) test is

$$\phi^* = \begin{cases} 1, & \text{if } \bar{x} > c \\ 0, & \text{o.w.} \end{cases}$$

where c is chosen so that

$$E_{\theta_0}(\phi^*) = \alpha$$
$$\Rightarrow P_{\theta_0}(\bar{X} > c) = \alpha$$
$$\Phi\left(\frac{\sqrt{n}(c-\theta_0)}{\sigma_0}\right) = \alpha$$

$$\Rightarrow c = \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha) + \theta_0$$

$\boxed{\text{Note:}}$ The above test is also UMP(α) test for testing

$$H_0: \theta = \theta_0$$
$$\text{vs } H_1: \theta > \theta_0$$

$\boxed{\text{Example}}$ Let $X_1, \dots, X_n$ be a random sample from $Bin(1, \theta)$, $\theta \in \Theta = (0, 1)$.

$f_\theta(\underline{x})$ belongs to one parameter exponential family with

$$T(\underline{X}) = \sum_{i=1}^{n} X_i \sim Bin(n, \theta) \text{ and } \eta(\theta) = \ln \frac{\theta}{1-\theta} \uparrow \text{ in } \theta \in (0, 1)$$

UMP(α) test for testing

$$H_0: \theta \geq \theta_0$$

$$\text{vs } H_1: \theta < \theta_0$$

is given by

$$\phi^*(\underline{X}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} X_i < c \\ \gamma, & \text{if } \sum_{i=1}^{n} X_i = c \\ 0, & \text{if } \sum_{i=1}^{n} X_i > c \end{cases}$$

where c and $\gamma \in [0, 1]$ are determined by

$$E_{\theta_0}(\phi^*(\underline{X})) = \alpha$$

$$\sum_{j=0}^{c-1} \binom{n}{j} \theta_0^j (1-\theta_0)^{n-j} + \gamma \binom{n}{c} \theta_0^c (1-\theta_0)^{n-c} = \alpha.$$

Obviously this test is also UMP(α) for testing $H_0: \theta = \theta_0$ vs $H_1: \theta < \theta_0$.

$\boxed{\text{Example}}$ Let $X_1, \dots, X_n$ be i.i.d. Poisson(θ), $\theta > 0$

$f_\theta(\underline{x})$ belongs to exponential family with $\eta(\theta) = \ln\theta \uparrow$ in $\theta \in (0, \infty)$,

$$T(\underline{X}) = \sum_{i=1}^{n} X_i \text{ and }$$

$$\sim \text{Poisson}(n\theta)$$

For testing

$$H_0: \theta \leq \theta_0$$

$$\text{vs } H_1: \theta > \theta_0$$

a UMP(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^n x_i > c \\ \gamma & \text{if } \sum_{i=1}^n x_i = c \\ 0, & \text{if } \sum_{i=1}^n x_i < c \end{cases}$$

where c and $\gamma \in [0, 1]$ satisfy

$$E_{\theta_0}[\phi^*(\underline{X})] = \alpha$$

$$\Leftrightarrow \quad P_{\theta_0}(T(\underline{X}) > c) + \gamma P_{\theta_0}(T(\underline{X}) = c) = \alpha$$

$$\Leftrightarrow \quad \sum_{j=c+1}^{\infty} \frac{e^{-n\theta_0}(n\theta_0)^j}{\lfloor j} + \gamma \frac{e^{-n\theta_0}(n\theta_0)^c}{\lfloor c} = \alpha$$

$\boxed{\text{Example}}$ Let $X_1, \ldots, X_n$ be a random sample from $U(0, \theta)$ where $\theta \in \Theta = (0, a)$. For a fixed $\theta_0 \in (0, a)$ consider testing

$$H_0: \theta \le \theta_0$$

$$\text{against} \quad H_1: \theta > \theta_0$$

$f_\theta(\underline{x})$ has MLR in $T(\underline{X}) = X_{(n)}$.

$$f_{X_{(n)}}(x|\theta) = \frac{n x^{n-1}}{\theta^n}, \quad 0 < x < \theta.$$

UMP(α) test for testing H_0: against H_1 is

$$\phi^*(\underline{x}) = \begin{cases} 1, & x_{(n)} > c \\ 0, & x_{(n)} < c \end{cases}$$

where c is given by

$$E_{\theta_0}(\phi^*(\underline{X})) = \alpha$$

$$P_{\theta_0}(X_{(n)} > c) = \alpha$$

$$\Rightarrow \frac{1}{\theta_0^n}(\theta_0^n - c^n) = \alpha, \quad \text{i.e.} \quad c = \theta_0(1-\alpha)^{1/n}.$$

The power function is

$$\beta_{\phi^*}(\theta) = P_\theta(X_{(n)} > c) = 1-\left(\frac{c}{\theta}\right)^n = 1-\left(\frac{\theta_0}{\theta}\right)^n(1-\alpha).$$

Another UMP(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & x_{(n)} > \theta_0 \\ \alpha & x_{(n)} \le \theta_0 \end{cases}$$

Obviously above tests are also UMP(α) tests for testing $H_0: \theta = \theta_0$ vs. $H_1: \theta > \theta_0$.

UMP Tests For Two Sided Hypotheses

The following hypotheses are referred to as two-sided hypotheses

- $H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs. $H_1: \theta_1 < \theta < \theta_2$ .... (1)
- $H_0: \theta_1 \le \theta \le \theta_2$ vs. $H_1: \theta < \theta_1$ or $\theta > \theta_2$ ... (2)
- $H_0: \theta = \theta_0$ vs. $H_1: \theta \ne \theta_0$, ... (3)

where θ_0, θ_1 and θ_2 ($\theta_1 < \theta_2$) are fixed constants.

Suppose that $\underline{X}$ has p.d.f. belonging to one parameter exponential family with p.d.f.

$$f_\theta(\underline{x}) = \exp\{\eta(\theta)T(\underline{x}) - B(\theta)\}h(\underline{x}); \quad \ldots (4),$$

$\theta \in \Theta \subseteq \mathbb{R}$

The p.d.f. of $T \equiv T(\underline{X})$ is given by

$$g_\theta(t) = \exp\{\eta(\theta)t - B(\theta)\}m(t), \quad \text{(see Lehmann (1986))} \quad \ldots (5)$$

for some function $m(\cdot)$; $\theta \in \Theta$.

Clearly $T \equiv T(\underline{X})$ is a sufficient statistic for $\theta \in \Theta$ and, for any test function $\phi(\cdot)$,

$$\phi^*(T) = E_\theta(\phi(\underline{X})|T)$$

is a proper test function ($0 \le \phi^* \le 1$ and it does not depend on $\theta \in \Theta$ by virtue of sufficiency of T). Moreover ϕ^* has the same power function as ϕ, i.e.

$$\beta_{\phi^*}(\theta) = E_\theta(E_\theta(\phi(\underline{X})|T))$$

$$= E_\theta(\phi(\underline{X})) = \beta_\phi(\theta), \quad \forall \theta \in \Theta,$$

and thus it suffices to consider only those test functions that depend on observations only through sufficient statistic T.

The following lemma will be useful in deriving UMP tests for two-sided alternatives.

$\boxed{\text{Lemma}}$ Suppose that a r.v. T has a p.d.f./p.m.f. in the family of p.d.f.s/p.m.f.s $\mathcal{P} = \{g_\theta : \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}$. Suppose that the family $\mathcal{P}$ has MLR in T.

Let $\psi(\cdot)$ be a function with single sign change i.e. there exists $x_0 \in \mathbb{R}$ such that $\psi(x) \le 0$ $\forall x < x_0$ and $\psi(x) \ge 0$, $\forall x \ge x_0$. (i) Then there exists $\theta_0 \in \Theta$ such that $E_\theta(\psi(T)) \le 0$ $\forall \theta < \theta_0$ and $E_\theta(\psi(T)) \ge 0$, $\forall \theta > \theta_0$, unless $E_\theta(\psi(T))$ is either positive for all $\theta \in \Theta$ or negative for all $\theta \in \Theta$.

(ii) Suppose that $g_\theta(x) > 0$ for all $x \in \mathbb{R}$ and $\theta \in \Theta$, that $g_{\theta_2}(t)/g_{\theta_1}(t)$ is strictly increasing in t, whenever $\theta_2 > \theta_1$, and that $P_\theta(\psi(T) \ne 0) > 0$, $\forall \theta \in \Theta$.

If $E_{\theta_0}(\psi(T)) = 0$, then $E_\theta(\psi(T)) \leq 0$ for $\theta < \theta_0$

and $E_\theta(\psi(T)) \geq 0$ for $\theta > \theta_0$.

Proof. (i) Suppose that there exists $x_0 \in \mathbb{R}$ such that $\psi(x) \leq 0$, $\forall x < x_0$ and $\psi(x) \geq 0$ $\forall x > x_0$.

Let $\theta_1, \theta_2 \in \Theta$ and $\theta_1 < \theta_2$.

Claim $E_{\theta_1}(\psi(T)) > 0 \Rightarrow E_{\theta_2}(\psi(T)) > 0$

Suppose that $E_{\theta_1}(\psi(T)) > 0$ and $\theta_1 < \theta_2$.

If $\dfrac{g_{\theta_2}(x_0)}{g_{\theta_1}(x_0)} = \infty$, then $g_{\theta_1}(t) = 0$ $\forall t \geq x_0$ and

therefore $E_{\theta_1}(\psi(T)) = \int_{-\infty}^{x_0} \psi(t)\, g_{\theta_1}(t)\,dt \leq 0$, which

is not true. Thus $g_{\theta_2}(x_0)/g_{\theta_1}(x_0) = c < \infty$ $(0 \leq c < \infty)$

Then $g_{\theta_2}(t)/g_{\theta_1}(t) \leq c$, $\forall t < x_0$ and $g_{\theta_2}(t)/g_{\theta_1}(t) \geq c$, $\forall t \geq x_0$;

$\psi(t) \geq 0$ on the set $A = \{t : g_{\theta_1}(t) = 0$ and $g_{\theta_2}(t) > 0\}$ implying that

(as on the set $g_{\theta_2}(t)/g_{\theta_1}(t) = \infty$ and $g_{\theta_2}(t)/g_{\theta_1}(t) \uparrow$

for every $t \in A$, $t \geq x_0$)

Thus

$$E_{\theta_2}(\psi(X)) = \int_A \psi(t)\, g_{\theta_2}(t)\,dt + \int_{A^c} \psi(t)\, g_{\theta_2}(t)\,dt$$

$$\geq \int_{A^c} \psi(t)\, g_{\theta_2}(t)\,dt$$

$$= \int_{\{t:\, g_{\theta_1}(t) > 0\} \cup \{t:\, g_{\theta_2}(t) = 0\}} \psi(t)\, g_{\theta_2}(t)\,dt$$

$$= \int_{\{t:\, g_{\theta_1}(t) > 0\}} \psi(t)\, g_{\theta_2}(t)\,dt$$

$$= \int_{\{t:\, g_{\theta_1}(t) > 0,\ t < x_0\}} \psi(t)\, \frac{g_{\theta_2}(t)}{g_{\theta_1}(t)}\, g_{\theta_1}(t)\,dt + \int_{\{t:\, g_{\theta_1}(t) > 0,\ t \geq x_0\}} \psi(t)\, \frac{g_{\theta_2}(t)}{g_{\theta_1}(t)}\, g_{\theta_1}(t)\,dt$$

$$\geq \int_{\{t:\, g_{\theta_1}(t)>0,\ t<\lambda_0\}} c\, \psi(t)\, g_{\theta_1}(t)\, dt + \int_{\{t:\, g_{\theta_1}(t)>0,\ t\geq\lambda_0\}} c\, \psi(t)\, g_{\theta_1}(t)\, dt$$

$$= c\left[\int \psi(t)\, g_{\theta_1}(t)\, dt\right] = c\, E_{\theta_1}(\psi(T)) \geq 0$$

The result now follows by letting

$$\theta_0 = \inf\{\theta \in \Theta : E_\theta(\psi(T)) > 0\}$$

(ii) Similar to proof of (i) by noting that $g_{\theta_2}(t)/g_{\theta_1}(t)$ is strictly increasing in t, whenever $\theta_1 < \theta_2$.

Now let us recall the Generalized Neyman Pearson Lemma stated on page 18

$\boxed{\text{Theorem}}$ Suppose that $\underline{X}$ has p.d.f/p.m.f. belonging one-parameter exponential family given by (4), where $\eta(\cdot)$ is a strictly increasing function of $\theta \in \Theta$.

(a) For testing (a) (i.e. $H_0: \theta \leq \theta_0$ or $\theta \geq \theta_1$ vs $H_1: \theta \in (\theta_0, \theta_1)$) a UMP($\alpha$) ($0<\alpha<1$) test is

$$\phi^*(T) = \begin{cases} 1, & \text{if } c_1 < T < c_2 \\ \gamma_i, & \text{if } T = c_i,\ i=1,2 \\ 0, & \text{if } T < c_1 \text{ or } T > c_2 \end{cases} \quad \cdots (6)$$

where $c_1 < c_2$ and γ_i's are determined by

$$\beta_{\phi^*}(\theta_1) = \beta_{\phi^*}(\theta_2) = \alpha \quad \cdots (7)$$

(b) The test ϕ^* defined in (a) minimizes $\beta_\phi(\theta)$ over all $\theta < \theta_1$, $\theta > \theta_2$ among all ϕ satisfying $\beta_\phi(\theta_1) = \beta_\phi(\theta_2) = \alpha$.

(c) If ϕ_1 and ϕ_2 are two tests of the form (6) (with possibly different θ_1 and θ_2) and $\beta_{\phi_1}(\theta_0) = \beta_{\phi_2}(\theta_0)$ and the region $\{\phi_2 = 1\}$ is to the right of $\{\phi_1 = 1\}$ then $\beta_{\phi_1}(\theta) < \beta_{\phi_2}(\theta)$ $\forall\, \theta < \theta_0$, $\forall\, \theta > \theta_0$ and $\beta_{\phi_1}(\theta) > \beta_{\phi_2}(\theta)$, (with possibly different θ_1 and θ_2) If ϕ_1 and ϕ_2 both satisfy (6) and (7) then $\phi_1 \equiv \phi_2$ a.s. P.

Proof. We consider $\alpha \in (0, 1)$, as the proof for $\alpha \in \{0, 1\}$ follows trivially. It suffices to consider tests based on sufficient statistic $T \equiv T(\underline{X})$. The p.d.f. of T is

$$g_\theta(t) = \exp\{\eta(\theta)t - B(\theta)\} m(t)$$

Let $\theta_3 \in (\theta_1, \theta_2)$. Consider problem of maximizing $\beta_\phi(\theta_3)$ subject to $\beta_\phi(\theta_1) = \beta_\phi(\theta_2) = \alpha$.

By generalized NP Lemma maximizing ϕ^* is of the form

$$\phi^*(t) = \begin{cases} 1, & g_{\theta_3}(t) > c_1 g_{\theta_1}(t) + c_2 g_{\theta_2}(t) \\ 0, & g_{\theta_3}(t) < c_1 g_{\theta_1}(t) + c_2 g_{\theta_2}(t) \end{cases}$$

Note that

$$g_{\theta_3}(t) > c_1 g_{\theta_1}(t) + c_2 g_{\theta_2}(t)$$

$$\Leftrightarrow \exp\{\eta(\theta_3)t - B(\theta_3)\} m(t) > c_1 \exp\{\eta(\theta_1)t - B(\theta_1)\} m(t) + c_2 \exp\{\eta(\theta_2)t - B(\theta_2)\} m(t)$$

$\Leftrightarrow \quad a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1,$ where $b_1 < 0 < b_2$,

$a_1 = c_1 e^{-B(\theta_1) + B(\theta_3)}$, $a_2 = c_2 e^{-B(\theta_2) + B(\theta_3)}$

Case I: $a_1 \le 0$, $a_2 \le 0$

$\phi^*(t) = 1 \quad \forall t$

$\rightarrow$ Can not have size $\alpha \in (0,1)$

Case II $\quad a_1 \le 0$, $a_2 > 0$ (or $a_1 \ge 0$, $a_2 < 0$)

$a_1 e^{b_1 t} + a_2 e^{b_2 t} \uparrow (\downarrow)$ in t

Thus $\quad a_1 e^{b_1 t} + a_2 e^{b_2 t} < 1 \Leftrightarrow t < (>) k,$

$$\phi^*(\underline{x}) = \begin{cases} 1, & T < (>) k \\ 0, & T > (<) k \end{cases} \quad \downarrow_{\wedge}^{(\uparrow)} T$$

If $E_{\theta_1}(\phi^*(\underline{x})) = \alpha$ (i.e. $E_{\theta_1}(\phi^*(\underline{x}) - \alpha) = 0$), then $E_{\theta_2}(\phi^*(\underline{x}) - \alpha) < (>) 0$ (by last lemma, part (iii))

Thus $\beta_{\theta_1}(\varphi) = \beta_{\theta_2}(\varphi) = \alpha$ Can not be satisfied.

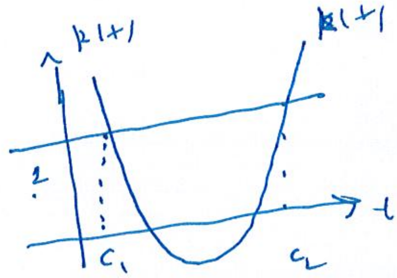
Case III $\quad a_1 > 0$, $a_2 > 0$

Let $k(t) = a_1 e^{b_1 t} + a_2 e^{b_2 t}$

$k'(t) = a_1 b_1 e^{b_1 t} + a_2 b_2 e^{b_2 t}$

$k'(t) > (<) 0 \Leftrightarrow (b_2 - b_1) t \underset{\wedge}{>}^{(<)} \ln\left(-\frac{a_1 b_1}{a_2 b_2}\right)$

$\Leftrightarrow t > (<) \frac{1}{b_2 - b_1} \ln\left(-\frac{a_1 b_1}{a_2 b_2}\right)$



Thus

$$\phi^*(\underline{x}) = \begin{cases} 1, & c_1 < T < c_2 \\ \gamma_i & T = c_i \\ 0 & T < c_1 \text{ or } T > c_2 \end{cases}$$

where $\nu_i \in [0,1]$, $i=1,2$, and c_is are real constants satisfying

$$E_{\theta_1}[\phi^*(\underline{X})] = E_{\theta_2}[\phi^*(\underline{X})] = \alpha$$

~~The second part of generalized NP lemma guarantees that~~

Thus ϕ^* is infact ~~UMP~~ most powerful test, for testing

$$H_0^*: \theta \in \{\theta_1, \theta_2\}$$

$$\text{vs } H_1^*: \theta = \theta_3$$

among all tests ϕ satisfying $E_{\theta_i}[\phi] = \alpha$, $i=1,2$. (as long as $\theta_3 \in (\theta_1, \theta_2)$)

Since ϕ^* does not depend on θ_3 for testing

ϕ^* is also UMP test of size α for testing

$$H_0^*: \theta \in \{\theta_1, \theta_2\}$$

$$\text{vs } H_1: \theta_1 < \theta < \theta_2$$

among all tests satisfying $E_{\theta_i}[\phi] = \alpha$, $i=1,2$.

Now comparing ϕ^* with the no-data test $\phi_0(\underline{x}) = \alpha$, $\forall \underline{x}$

and using (b), we conclude that

$$\beta_{\phi^*}(\theta) \le \beta_{\phi_0}(\theta) = \alpha, \quad \forall \theta \in (-\infty, \theta_1) \cup (\theta_2, \infty)$$

$$\Rightarrow \beta_{\phi^*}(\theta) \le \alpha, \quad \forall \theta < \theta_1 \text{ or } \theta > \theta_2$$

$\Rightarrow$ $\sup_{\theta \in (-\infty, \theta_1] \cup [\theta_2, \infty)} \beta_{\phi^*}(\theta) = \alpha$, i.e. ϕ^* is size α test for $H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs $H_1: \theta \in (\theta_1, \theta_2)$

Consequently ϕ^* is UMP test of size α for testing

$H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs $H_1: \theta \in (\theta_1, \theta_2)$

(b) Let $\theta_4 < \theta_1$. Consider minimizing $\beta_\phi(\theta_4) = E_{\theta_4}[\phi]$ (or maximizing $E_{\theta_4}(1-\phi)$) among all ϕ's s.t. $E_{\theta_1}(\phi) = E_{\theta_2}(\phi) = \alpha$. (or $E_{\theta_1}(1-\phi) = E_{\theta_2}(1-\phi) = 1-\alpha$). By generalised NP lemma the minimizing ϕ^{**} is such that

$$1-\phi^{**} = 1 \Leftrightarrow g_{\theta_4}(t) > c_1 g_{\theta_1}(t) + c_2 g_{\theta_2}(t)$$

$$1-\phi^{**} = 0 \Leftrightarrow g_{\theta_4}(t) < c_1 g_{\theta_1}(t) + c_2 g_{\theta_2}(t)$$

$$g_{\theta_4}(t) < c_1\, g_{\theta_1}(t) + c_2\, g_{\theta_2}(t)$$

$$\Leftrightarrow \quad e^{(\eta(\theta_4) - \eta(\theta_1))t + B(\theta_1) - B(\theta_2)} - c_2\, e^{(\eta(\theta_2) - \eta(\theta_1))t + B(\theta_1) - B(\theta_2)} < c_1$$

$$\Leftrightarrow \quad a_1 e^{b_1 t} + a_2 e^{b_2 (t)} < c_1 \quad \text{where} \quad a_1 = e^{B(\theta_1) - B(\theta_2)} > 0,$$

$$b_1 = \eta(\theta_4) - \eta(\theta_1) < 0$$

$$a_2 = -c_2\, e^{B(\theta_1) - B(\theta_2)} \in \mathbb{R}$$

$$b_2 = \eta(\theta_2) - \eta(\theta_1) > 0.$$

As in (a), the only possibility is $a_1 > 0$, $a_2 > 0$, $c_1 > 0$, in which case $\phi^{**} = \phi^*$.

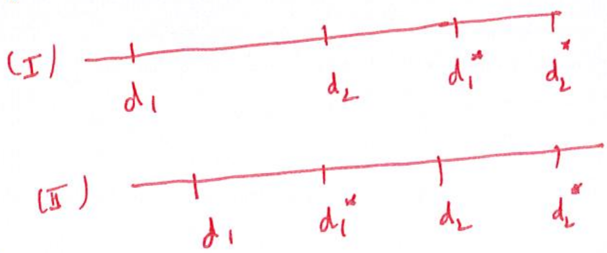
Similarly one may minimize $\beta_\phi(\theta_5)$, for $\theta_5 > \theta_2$, and get ϕ^* as the desired test.

(C) Let

$$\phi_1(t) = \begin{cases} 1 & d_1 < t < d_2 \\ \gamma_i & t = d_i,\ i = 1,2 \\ 0, & t < d_1 \text{ or } t > d_2 \end{cases},$$

$$\phi_2(t) = \begin{cases} 1 & d_1^* < t < d_2^* \\ \gamma_i^* & t = d_i^*,\ i = 1,2 \\ 0, & t < d_1^* \text{ or } t > d_2^* \end{cases},$$

$\beta_{\phi_1}(\theta_0) = \beta_{\phi_2}(\theta_0)$ and the region $\{\phi_2 = 1\}$ is to right of the region $\{\phi_1 = 1\}$



The following two cases I and (II) arise. Consider

$$\phi_2(t) - \phi_1(t) = \begin{cases} 0\ (0) & t < d_1 \\ \leq 0\ (-) & t = d_1 \\ -\ (-) & d_1 < t < d_2\ (d_1 < t < d_1^*) \\ \leq 0\ (-) & t = d_2\ (t = d_1^*) \\ -\ (0) & d_2 < t < d_1^*\ (d_1^* < t < d_2) \\ \geq 0\ (+) & t = d_1^*\ (t = d_2) \\ +\ (+) & d_1^* < t < d_2^*\ (d_2 < t < d_2^*) \\ \geq 0\ (+) & t = d_2^*\ (\\ 0 & t > d_2^* \end{cases}$$

⇒ $\phi_2(t) - \phi_1(t)$ has ~~at most~~ one change in sign (from $-$ to $+$)

~~⇒ $E_\theta[\phi_2(t) - \phi_1(t)]$ has at most one change in sign (from $-$ to $+$ve).~~

Since $E_{\theta_0}[\phi_2(T) - \phi_1(T)] = 0$, we get from last lemma

$$E_\theta[\phi_2 - \phi_1] < 0, \quad \forall\, \theta < \theta_0$$

$$E_\theta[\phi_2 - \phi_1] > 0, \quad \forall\, \theta > \theta_0$$

$$\Rightarrow \quad \beta_{\phi_2}(\theta) < \beta_{\phi_1}(\theta), \quad \forall\, \theta < \theta_0$$

and $\beta_{\phi_2}(\theta) > \beta_{\phi_1}(\theta) \quad \forall\, \theta > \theta_0$

Now suppose $E_{\theta_i}[\phi_1] = E_{\theta_i}[\phi_2] = \alpha$, $i = 1, 2$.

$E_{\theta_1}[\phi_1] = E_{\theta_1}[\phi_2]$

⇒ the region $\{\phi_2 = 1\}$ lies either to the left or to the right of $\{\phi_1 = 1\}$ (as the possibility 

is ruled out)

Suppose $\{\phi_2 = 1\}$ lies to the right of $\{\phi_1 = 1\}$. Then $\phi_2 - \phi_1$ has at most one change in sign (from $-$ to $+$) and $E_{\theta_2}[\phi_2 - \phi_1] = 0$. This implies that

$$E_\theta[\phi_2 - \phi_1] < 0, \quad \forall\, \theta < \theta_1$$

$$E_\theta[\phi_2 - \phi_1] > 0, \quad \forall\, > \theta_1$$

$$\Rightarrow \quad \beta_{\phi_2}(\theta_1) \neq \beta_{\phi_1}(\theta_1), \quad \text{a contradiction.}$$

Thus $\phi_2 = \phi_1$ a.s. P

Remark: (a) From part (c) of above theorem we conclude that c_i's and γ_i's are uniquely determined.

(b) One can start with a trial test $\phi_1^{(0)}$ with trial values $c_1^{(0)}$ and $\gamma_1^{(0)}$. Then find $c_2^{(0)}$ and $\gamma_2^{(0)}$ s.t. $\beta_{\phi_1^{(0)}}(\theta_1) = \alpha = \beta_{\phi^*}(\theta_1)$ and compute $\beta_{\phi_1^{(0)}}(\theta_2)$. If $\beta_{\phi_1^{(0)}}(\theta_2) < \alpha = \beta_{\phi^*}(\theta_2)$, then $\phi_1^{(0)}$ is to the left of ϕ^* (i.e. ϕ^* is to the right of $\phi_1^{(0)}$). Therefore one should try $c_1^{(1)} > c_1^{(0)}$ or $c_1^{(1)} = c_1^{(0)}$ and $\gamma_1^{(1)} > \gamma_1^{(0)}$. The converse holds if $\beta_{\phi_1^{(0)}}(\theta_2) > \alpha$.

(c) When the distribution of X does not belong to exponential family, UMP tests for testing $H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs $H_1: \theta_1 < \theta < \theta_2$ exist in some cases. (Ex: $h_\theta(x) = \theta f_0(x) + (1-\theta) g_0(x)$, $-\infty < x < \infty$, $\theta \in (0,1) = \Theta$, where f_0 and g_0 are given pdfs. In this case it can be shown that $\phi^*(\underline{x}) = \alpha$ is UMP(α) test for testing $H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs. $H_1: \theta_1 < \theta < \theta_2$)

(d) UMP tests, in general, do not exist for testing

$$H_0: \theta_1 \le \theta \le \theta_2 \quad \text{vs.} \quad H_1: \theta < \theta_1 \text{ or } \theta > \theta_2$$

or

$$H_0: \theta = \theta_0 \quad \text{vs} \quad H_1: \theta \ne \theta_0.$$

Example (Non-Existence of UMP tests)

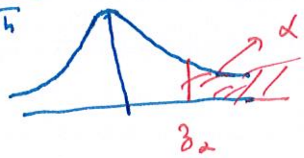
Let $X_1, \dots, X_n$ be a random sample from $N(\theta, 1)$, where $\theta \in \mathbb{R} = \Theta$, say. Show that UMP($\alpha$) test does not exist for testing

$$H_0: \theta = 0$$

$$\text{vs} \quad H_1: \theta \ne 0$$

Solution On contrary suppose that there exists a UMP(α) test, say ϕ^*. Consider testing $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$, where $\theta_1 > \theta_0$ is a fixed constant. The UMP(α) test is

$$\phi_1(\underline{x}) = \begin{cases} 1 & \bar{x} > \dfrac{z_\alpha}{\sqrt{n}} \\ 0 & \text{o.w.} \end{cases}$$



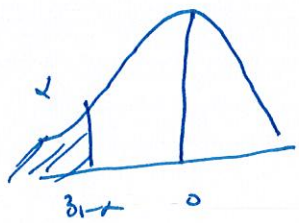
and $\quad \beta_{\phi_1}(\theta_1) = \beta_{\phi^*}(\theta_1)$

By uniqueness of MP(α) test we have $\phi_1 \equiv \phi^*$, a.e..

Now consider testing $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_2$ where $\theta_2 < \theta_0$.

A MP(α) test is

$$\phi_2(\underline{x}) = \begin{cases} 1, & \bar{x} < \dfrac{z_{1-\alpha}}{\sqrt{n}} \\ 0 & \text{o.w} \end{cases}$$



and $\quad \beta_{\phi_2}(\theta_2) = \beta_{\phi^*}(\theta_2)$.

By uniqueness of MP(α) test we have

$$\phi_2 \equiv \phi^* \quad \text{a.e.}$$

Thus we have

$$\phi^* = \phi_1 = \phi_2 \quad \text{a.e.},$$

leading to contradiction.

Remark: (1) One can carry out the analysis of above example using completeness of $\{N(\theta,1): \theta > 0\}$ and $\{N(\theta,1): \theta < 0\}$ and consider testing problems (a) $H_0: \theta = 0$ vs $H_1: \theta > 0$; (b) $H_0: \theta = 0$ vs $H_1: \theta < 0$. In the above analysis completeness of families $\{N(\theta,1): \theta > 0\}$ and $\{N(\theta,1): \theta < 0\}$ is crucial and the analysis will break down if any of these subfamilies is not complete, as the following example illustrates.

Example Let $X_1, \dots, X_n$ be a random sample from $N(\theta,1)$ where $\theta \in \Theta = \mathbb{R}$. Let θ_1 and θ_2 $(\theta_1 < \theta_2)$ be fixed real constants. Consider testing

$$H_0: \theta \le \theta_1 \text{ or } \theta \ge \theta_2$$

$$\text{vs } H_1: \theta_1 < \theta < \theta_2.$$

A UMP(α) test of size α is

$$\phi_*(\underline{x}) = \begin{cases} 1, & c_1 < \bar{x} < c_2 \\ 0, & \text{Otherwise} \end{cases}$$

where $c_1 < c_2$ is determined by

$E_{\theta_1}(\phi_*(\underline{X})) = E_{\theta_2}(\phi_*(\underline{X})) = \alpha \iff$ $\Phi(\sqrt{n}(c_2 - \theta_1)) - \Phi(\sqrt{n}(c_1 - \theta_1)) = \alpha$, $\quad \Phi(\sqrt{n}(c_2 - \theta_2)) - \Phi(\sqrt{n}(c_1 - \theta_2)) = \alpha$

Uniformly Most Powerful Unbiased (UMPU) Tests

When the UMP test does not exist, we may restrict to a smaller class of reasonable tests to find optimal test in this class of tests. One such restricted class of tests is the class of unbiased tests.

Note that a UMP(α) test ϕ^* for testing $H_0: f \in \mathcal{P}_0$ vs $H_1: f \in \mathcal{P}_1$, has the property

$$\beta_{\phi^*}(f) \le \alpha, \quad \forall f \in \mathcal{P}_0 \quad \text{and} \quad \beta_{\phi^*}(f) \ge \alpha \quad \forall f \in \mathcal{P}_1 \quad \ldots (I)$$

The later inequality above follows on comparing ϕ^* with no-data test $\phi_0(\underline{x}) = \alpha$ $\forall \underline{x}$.

Note that (I) is equivalent to saying that ϕ^* is at least as good as no-data test ϕ_0. (or power $\ge$ level of significance or size of the test)

Definition Consider testing

$$H_0: f \in \mathcal{P}_0 \quad \ldots \quad (A)$$
$$\text{vs } H_1: f \in \mathcal{P}_1,$$

at α ($\alpha \in (0,1)$) level of significance, where $\mathcal{P}_0$ and $\mathcal{P}_1$ are non-empty classes of p.d.f.s such that $\mathcal{P}_0 \cap \mathcal{P}_1 = \phi$ and $\mathcal{P}_0 \cup \mathcal{P}_1 = \mathcal{P}$.

(i) A test ϕ for testing (A) is said to be unbiased if $\beta_\phi(f) \le \alpha$, $\forall f \in \mathcal{P}_0$ and $\beta_\phi(f) \ge \alpha$ $\forall f \in \mathcal{P}_1$. (power $\ge$ level/size)

(ii) A test of level α is called a uniformly most powerful unbiased (UMPU) test (UMPU(α)) iff it is UMP within the class of unbiased tests of level α.

Let $\mathcal{P} = \{f_\theta : \theta \in \Theta\}$ where $\Theta \subseteq \mathbb{R}^k$. Let $\Theta_0 \subseteq \Theta$ and $\Theta_1 \subseteq \Theta$ be such that $\Theta_i \ne \phi$, $i=0,1$, $\Theta_0 \cap \Theta_1 = \phi$ and $\Theta_0 \cup \Theta_1 = \Theta$. Consider testing

$$H_0: \theta \in \Theta_0 \quad \ldots \ldots (II)$$
$$\text{vs } H_1: \theta \in \Theta_1$$

at α ($\alpha \in (0,1)$) level of significance.

Definition: Consider the testing problem (II). Let $\alpha \in (0,1)$ be a given level of significance and let $\overline{\Theta}_{01}$ be the common boundary of Θ_0 and Θ_1, i.e. the set of points that are ~~points or~~ limit points of both Θ_0 and Θ_1. (a) A test ϕ is said to be similar (or α-similar) on the boundary $\overline{\Theta}_{01}$ iff

$$\beta_\phi(\theta) = \alpha, \quad \forall\ \theta \in \overline{\Theta}_{01}$$

(b) For a given test ϕ, the power function $\beta_\phi(\theta) = E_\theta(\phi(X))$ is said to be continuous in $\theta \in \Theta$ iff for any sequence $\{\theta_m\}_{m\geq 1} \in \Theta$ with $\theta_m \to \theta_0$ as $m \to \infty$ $(\theta_0 \in \Theta)$, we have $\beta_\phi(\theta_m) \to \beta_\phi(\theta_0)$.

Define

$\mathcal{C}_S$ = class of α-similar tests

$= \{\phi : 0 \leq \phi \leq 1,\ \beta_\phi(\theta) = \alpha,\ \forall\ \theta \in \overline{\Theta}_{01}\}$

$\mathcal{C}_U$ = class of all unbiased tests

$= \{\phi : 0 \leq \phi \leq 1,\ \beta_\phi(\theta) \leq \alpha\ \forall\ \theta \in \Theta_0$ and $\beta_\phi(\theta) \geq \alpha,\ \forall\ \theta \in \Theta_1\}$.

Lemma If for every test function ϕ, $\beta_\phi(\theta)$ is continuous in $\theta \in \Theta$, then $\mathcal{C}_U \subseteq \mathcal{C}_S$.

Proof. Let $\phi \in \mathcal{C}_U$. Then

$\beta_\phi(\theta) \leq \alpha,\ \forall\ \theta \in \Theta_0$ and $\beta_\phi(\theta) \geq \alpha,\ \forall\ \theta \in \Theta_1$.

Let $\theta_0 \in \overline{\Theta}_{01}$. Then θ_0 is a limit point of Θ_0 and Θ_1 both. Thus there exist sequences $\{\theta_m^{(0)}\}_{m\geq 1} \in \Theta_0$ and $\{\theta_m^{(1)}\}_{m\geq 1} \subseteq \Theta_1$ such that

$$\lim_{m\to\infty} \theta_m^{(0)} = \lim_{m\to\infty} \theta_m^{(1)} = \theta_0$$

Then

$$\beta_\phi(\theta_m^{(0)}) \leq \alpha,\ \forall\ m \geq 1 \quad \text{and} \quad \beta_\phi(\theta_m^{(1)}) \geq \alpha,\ \forall\ m\geq 1 \quad (\phi \text{ is unbiased})$$

$$\lim_{m\to\infty} \beta_\phi(\theta_m^{(0)}) \leq \alpha \quad \text{and} \quad \lim_{m\to\infty} \beta_\phi(\theta_m^{(1)}) \geq \alpha$$

$\Rightarrow \beta_\phi(\theta_0) \leq \alpha$ and $\beta_\phi(\theta_0) \geq \alpha$ (Continuity of $\beta_\phi(\theta)$)

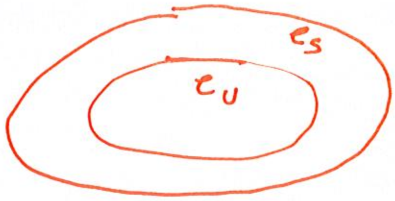
$\Rightarrow \beta_\phi(\theta_0) = \alpha$

Since $\theta_0 \in \bar{\Theta}_{01}$ was arbitrary, we have

$$\beta_\phi(\theta) = \alpha, \quad \forall\, \theta \in \bar{\Theta}_{01}$$

$\Rightarrow \phi \in e_S$.

Consequently $e_U \subseteq e_S$.



Theorem Consider testing

$$H_0: \theta \in \Theta_0$$
$$\text{vs} \quad H_1: \theta \in \Theta_1$$

at α ($\alpha \in (0,1)$) level of significance. Suppose that, for every test function ϕ, $\beta_\phi(\theta) = E_\theta(\phi(X))$ is a continuous function of $\theta \in \Theta$. Let ϕ^* be a UMP test among all tests ϕ satisfying $\beta_\phi(\theta) = \alpha$, $\forall\, \theta \in \bar{\Theta}_{01}$ and has level α.
(i.e. ϕ^* is UMP(α) test among all α-similar tests)
Then ϕ^* is UMPU(α) test.

Proof By the last lemma $e_U \subseteq e_S$. Thus it suffices to show that $\phi^* \in e_U$. Consider the no-data test $\phi_0(\underline{x}) \equiv \alpha$, $\forall\, \underline{x}$. Then $\phi_0 \in e_S$ and therefore

$$\beta_{\phi^*}(\theta) \geq \beta_{\phi_0}(\theta), \quad \forall\, \theta \in \Theta, \quad (\phi^* \text{ is UMP within } e_S)$$

$$\beta_{\phi^*}(\theta) \geq \alpha, \quad \forall\, \theta \in \Theta_1 \quad \ldots (A)$$

Since ϕ^* has level α, we have

$$\beta_{\phi^*}(\theta) \leq \alpha, \quad \forall\, \theta \in \Theta_0 \quad \ldots (B)$$

$$(A) + (B) \Rightarrow \phi^* \in e_U.$$

Hence the result follows.

Remark Continuity of $\beta_\phi(\theta)$, for any test function ϕ, is satisfied for distributions belonging to exponential family (to be discussed later) a sufficient statistic for $\theta \in \bar{\Theta}_{01}$. Define, for a test function ϕ,

~~for a sufficient statistic (under $\theta \in \bar{\Theta}_{01}$)~~

$$K_\phi(U) = E_\theta(\phi(X) \mid \underline{U}), \quad \theta \in \bar{\Theta}_{01}; \quad \text{(does not depend on } \theta \in \bar{\Theta}\text{)}$$

~~= family of distributions of U as θ varies over $\bar{\Theta}_{01}$~~

$K_\phi(U)$ does not depend on $\theta \in \bar{\Theta}_{01}$ (as U is sufficient for $\theta \in \bar{\Theta}_{01}$)

~~ϕ be a test function satisfying~~ Let ϕ be a test function satisfying

$$\cancel{E(\phi(X) \mid U) = \alpha} \text{ a.s. } P_U, \qquad P_\theta(K_\phi(U) = \alpha) = 1 \quad \forall \theta \in \bar{\Theta}_{01}. \quad \ldots \quad (c)$$

Then

$$E_\theta(\phi(X)) = E_\theta(K_\phi(U)) = \alpha \quad \forall \theta \in \bar{\Theta}_{01},$$

i.e. ϕ is α-similar on $\bar{\Theta}_{01}$ (i.e. $E_\theta(\phi(X)|U) = \alpha$, $\forall \theta \in \bar{\Theta}_{01}$)

A test ϕ satisfying (c) (i.e. E_θ...) is said to have Neyman structure w.r.t. U. Define

$\mathcal{C}_N$ = class of all tests having Neyman structure. Then

$$\mathcal{C}_N \subseteq \mathcal{C}_S$$

Thus if all α-similar tests on $\bar{\Theta}_{01}$ have Neyman structure (i.e. $\mathcal{C}_S \subseteq \mathcal{C}_N$) wrt U, then $\mathcal{C}_S = \mathcal{C}_N$ and working with $\mathcal{C}_S$ is the same as working with $\mathcal{C}_N$.

Theorem Let $U(\underline{X})$ be a sufficient statistic for $\theta \in \bar{\Theta}_{01}$. Then a necessary and sufficient condition for $\mathcal{C}_S = \mathcal{C}_N$ is that U is boundedly complete for $\theta \in \bar{\Theta}_{01}$.

Proof. First suppose that U is sufficient and boundedly complete for $\theta \in \bar{\Theta}_{01}$. Let $\phi \in \mathcal{C}_S$. Then

$$E_\theta(\phi(X)) = \alpha, \quad \forall \theta \in \bar{\Theta}_{01}$$

$$\Rightarrow \quad E_\theta(\phi(\underline{X}) - \alpha) = 0 \quad \forall \theta \in \bar{\Theta}_{01}$$

Consider

$$\psi(U) = E_\theta((\phi(X) - \alpha) \mid U) = E_\theta(\phi(X) \mid U) - \alpha = K_\phi(U) - \alpha, \quad \theta \in \bar{\Theta}_{01}$$

Then ψ is bounded ($-\alpha \le \psi \le 1-\alpha$) and

$$E_\theta(\psi(U)) = E_\theta(E_\theta(\phi(X) - \alpha) \mid U) = 0, \quad \forall \theta \in \bar{\Theta}_{01}$$

$$\Rightarrow \quad P_\theta(\psi(U) = 0) = 1 \quad \forall \theta \in \bar{\Theta}_{01}$$

~~$\Rightarrow \psi(U)=0$ a.s. P_U~~ $P_\theta(K_\phi(U)=\alpha)=1 \quad \forall\, \theta \in \bar{\Theta}_{01}$

$\Rightarrow$ ~~$E_\theta[\phi(\underline{X})\mid U]=\alpha$ a.s. P_U~~

$\Rightarrow \quad \phi \in \mathcal{C}_r$

$\Rightarrow \quad \mathcal{C}_s \subseteq \mathcal{C}_r$ and $\mathcal{C}_s = \mathcal{C}_r$

Conversely suppose that $\mathcal{C}_s \subseteq \mathcal{C}_r$. On contrary suppose that U is not boundedly complete for $\theta \in \bar{\Theta}_{01}$. Then there exists a function B s.t. $|B(U)| \leq c < \infty$ for some constant $0<c<\infty$, $E_\theta(B(U))=0 \ \forall\, \theta \in \bar{\Theta}_{01}$ and $P_\theta(B(U)\neq 0)>0$, for some $\theta \in \bar{\Theta}_{01}$.

Define

$$\phi(\underline{X}) = \alpha + d\,B(U),$$

where $d = \dfrac{\min(\alpha, 1-\alpha)}{c}$. Then $0 \leq \phi \leq 1$ and

$$E_\theta(\phi(\underline{X})) = \alpha, \quad \forall\, \theta \in \bar{\Theta}_{01},$$

i.e. $\phi \in \mathcal{C}_s$. By hypothesis, then $\phi \in \mathcal{C}_r$ i.e. $P_\theta(K_\phi(U)=\alpha)=1, \ \forall\, \theta \in \bar{\Theta}_{01}$ $\Rightarrow P_\theta(B(U)=0)=1 \quad \forall\, \theta \in \bar{\Theta}_{01}$

~~$E_\theta(\phi(\underline{X})\mid U)=\alpha$, a.s. P_U~~ $\Rightarrow$

~~But, we have $E_\theta(\phi(\underline{X})\mid U) = \alpha + d\,B(U) \neq \alpha$, a.s. P_U~~

leading to a contradiction.

Hence U is boundedly complete for $\theta \in \bar{\Theta}_{01}$.

Theorem Let $\underline{X}$ be a random vector having pdf/pmf ~~canonical~~ Canonical form belonging to exponential family having

$$f_\eta(\underline{x}) = \exp\left\{\sum_{i=1}^{p} \eta_i T_i(\underline{x}) - \psi(\underline{\eta})\right\} h(\underline{x}), \qquad \eta \in \Theta_N,$$

where Θ_N is the natural parameter space.

(a) The r.v. $\underline{T} = (T_1(\underline{X}), \ldots, T_p(\underline{X}))$ has the following p.d.f. in an exponential family

$$g_\eta(\underline{t}) = \exp\left\{\sum_{i=1}^{p} \eta_i t_i - \psi(\underline{\eta})\right\} m(\underline{t}), \qquad \eta \in \Theta_N$$

where $m(\underline{t})$ is a non-negative function

(ii) If η_0 is an interior point of natural parameter space Θ_N and K is a function such that $E_{\eta_0}(K(\underline{X})) \in (-\infty, \infty)$, then the function $E_\eta(K(\underline{X}))$ is infinitely differentiable in a neighborhood of η_0 and the derivatives may be computed by differentiating under the integral sign.

UMPU Tests For Exponential Family

Suppose that $\underline{X}$ has a p.d.f. / p.m.f. belonging to multiparameter exponential family with p.d.f. of $\underline{X}$ as

$$f_{\theta, \underline{\psi}}(\underline{x}) = \exp\left\{\theta Y(\underline{x}) + \sum_{i=1}^{k} \psi_i U_i(\underline{x}) - \xi(\theta, \underline{\psi})\right\} h(\underline{x}) \quad \ldots\ldots (A),$$

where θ and Y are real-valued, $\underline{\psi} = (\psi_1, \ldots, \psi_k)^t$ and $\underline{U} = (U_1, \ldots, U_k)^t$ are vector-valued. Let the range of $\underline{\psi} = (\psi_1, \ldots, \psi_k)$ contain a k-dimensional rectangle in $\mathbb{R}^k$ so that for any fixed θ $\underline{U} = (U_1, \ldots, U_k)$ is a C-S statistic

Clearly $(Y, \underline{U})$ is a sufficient statistic with p.d.f. of $(Y, \underline{U})$ given by

$$h_{\theta, \underline{\psi}}(y, \underline{u}) = \exp\left\{\theta y + \sum_{i=1}^{k} \psi_i u_i - \xi(\theta, \underline{\psi})\right\} m(y, \underline{u})$$

and the conditional p.d.f. of Y given $\underline{U} = \underline{u}$ has p.d.f.

$$g_\theta(y \mid \underline{u}) = e^{\theta y}\, l(\underline{u}, y)$$

Also, note that, for any fixed θ, $\underline{U} = (U_1, \ldots, U_k)$ is sufficient for $\underline{\psi}$.

For exponential family, continuity assumption of any test function ϕ holds and thus $\mathcal{C}_S = \mathcal{C}_N$, i.e. working with $\mathcal{C}_S$ is the same as working with $\mathcal{C}_N$.

Theorem Suppose that the distribution of $\underline{X}$ is in multiparameter natural exponential family with pdf/pmf

$$f_{\theta,\underline{\psi}}(\underline{x}) = \exp\left\{\theta Y(\underline{x}) + \sum_{i=1}^{k} \psi_i U_i(\underline{x}) - \xi(\theta,\underline{\psi})\right\} h(\underline{x}) \quad \cdots (A),$$

where θ and Y are real-valued, $\underline{\psi} = (\psi_1, \ldots, \psi_k)^t$ and $\underline{U} = (U_1, \ldots, U_k)^t$ are vector-valued and for any fixed θ, the range of $\underline{\psi} = (\psi_1, \ldots, \psi_k)$ contains a k-dimensional rectangle in $\mathbb{R}^k$. Let θ_0, θ_1, θ_2, be fixed constants.

(a) For testing $H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$, a UMPU(α) test having size α is

$$\phi_1^*(Y, \underline{U}) = \begin{cases} 1, & Y > c(\underline{U}) \\ \gamma(\underline{U}), & Y = c(\underline{U}) \\ 0, & Y < c(\underline{U}) \end{cases},$$

where $c(u)$ and $\gamma(u)$ are functions determined by

$$E_{\theta_0}(\phi_1^*(Y, \underline{U}) \mid \underline{U} = \underline{u}) = \alpha,$$

for every $\underline{u}$, and $E_{\theta_0}(\cdot)$ is the expectation wrt $f_{\theta_0,\underline{\psi}}$

(b) For testing $H_0: \theta \le \theta_1$ or $\theta \ge \theta_2$ vs. $H_1: \theta_1 < \theta < \theta_2$, a UMPU($\alpha$) test, having size α, is

$$\phi_2^*(Y, \underline{U}) = \begin{cases} 1, & \text{if } c_1(\underline{U}) < Y < c_2(\underline{U}). \\ \gamma_i(\underline{U}), & \text{if } Y = c_i(\underline{U}) \quad i=1,2 \\ 0, & \text{if } Y < c_1(\underline{U}) \text{ or } Y > c_2(\underline{U}), \end{cases}$$

where $c_i(\underline{u})$'s and $\gamma_i^*(u)$'s are determined by

$$E_{\theta_1}(\phi_2^*(Y, \underline{U}) \mid \underline{U} = \underline{u}) = E_{\theta_2}(\phi_2^*(Y, \underline{U}) \mid \underline{U} = \underline{u}) = \alpha,$$

for every $\underline{u}$

(c) For testing $H_0: \theta_1 \le \theta \le \theta_2$ vs $H_1: \theta < \theta_1$ or $\theta > \theta_2$ a UMPU(α) test, having size α, is

$$\phi_3^*(Y, \underset{\sim}{U}) = \begin{cases} 1, & \text{if } Y < c_1(\underset{\sim}{U}) \text{ or } Y > c_2(\underset{\sim}{U}) \\ \gamma_i(\underset{\sim}{U}) & \text{if } Y = c_i(\underset{\sim}{U}), \\ 0 & \text{if } c_1(\underset{\sim}{U}) < Y < c_2(\underset{\sim}{U}) \end{cases} \quad \cdots (B)$$

where $c_i(\underset{\sim}{U})$'s and $\gamma_i(\underset{\sim}{U})$'s are determined by

$$E_{\theta_1}(\phi_3^*(Y, \underset{\sim}{U}) \mid \underset{\sim}{U} = \underset{\sim}{u}) = E_{\theta_2}(\phi_3^*(Y, \underset{\sim}{U}) \mid \underset{\sim}{U} = \underset{\sim}{u}) = \alpha,$$

for every $\underset{\sim}{u}$.

(d) For testing $H_0: \theta = \theta_0$ vs $H_1: \theta \ne \theta_0$ a UMPU(α) test having size α, is given by (B) where $c_i(\underset{\sim}{U})$'s and $\gamma_i(\underset{\sim}{U})$'s are determined by

$$E_{\theta_0}(\phi_3^*(Y, \underset{\sim}{U}) \mid \underset{\sim}{U} = \underset{\sim}{u}) = \alpha$$

and $$E_{\theta_0}(\phi_3^*(Y, \underset{\sim}{U})Y \mid \underset{\sim}{U} = \underset{\sim}{u}) = \alpha\, E_{\theta_0}(Y \mid \underset{\sim}{U} = \underset{\sim}{u}),$$

for every $\underset{\sim}{u}$.

Proof. Note that $(Y, \underset{\sim}{U})$ is a sufficient statistic for $(\theta, \underset{\sim}{\psi})$. Thus it suffices to consider only those test functions that depend on $\underset{\sim}{X}$ only through $(Y, \underset{\sim}{U})$.

(a) We have $\bar{\Theta}_{01} = \{(\theta, \underset{\sim}{\psi}): \theta = \theta_0\}$. Clearly $\underset{\sim}{U}$ is C-S for $\underset{\sim}{\theta} \in \bar{\Theta}_{01}$ and consequently $e_S = e_N$ and power functions of all tests are continuous function of $\underset{\sim}{\theta}$. Thus if ϕ^* is UMP α-similar test and has level α, then ϕ^* is UMPU(α) test. It is enough to show that ϕ_1^* is UMP among all tests having Neyman-structure wrt $\underset{\sim}{U}$. Conditional p.d.f. of Y given $\underset{\sim}{U} = \underset{\sim}{u}$ is

$$g_\theta(y \mid \underset{\sim}{u}) = e^{\theta y}\, l(y, \underset{\sim}{u})$$

The assertion follows from a result done before.

(b) Here $\bar{\Theta}_{01} = \{(\theta, \underline{\psi}) : \theta \in \{\theta_1, \theta_2\}\}$. Let $\bar{\Theta}_1^* = \{(\theta, \underline{\psi}) : \theta = \theta_1\}$ and $\bar{\Theta}_2^* = \{(\theta, \underline{\psi}) : \theta = \theta_2\}$. Then $\underline{U}$ is C-S under $\bar{\Theta}_1^*$ and $\bar{\Theta}_2^*$ and it suffices to find UMP test ~~based~~ among tests having Neyman structure. Using the result done before the assertion follows.

(C) From a result done before we know that ϕ_2^* of ~~(b)~~ (b) minimizes $E_\theta(\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u})$ over all $\theta < \theta_1$, $\theta > \theta_2$ and ϕ satisfying $E_{\theta_1}(\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}) = E_{\theta_2}(\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u})$ $= 1-\alpha$ (~~replace~~ using (b) with α replaced by $1-\alpha$). Also $E_\theta(\phi_2^*(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}] \geq 1-\alpha \; \forall \; \theta_1 < \theta < \theta_2$

$\Rightarrow$ $1 - \phi_2^*(\underline{Y}, \underline{U})$ maximizes $E_\theta[1 - \phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}]$ over all $\theta < \theta_1$, $\theta > \theta_2$ and ϕ satisfying $E_{\theta_1}[1 - \phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}] = E_{\theta_2}[1 - \phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}]$ $= \alpha$. Also $E_\theta[1 - \phi_2^*(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}] \leq \alpha, \; \forall \; \theta_1 < \theta < \theta_2$

Since $\{\phi : 0 \leq \phi \leq 1\} = \{1 - \phi : 0 \leq \phi \leq 1\}$, we conclude that the test in (c) maximizes $E_\theta[\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}]$ over all $\theta < \theta_1$, $\theta > \theta_2$ and ϕ satisfying

$$E_{\theta_1}[\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}] = E_{\theta_2}[\phi(\underline{Y}, \underline{U}) \mid \underline{U} = \underline{u}] = \alpha.$$

Also ϕ_3^* has level α.

(d) Let us first derive conditions for a test ϕ to be unbiased for testing $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$. Here $\bar{\Theta}_{01} = \{(\theta, \underline{\psi}) : \theta = \theta_0\}$ and $\underline{U}$ is C-S for $\bar{\Theta}_{01}$. ~~Any unbiased test ϕ satisfies~~ A test ϕ is unbiased iff

$$E_{\theta, \underline{\psi}}(\phi(\underline{Y}, \underline{U})) \geq \alpha \geq E_{\theta_0, \underline{\psi}}(\phi(\underline{Y}, \underline{U})), \quad \forall \; \cancel{\theta \neq \theta_0 \; \forall \psi} \; (\theta_0, \underline{\psi}) \in \bar{\Theta}_{01}, \; (\theta, \underline{\psi}) \notin \bar{\Theta}_{01}$$

By continuity of power function, ~~we have~~, a test ϕ is unbiased iff

$$\begin{cases} E_{\theta_0, \underline{\psi}}(\phi(\underline{Y}, \underline{U})) = \alpha, \quad \forall \; (\theta_0, \underline{\psi}) \in \bar{\Theta}_{01} \\ \text{and} \quad E_{\theta, \underline{\psi}}(\phi(\underline{Y}, \underline{U})) \geq E_{\theta_0, \underline{\psi}}(\phi(\underline{Y}, \underline{U})) = \alpha \; \forall \; \cancel{\theta \neq \theta_0, \; \forall \underline{\psi}} \; (\theta_0, \underline{\psi}) \in \bar{\Theta}_{01} \text{ and } (\theta, \underline{\psi}) \notin \bar{\Theta}_{01} \end{cases}$$

$$\Rightarrow \Leftarrow \begin{cases} E_{\theta_0,\psi}(\phi(Y,\underline{U})) = \alpha & \forall\, (\theta_0,\underline{\psi}) \in \overline{\Theta}_{01} \\ \text{and } E_{\theta,\underline{\psi}}(\phi(Y,\underline{U})) \text{ has a local minimum at every } (\theta_0,\underline{\psi}) \in \overline{\Theta}_\alpha \ \cancel{\theta=\theta_0, \forall \underline{\psi}}. \end{cases}$$

Thus ~~any unbiased estimator φ satisfies~~ a test ϕ is unbiased iff

$$E_{\theta_0,\psi}(\phi(Y,\underline{U})) = \alpha, \quad \forall\, \underline{\psi}\ (\theta_0,\underline{\psi}) \in \overline{\Theta}_{01}$$

and
$$\frac{\partial}{\partial\theta} E_{\underline{\theta},\underline{\psi}}(\phi(Y,\underline{U})) = 0, \quad \forall\, (\underline{\theta},\psi) \in \overline{\Theta}_{01}$$

$$\Rightarrow \frac{\partial}{\partial\theta} \int \phi(y,\underline{u})\, e^{\theta y + \sum_{i=1}^{k} \psi_i u_i - S(\underline{\theta},\underline{\psi})}\, h(\underline{x})\, d\underline{x} = 0, \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

$$\Rightarrow \int \frac{\partial}{\partial\theta} [\quad]\, d\underline{x} = 0, \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

$$\Rightarrow \int \phi(y,\underline{u})\, e^{\theta y + \sum_{i=1}^{k} \psi_i u_i - S(\underline{\theta},\underline{\psi})}\, h(\underline{x}) \left[y - \frac{\partial}{\partial\theta} S(\underline{\theta},\underline{\psi})\right] = 0, \quad \forall\, (\underline{\theta},\psi) \in \overline{\Theta}_{01}$$

$$\Rightarrow E_{\underline{\theta},\psi}[Y\phi(Y,\underline{U})] = \alpha \frac{\partial}{\partial\theta} S(\underline{\theta},\underline{\psi}), \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

We have
$$\int e^{\theta y + \sum \psi_i u_i - S(\underline{\theta},\psi)}\, h(\underline{x})\, d\underline{x} = 1, \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

$$\Rightarrow \frac{\partial}{\partial\theta} \int (\quad) = 0, \quad ''$$

$$\Rightarrow \int \frac{\partial}{\partial\theta} [\quad] = 0, \quad ''$$

$$\Rightarrow \int e^{\theta y + \sum \psi_i u_i - S(\underline{\theta},\underline{\psi})}\, h(\underline{x}) \left[y - \frac{\partial}{\partial\theta} S(\underline{\theta},\underline{\psi})\right] = 0, \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

$$\Rightarrow E(Y) = \frac{\partial}{\partial\theta} S(\underline{\theta},\underline{\psi}), \quad \forall\, (\underline{\theta},\underline{\psi}) \in \overline{\Theta}_{01}$$

Thus, for (d), any unbiased test of level α will satisfy

$$\begin{cases} E_{\theta_0,\underline{\psi}}(\phi(Y,\underline{U})) = \alpha, & \forall\, (\theta_0,\underline{\psi}) \in \overline{\Theta}_{01} \\ \text{and } E_{\theta_0,\psi}(Y\phi(\underline{Y},\underline{U})) = \alpha\, E_{\theta_0,\psi}(\underline{Y}), & \forall\, (\theta_0,\underline{\psi}) \in \overline{\Theta}_{\alpha} \end{cases}$$

$$\Leftrightarrow \begin{cases} E_{\theta_0,\underline{\psi}}\left(E_{\theta_0,\underline{\psi}}(\phi(Y,\underline{U}) - \alpha \mid \underline{U}) \right) = 0 & \forall (\theta_0,\underline{\psi}) \in \overline{\Theta}_{01} \\ E_{\theta_0,\underline{\psi}}\left(E(Y\phi(Y,\underline{U}) - \alpha Y \mid \underline{U}) \right) = 0 & \forall (\theta_0,\underline{\psi}) \in \overline{\Theta}_{01} \end{cases}$$

By virtue of completeness of $\underline{U}$, it follows that

$$\begin{cases} E_{\theta_0\psi}(\phi(Y,\underline{U}) \mid \underline{U}) = \alpha, \\ \text{and } E_{\theta_0,\underline{\psi}}(Y\phi(Y,\underline{U}) \mid \underline{U}) = \alpha E_{\theta_0}(Y\mid\underline{U}) \end{cases} \quad \cdots\cdots \quad (*)$$

does not depend on $(\theta_0,\psi) \in \overline{\Theta}_{01}$

Thus, for (d), it suffices to show that ϕ_3^* is UMP among all tests satisfying (*)

~~Under conditional distribution (given $\underline{U}$)~~ The power function of any test $\phi(Y, \underline{U})$ is

$$\beta_\phi(\theta,\underline{\psi}) = E_{\theta,\underline{\psi}}\left(E_{\theta,\psi}(\phi(Y,\underline{U}) \mid \underline{U}) \right)$$

and thus it suffices to show that for any $(\theta,\underline{\psi}) \in \Theta_1$, and any fixed $\underline{U}$, ϕ_3^* maximizes

$$E_{\theta,\underline{\psi}}(\phi(Y,\underline{U}) \mid \underline{U})$$

over all ϕ satisfying (*)

Now the conditional p.d.f. of Y given $\underline{U}=\underline{u}$ is

$$g_\theta(y\mid\underline{u}) = e^{\theta y}\, l(\underline{u}, y)$$

Clearly $\{ g_\theta : \theta \in \Theta \}$ has MLR in Y, for every fixed $\underline{u}$,

We now omit $\underline{U}$ in the following discussion, as whole discussion is in reference to conditional dist. of Y given $\underline{U}$, for fixed $\underline{U}$.

Consider testing $H_0^*: \theta=\theta_0$ vs $H_1^*: \theta=\theta_1$. Then the rejection region of UMP tests satisfying (*) is

$$g_{\theta_1}(y\mid\underline{u}) > k_1 e^{\theta_0 y} + k_2 y e^{\theta_0 y}$$

$$g_{\theta_1}(y|\underline{u}) > k_1 \, g_{\theta_0}(y) + k_2 \; y \, g_{\theta_0}(y)$$

$$\Longleftrightarrow \quad e^{\theta_1 y} > k_1 e^{\theta_0 y} + k_2 \, y \, e^{\theta_0 y},$$

where k_i's are constants.

$$\Longleftrightarrow \quad a_1 + a_2 y < e^{by} \quad (a_1 = k_1,\ a_2 = k_2,\ b = \theta_1 - \theta_0),$$

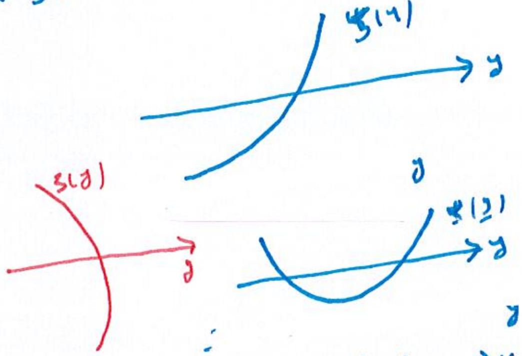
for some constants a_1, a_2 and b.

Consider

$$s(y) = e^{by} - a_2 y - a_1$$

$$s'(y) = b e^{by} - a_2$$

$$s''(y) = b^2 e^{by} > 0$$



Thus the C.R. $\{y : s(y) > 0\}$ is either one-sided or two outside an interval. But a one-sided test has strictly monotone power function and therefore can not be unbiased. Thus this test must have the form of the test given in (c). Since this test does not depend on θ_1, it is UMP among all tests satisfying the required condition.

Lemma: Suppose that $\underline{X}$ has pdf/pmf

$$f_{\theta,\underline{\psi}}(\underline{x}) = \exp\{\theta Y(\underline{x}) + \sum_{i=1}^{k} \psi_i U_i(\underline{x}) - \zeta(\theta, \underline{\psi})\} h(\underline{x}),$$

θ_0, θ_1 and θ_2 are fixed constants as as defined in (A). Let $V(\underline{Y}, \underline{U})$ defined in the hypotheses of (a)-(d) last theorem. Let $V(Y, \underline{U})$ be a statistic such that $V(Y, \underline{U})$ and $\underline{U}$ are independent when $(\theta, \underline{\psi}) \in \bar{\Theta}_{01}$.

(a) If $V(Y, \underline{U})$ is increasing in Y, for each $\underline{U}$, then UMPU tests in (a)-(c) of above theorem are equivalent to those tests in (a)-(c) replaced by V and with in (a)-(c) with $\underline{Y}$ and $V(Y, \underline{U})$

$c_i(u)$'s and $\gamma_i(u)$'s replaced by c_i's and γ_i's (independent of $\underline{u}$).

(b) If there are functions $a(\underline{u}) > 0$ and $b(\underline{u})$ such that $V(Y, \underline{u}) = a(\underline{u}) Y + b(\underline{u})$, then the UMPU test in (d) of above theorem is equivalent to the one obtained by replacing Y and $(Y, \underline{u})$ by V and $c_i(\underline{u})$'s and $\gamma_i(\underline{u})$'s replaced by c_i's and γ_i's.

Proof (a) Since, for every fixed $\underline{u}$, $V(Y, \underline{u}) \uparrow$

$Y > c_i(\underline{u}) \Leftrightarrow V > d_i(\underline{u})$, for some $d_i(\cdot)$.

The assertion now follows from the fact that V and $\underline{u}$ are statistically independent so that d_i's and γ_i's do not depend on u, when Y is replaced by V.

(b) (a) Since $V = a(\underline{u}) Y + b(\underline{u})$, the UMPU test of (d) is the same as

$$\phi^*_{\underline{u}}(V, \underline{u}) = \begin{cases} 1, & V < d_1(\underline{u}) \text{ or } V > d_2(\underline{u}) \\ \gamma_i(\underline{u}), & V = d_i(\underline{u}),\ i = 1,2 \\ 0, & d_1(u) < V < d_2(u) \end{cases}$$

subject to

$$\begin{cases} E_{\theta_0, \underline{\psi}}(\phi^*(V, \underline{u}) \mid \underline{u}) = \alpha, & \forall (\theta_0, \underline{\psi}) \in \overline{\Theta}_{01} \\ \text{and } E_{\theta_0 \underline{\psi}}\left(\phi^*(V, \underline{u}) \dfrac{V - b(\underline{u})}{a(\underline{u})} \,\Big|\, \underline{u}\right) = \alpha\, E_{\theta_0}\left(\dfrac{V - b(\underline{u})}{a(\underline{u})} \,\Big|\, \underline{u}\right) & \forall (\theta_0, \underline{\psi}) \in \overline{\Theta}_{01} \end{cases}$$

$$\Leftrightarrow \begin{cases} E_{\theta_0 \psi}(\phi^*(V, \underline{u}) \mid \underline{u}) = \alpha, & \forall (\theta_0, \underline{\psi}) \in \overline{\Theta}_{01} \\ \text{and } E_{\theta_0}(\phi^*(V, \underline{u}) V \mid \underline{u}) = \alpha\, E_{\theta_0}(V \mid \underline{u}), & \forall (\theta_0, \underline{\psi}) \in \overline{\Theta}_{01} \end{cases}$$

Since V and $\underline{u}$ are independent, $c_i(\underline{u})$'s and $\gamma_i(\underline{u})$'s do not depend on $\underline{u}$ and therefore $\phi^*_{\underline{u}}$ does not depend on $\underline{u}$.

One Sample Problem From Normal Family

Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$ where $n \geq 2$ and $\mu \in (-\infty, \infty)$ and $\sigma > 0$ are unknown parameters

Case I: Hypothesis Concerning μ with $\sigma = \sigma_0$ known

$$H_0: \mu = \mu_0$$
$$\text{vs } H_1: \mu \neq \mu_0,$$

for known $\mu_0 \in \mathbb{R}$.

$$f_\theta(x) = \frac{1}{(2\pi\sigma_0^2)^{\frac{n}{2}}} e^{\frac{n}{\sigma_0^2}(\mu - \mu_0)\bar{x}} e^{\frac{n\mu_0\bar{x}}{\sigma_0^2} - \frac{1}{2\sigma_0^2}\sum x_i^2 - \frac{n\mu_0^2}{2\sigma_0^2}}$$

$$\theta = \frac{n}{\sigma_0^2}(\mu - \mu_0), \qquad H_0: \theta = 0 \quad \text{vs } H_1: \theta \neq 0, \qquad Y = \bar{X}$$

There are no nuisance parameters ψ's and $\underline{U}$'s so one should use the above results with condition distribution of Y given $\underline{U} = \underline{u}$ just replaced by distribution of Y.

The UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & Y < c_1 \text{ or } Y > c_2 \\ 0, & c_1 < Y < c_2 \end{cases},$$

Where $E_{\theta=0}(\phi^*(\underline{X})) = \alpha$

and $E_{\theta=0}(\phi^*(\underline{X})Y) = \alpha E_{\theta=0}(Y)$

$$P_{\mu_0}(\bar{X} < c_1) + P_{\mu_0}(\bar{X} > c_2) = \alpha$$

$$E_{\mu_0}(\bar{X} I(\bar{X} < c_1)) + E_{\mu_0}(\bar{X} I(\bar{X} > c_2)) = \alpha E_{\mu_0}(\bar{X})$$

~~Let~~ $Z = \dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma_0} \sim N(0,1)$, under $\mu = \mu_0$

$$\Leftrightarrow \quad \Phi\left(\frac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}\right) + 1 - \Phi\left(\frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right) = \alpha \quad \ldots \quad (a)$$

and
$$E\left[\left(\frac{\sigma_0 Z}{\sqrt{n}} + \mu_0\right) I\left(Z < \frac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}\right)\right] + E\left[\left(\frac{\sigma_0 Z}{\sqrt{n}} + \mu_0\right) I\left(Z > \frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right)\right] = \alpha\mu_0 \quad \ldots (b)$$

$$(b) \Leftrightarrow \frac{\sigma_0}{\sqrt{n}}\left[E\left(Z\, I\left(Z < \frac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}\right)\right) + E\left(Z\, I\left(Z > \frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right)\right)\right] = 0 \quad \ldots (c)$$

Since $Z \overset{d}{=} -Z$, we have

$$E\left(Z\, I\left(Z > \frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right)\right) = -E\left[Z\, I\left(Z < -\frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right)\right]$$

Thus by taking $\dfrac{\sqrt{n}(c_2-\mu_0)}{\sigma_0} = -\dfrac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}$, (c) (and hence (b)) is satisfied. Putting this in (a) we get

$$\Phi\left(\frac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}\right) = \frac{\alpha}{2}$$

$$\Rightarrow \quad c_1 = \mu_0 + \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}\left(\frac{\alpha}{2}\right) = \mu_0 - \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha/2)$$

$$c_2 = \mu_0 + \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha/2)$$
(Note that $\Phi^{-1}(\alpha/2) < 0$ and $\Phi^{-1}(1-\alpha/2) > 0$)

and UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & \bar{X} < \mu_0 - \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha/2) \text{ or } \bar{X} > \mu_0 + \frac{\sigma_0}{\sqrt{n}}\Phi^{-1}(1-\alpha/2) \\ 0, & \text{o.w.} \end{cases}$$

$$= \begin{cases} 1, & \left|\frac{\sqrt{n}(\bar{X}-\mu_0)}{\sigma_0}\right| > \Phi^{-1}(1-\alpha/2) \\ 0, & \text{otherwise.} \end{cases}$$

Under Case (I), now consider testing

$$H_0: \; \cancel{\mu \le \mu_1 \text{ or } \mu \ge \mu_2} \quad \mu_1 \le \mu \le \mu_2$$

$$\text{vs} \quad H_1: \; \mu < \mu_1 \text{ or } \mu > \mu_2$$

where μ_1 and μ_2 $(\mu_1 < \mu_2)$ are pre-specified.

UMPU(α) test is

$$\phi^*(\underline{X}) = \begin{cases} 1, & Y < c_1 \text{ or } Y > c_2 \\ 0, & c_1 < Y < c_2 \end{cases},$$

where

$$E_{\mu_1}[\phi^*(\underline{X})] = E_{\mu_2}[\phi^*(\underline{X})] = \alpha$$

$$P_{\mu_i}(\bar{X} < c_1) + P_{\mu_i}(\bar{X} > c_2) = \alpha, \quad i = 1, 2$$

$$\Phi\left(\frac{\sqrt{n}(c_1-\mu_i)}{\sigma_0}\right) + 1 - \Phi\left(\frac{\sqrt{n}(c_2-\mu_i)}{\sigma_0}\right) = \alpha, \quad i = 1, 2$$

i.e.

$$\Phi\left(\frac{\sqrt{n}(c_2-\mu_i)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(c_1-\mu_i)}{\sigma_0}\right) = 1-\alpha, \quad i = 1, 2$$

$$\Leftrightarrow \quad \Phi\left(\frac{\sqrt{n}(c_2-\mu_1)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(c_1-\mu_1)}{\sigma_0}\right) = 1-\alpha$$

and

$$\Phi\left(\frac{\sqrt{n}(c_2-\mu_2)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(c_1-\mu_2)}{\sigma_0}\right) = 1-\alpha$$

Note: For testing $H_0: |\mu| \le \mu_0$ (where $\mu_0 > 0$ is pre-specified) vs $H_1: |\mu| > \mu_0$ (i.e. $\mu_2 = \mu_0$ and $\mu_1 = -\mu_0$), we get

$$\Phi\left(\frac{\sqrt{n}(c_2+\mu_0)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(c_1+\mu_0)}{\sigma_0}\right) = 1-\alpha$$

$$\Phi\left(\frac{\sqrt{n}(c_2-\mu_0)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(c_1-\mu_0)}{\sigma_0}\right) = 1-\alpha$$

$$\Leftrightarrow \Phi\left(\frac{\sqrt{n}(-c_1+\mu_0)}{\sigma_0}\right) - \Phi\left(\frac{\sqrt{n}(-c_2+\mu_0)}{\sigma_0}\right) = 1-\alpha$$

$\Rightarrow c_1 = -c_2$ and $\Phi\left(\frac{\sqrt{n}(c_2+\mu_0)}{\sigma}\right) - \Phi\left(\frac{\sqrt{n}(-c_2+\mu_0)}{\sigma_0}\right) = 1-\alpha$ — $(*)$

Clearly, LHS of (*) is a strictly increasing function of c_2 and LHS $\in (0, \infty)$ as c_2 varies from 0 to ∞. Thus there is unique c_2 satisfying (*).

Case II: $\sigma > 0$ is unknown.

$$H_0: \mu \le \mu_0$$
$$\text{vs} \quad H_1: \mu > \mu_0$$

$$f_{\theta, \psi}(\underline{x}) = e^{\frac{n}{\sigma^2}(\mu-\mu_0)(\bar{x}-\mu_0) - \frac{1}{2\sigma^2}\sum(x_i-\mu_0)^2} \cdots$$

$$\theta = \frac{n(\mu-\mu_0)}{\sigma^2}, \quad \psi = -\frac{1}{2\sigma^2}, \quad Y = \bar{X}-\mu_0, \quad U = \sum_{i=1}^{n}(x_i-\mu_0)^2$$

By Basu's Theorem Y and U are independent under $\mu = \mu_0$.

Take $V(Y, U) \equiv V = \dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{\dfrac{\sum(x_i-\mu_0)^2}{n}}}$ $\uparrow$ $Y = \bar{X}$ for every fixed U.

UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & V > c \\ 0, & \text{o.w} \end{cases}$$

Where

$$E_{\mu_0}(\phi^*(\underline{X})) = \alpha$$

$$\Rightarrow \quad P_{\mu_0, \sigma^2}(V > c) = \alpha$$

Under $\mu = \mu_0$ and $\sigma > 0$, $V \sim t_n$ (Student t dist with n d.f.)

$\Rightarrow$ $c = (1-\alpha)$th quantile of t dist with n d.f.

Note: $\dfrac{V}{\sqrt{n}} = \dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{\sum(x_i-\mu_0)^2}} = \dfrac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{(n-1)S_{n-1}^2 + n(\bar{X}-\mu_0)^2}} = \dfrac{T_{n-1}}{\sqrt{n-1+T_{n-1}^2}}$, Where $T_{n-1} = \dfrac{\sqrt{n}(\bar{X}-\mu_0)}{S_{n-1}}$ $\quad \left(S_{n-1} = \dfrac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \right)$

$\sim t_{n-1}$ (Student t-distn with $(n-1)$ d.f.) Under $\mu = \mu_0$

$\dfrac{V}{\sqrt{n}} \uparrow T_{n-1}$ $\quad \left[\dfrac{d}{dx} \dfrac{x}{\sqrt{n-1+x^2}} = \dfrac{n-1}{(n-1+x^2)^{3/2}} > 0 \right]$

An equivalent UMPU(α) test is

$$\phi_1^*(\underline{x}) = \begin{cases} 1, & T_{n-1} > c^* \\ 0, & \text{o.w.} \end{cases}$$

where $E_{\mu_0,\sigma}(\phi_1^*(\underline{x})) = \alpha$

$P_{\mu_0,\sigma}(T_{n-1} > c^*) = \alpha$

$\Rightarrow c^* = (1-\alpha)^{th}$ quantile of t-distribution with $(n-1)$ d.f.

Now consider

$H_0: \mu = \mu_0$ vs $H_1: \mu \neq \mu_0$ $\iff$ $H_0: \theta = 0$ vs $H_1: \theta \neq 0$

$$V \equiv V(\underline{Y}, U) = \frac{\sqrt{n}(\bar{X}-\mu_0)}{\sqrt{\sum_{i=1}^{n}(X_i-\mu_0)^2/n}} = \frac{n}{\sqrt{U}}\bar{X} - \frac{n\mu_0}{U}$$

$= a(U)Y + b(U)$, $\quad a(U) > 0$.

Under $\mu = \mu_0$, $V \sim T_n$ (Student t-distn with n d.f.)

UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & V < c_1 \text{ or } V > c_2 \\ 0, & c_1 < \bar{X} < c_2 \end{cases}$$

where $E_{\mu_0,\sigma}(\phi^*) = \alpha$ $\quad \cdots$ (a)

$E_{\mu_0,\sigma}(\phi^* V) = \alpha E_{\mu_0,\sigma}(V)$ $\quad \cdots$ (b)

$\Rightarrow \quad P(T_n < c_1) + P_{:}(T_n > c_2) = \alpha$

$$E(T_n\, I(T_n < c_1)) + E(T_n\, I(T_n > c_2)) = \alpha$$

$T_n \overset{d}{=} -T_n \quad \Rightarrow \quad c_1 = -c_2.$

The UMPU(α) test is

$$\phi'(\underline{x}) = \begin{cases} 1, & |T| > c \\ 0, & \text{o.w} \end{cases}$$

where $P(T_n \le c) = 1 - \frac{\alpha}{2}$

$\Rightarrow \; c = (1-\frac{\alpha}{2})^{th}$ quantile of t dist. with n d.f.

As earlier an equivalent test is

$$\phi_1^*(\underline{x}) = \begin{cases} 1, & \text{if } |T_{n-1}| > c^* \\ 0, & \text{o.w.} \end{cases}$$

where $T_{n-1} = \dfrac{\sqrt{n}(\bar{X} - \mu_0)}{S_{n-1}}$ and $c^* = (1-\frac{\alpha}{2})^{th}$ quantile of Student t-distribution with $(n-1)$ d.f.

Case III hypothesis concerning σ^2

Case III A: $\mu = \mu_0$ is known

$$H_0: \sigma^2 = \sigma_0^2$$
$$\text{vs} \quad H_1: \sigma^2 \neq \sigma_0^2$$

$$f_\sigma(\underline{x}) = e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu_0)^2 - \frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2}$$

$\theta = -\frac{1}{2\sigma^2}, \quad Y = \sum_{i=1}^n (X_i - \mu_0)^2 \sim \sigma^2 \chi^2_n$ under H_0

$\rightarrow$ has MLR in Y.

UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & Y < c_1 \text{ or } Y > c_2 \\ 0, & c_1 < Y < c_2 \end{cases},$$

Where $E_{\sigma_0}(\phi^*) = \alpha$ and $E_{\sigma_0}(\phi^* V) = \alpha E_{\sigma_0}(V)$

$E_{\sigma_0}((1-\phi^*) V) = (1-\alpha) E_{\sigma_0}(V)$

$\Rightarrow$ $P_{\sigma_0}(c_1 < Y < c_2) = 1-\alpha$ and

$$\Rightarrow \int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} \frac{e^{-y/2} y^{\frac{n}{2}-1}}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} dy = 1-\alpha \quad \text{and} \quad \int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} \frac{e^{-y/2} y^{\frac{n+2}{2}-1}}{2^{\frac{n}{2}} \Gamma\frac{n}{2}} dy = n(1-\alpha)$$

(Under the $\frac{Y}{\sigma_0^2} \sim \chi^2_n$)

$$\int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_n}(y)\, dy = 1-\alpha \quad \text{and} \quad \int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_{n+2}}(y)\, dy = 1-\alpha$$

i.e. $$\int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_n}(y)\, dy = \int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_{n+2}}(y)\, dy = 1-\alpha$$

When n is large (i.e. $n-1 \approx n+1$) then $\frac{c_1}{\sigma_0^2}$ and $\frac{c_2}{\sigma_0^2}$ are nearly $(\frac{\alpha}{2})^{th}$ and $(1-\frac{\alpha}{2})^{th}$ quantiles of χ^2_n distribution

Similarly for testing

$$H_0: \sigma_1^2 \le \sigma^2 \le \sigma_2^2$$

$$\text{vs } H_1: \sigma^2 < \sigma_1^2 \text{ or } \sigma^2 > \sigma_2^2,$$

When $0 < \sigma_1 < \sigma_2 < \infty$ are fixed constants is,

$$\phi_1^*(\underline{x}) = \begin{cases} 1, & Y < d_1 \text{ or } Y > d_2 \\ 0, & \text{Otherwise} \end{cases},$$

Where $E_{\sigma_i}(\phi_i^*(\underline{x})) = \alpha,\quad i = 1, 2$

$\Leftrightarrow$ $E_{\sigma_i}((1-\phi_i^*(\underline{x}))) = 1-\alpha,\quad i = 1, 2$

$$\Leftrightarrow \int_{d_1/\sigma_0^2}^{d_2/\sigma_0^2} f_{\chi^2_{n-1}}(y)\,dy = \int_{d_1/\sigma_0^2}^{d_2/\sigma_0^2} f_{\chi^2_{n-1}}(y)\,dy = 1-\alpha.$$

Case III B: μ is unknown

$$H_0: \sigma^2 = \sigma_0^2$$
$$\text{vs } H_1: \sigma^2 \neq \sigma_0^2$$

$$f_{\theta,\psi}(\underline{x}) = e^{-\frac{1}{2\sigma^2}\sum x_i^2 + \frac{n\mu}{\sigma^2}\bar{x} - \frac{n\mu^2}{2\sigma^2} - \frac{n}{2}\ln 2\pi - \frac{n}{2}\ln\sigma^2}$$

$$\theta = -\frac{1}{2\sigma^2},\quad Y = \sum_{i=1}^{n} x_i^2,\quad \psi = \frac{n\mu}{\sigma^2},\quad U = \bar{x}$$

By Basu's Theorem

$V = (n-1)S^2$ and $U = \bar{X}$ are independent

$$V = Y - nU^2 = a(\underline{U})Y + b(\underline{U}), \quad a(\underline{U}) = 1 > 0.$$

Under H_0 $\frac{V}{\sigma_0^2} \sim \chi^2_{n-1}$

$$\begin{array}{ll} H_0: \sigma^2 = \sigma_0^2 \\ \text{vs } H_1: \sigma^2 \neq \sigma_0^2 \end{array} \Leftrightarrow \begin{array}{ll} H_0: \theta = \theta_0 \\ \text{vs } H_1: \theta \neq \theta_0 \end{array}$$

UMPU(α) test is

$$\phi^*(\underline{X}) = \begin{cases} 1, & V < c_1 \text{ or } V > c_2 \\ 0, & \text{o.w.} \end{cases}$$

When $E_{\sigma_0}(\phi^*) = \alpha$ and $E_{\sigma_0}(\phi^* V) = \alpha E_{\sigma_0}(V)$

As before we get

$$\int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_{n-1}}(y)\,dy = \int_{c_1/\sigma_0^2}^{c_2/\sigma_0^2} f_{\chi^2_{n+1}}(y)\,dy = 1-\alpha$$

When $n-1 \approx n+1$ (i.e. n is large), then $\frac{c_1}{\sigma_0^2}$ and $\frac{c_2}{\sigma_0^2}$ are nearly $\left(\frac{\alpha}{2}\right)$th and $(1-\frac{\alpha}{2})^{th}$ quantiles of χ^2_n distribution.

The hypothesis testing of

$$H_0: \sigma_1^2 \le \sigma^2 \le \sigma_2^2$$

$$\text{vs } H_1: \sigma^2 < \sigma_1^2 \text{ or } \sigma^2 > \sigma_2^2$$

can be dealt with similarly.

Two Sample Problems

$$\left.\begin{array}{l} X_{11}, \dots, X_{1n_1} \overset{iid}{\sim} N(\mu_1, \sigma_1^2) \\ X_{21}, \dots, X_{2n_2} \overset{iid}{\sim} N(\mu_2, \sigma_2^2) \end{array}\right\} \text{independent}, \quad n_i \ge 2, \ i=1,2.$$

Consider testing

$$H_0: \frac{\sigma_2^2}{\sigma_1^2} \le \Delta_0 \quad \text{or} \quad H_0: \frac{\sigma_2^2}{\sigma_1^2} = \Delta_0$$

$$\text{vs } H_1: \frac{\sigma_2^2}{\sigma_1^2} > \Delta_0 \qquad \text{vs } H_1: \frac{\sigma_2^2}{\sigma_1^2} \ne \Delta_0,$$

where $\Delta_0 > 0$ is a fixed constant

$$f_{\theta,\underline{\psi}}(\underline{x}) = \exp\left\{ -\frac{1}{2\sigma_1^2}\sum_{j=1}^{n_1} x_{1j}^2 - \frac{1}{2\sigma_2^2}\sum_{j=1}^{n_2} x_{2j}^2 + \frac{n_1\mu_1\bar{x}_1}{\sigma_1^2} + \frac{n_2\mu_2}{\sigma_2^2}\bar{x}_2 \right.$$
$$\left. - \frac{n_1\mu_1^2}{\sigma_1^2} - \frac{n_2\mu_2^2}{\sigma_2^2} - \frac{n_1+n_2}{2}\ln 2\pi - \frac{n_1}{2}\ln\sigma_1^2 - \frac{n_2}{2}\ln\sigma_2^2 \right\}$$

$$= \exp\left\{ -\frac{1}{2}\sum_{j=1}^{n_2} x_{2j}^2 \left\{\frac{1}{\sigma_2^2} - \frac{1}{\Delta_0\sigma_1^2}\right\} - \frac{1}{2\sigma_1^2}\left\{ \sum_{j=1}^{n_1} x_{1j}^2 + \frac{\sum_{j=1}^{n_2} x_{2j}^2}{\Delta_0} \right\} \right.$$
$$+ \frac{n_1\mu_1}{\sigma_1^2}\bar{x}_1 + \frac{n_2\mu_2}{\sigma_2^2}\bar{x}_2 - \frac{n_1\mu_1^2}{\sigma_1^2} - \frac{n_2\mu_2^2}{\sigma_2^2} - \frac{n_1+n_2}{2}\ln 2\pi$$
$$\left. - \frac{n_1}{2}\ln\sigma_1^2 - \frac{n_2}{2}\ln\sigma_2^2 \right\}$$

$$\theta = \frac{1}{2}\left(\frac{1}{\Delta_0 \sigma_1^2} - \frac{1}{\sigma_2^2}\right), \quad Y = \sum_{j=1}^{n_2} X_{2j}^2$$

$$\underline{\psi} = \left(-\frac{1}{2\sigma_1^2}, \frac{n_1\mu_1}{\sigma_1^2}, \frac{n_2\mu_2}{\sigma_2^2}\right)$$

$$\underline{U} = \left(\sum_{j=1}^{n_1} x_{1j}^2 + \frac{1}{\Delta_0}\sum_{j=1}^{n_2} x_{2j}^2, \bar{x}_1, \bar{x}_2\right) = (U_1, U_2, U_3)$$

Define

$$V(\underline{Y}, \underline{U}) = \frac{(n_2-1)S_2^2/\Delta_0}{(n_1-1)S_1^2 + (n_2-1)S_2^2/\Delta_0} = \frac{Y - n_2 U_3^2}{U_1 - n_1 U_2^2 - n_2 U_3^2/\Delta_0} \uparrow Y$$

By Basu's Theorem V and $\underline{U}$ are independent under $\sigma_2^2 = \Delta_0 \sigma_1^2$. Thus a UMPU($\alpha$) test for

$$H_0: \frac{\sigma_2^2}{\sigma_1^2} \le \Delta_0 \quad \text{vs} \quad H_1: \frac{\sigma_2^2}{\sigma_1^2} > \Delta_0 \iff H_0: \theta \le 0 \quad \text{vs} \quad H_1: \theta > 0$$

is

$$\phi^*(X) = \begin{cases} 1, & V > c_1 \\ 0, & \text{otherwise} \end{cases}$$

where

$$E_{\sigma_2^2 = \sigma_1^2\Delta_0}(\phi^*) = \alpha$$

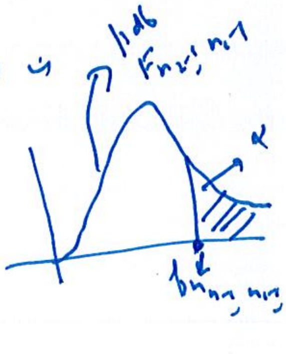
$$P_{\sigma_2^2 = \sigma_1^2\Delta_0}(V > c_1) = \alpha$$

Under $\sigma_2^2 = \sigma_1^2 \Delta_0$

$$V = \frac{(n_2-1)F}{(n_1-1) + (n_2-1)F}, \quad \text{where } F = \frac{S_2^2/\Delta_0}{S_1^2/\Delta_0} \sim F_{n_2-1, n_1-1} \text{ and}$$

Since $V \uparrow$ in F, an equivalent UMPU(α) test is

$$\phi_1^*(X) = \begin{cases} 1, & \text{if } F > f_{n_2-1, n_1-1, \alpha} \\ 0, & \text{if } F < f_{n_2-1, n_1-1, \alpha} \end{cases}$$



Where $f_{a,b,\alpha} = (1-\alpha)^{th}$ quantile for $F_{a,b}$

Now consider testing

$H_0 : \frac{\sigma_2^2}{\sigma_1^2} = \Delta_0$ $\Leftrightarrow$ $H_0 : \theta = 0$

vs $H_1 : \frac{\sigma_2^2}{\sigma_1^2} \neq \Delta_0$ vs $H_1 : \theta \neq 0$

Here

$V = a(\underline{U})Y + b(\underline{U})$

Thus a UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } V < c_1 \text{ or } V > c_2 \\ 0, & \text{if } c_1 < V < c_2 \end{cases}$$

where

$E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(\phi^*) = \alpha$ and $E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(\phi^* V) = \alpha\, E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(V)$

Under $\sigma_2^2 = \sigma_1^2 \Delta_0$, $V \sim Beta\left(\frac{n_2-1}{2}, \frac{n_1-1}{2}\right)$

$$E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(V) = \frac{n_2-1}{n_1+n_2-2}$$

$$E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(\phi^* V) = \alpha\, E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(V)$$

$$\Leftrightarrow \quad E_{\sigma_2^2 = \sigma_1^2 \Delta_0}((1-\phi^*)V) = (1-\alpha)\, E_{\sigma_2^2 = \sigma_1^2 \Delta_0}(V)$$

$$\Leftrightarrow \quad \int_{c_1}^{c_2} v\, f_{\frac{n_2-1}{2}, \frac{n_1-1}{2}}(v)\, dv = \frac{(1-\alpha)(n_2-1)}{n_1+n_2-2},$$

where $f_{a,b}$ is the p.d.f. of Beta(a,b)

$$v\, f_{\frac{n_2-1}{2}, \frac{n_1-1}{2}}(v) = \frac{n_2-1}{n_1+n_2-2} f_{\frac{n_2+1}{2}, \frac{n_1-1}{2}}(v)$$

Thus, we have

$$\int_{c_1}^{c_2} f_{\frac{n_2+1}{2}, \frac{n_1-1}{2}}(v)\, dv = 1-\alpha = \int_{c_1}^{c_2} f_{\frac{n_2-1}{2}, \frac{n_1-1}{2}}(v)\, dv$$

If $n_2-1 \approx n_2+1$ (i.e. if n_2 is large) then equal tail test can be used.

Now consider testing

$H_0: \mu_1 \geq \mu_2$ vs H_1 $\mu_1 < \mu_2$ or $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 \neq \mu_2$

Case I $\sigma_1^2 \neq \sigma_2^2$

→ Behren's Fisher Problem and is not accessible by UMPU test method.

→ Two stage procedure (Stein)

Case II: $\sigma_1^2 = \sigma_2^2 = \sigma^2$, say

$$f_{\sigma,\underline{\psi}}(\underline{x}) = \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} x_{ij}^2 + \frac{n_1\mu_1}{\sigma^2}\bar{x}_1 + \frac{n_2\mu_2}{\sigma^2}\bar{x}_2 - \frac{n_1\mu_1}{\sigma_1^2} - \frac{n_2\mu_2}{\sigma_2^2} - \frac{n_1+n_2}{2}\ln(2\pi) - \frac{n_1+n_2}{2}\ln\sigma^2 \right\}$$

$$= \exp\left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{2} \sum_{j=1}^{n_i} x_{ij}^2 + \left(\frac{\mu_2}{\frac{\sigma^2}{n_2}+\frac{\sigma^2}{n_1}} - \frac{\mu_1}{\frac{\sigma^2}{n_2}+\frac{\sigma^2}{n_1}} \right)(\bar{x}_2 - \bar{x}_1) + \frac{n_1\mu_1+n_2\mu_2}{(n_1+n_2)\sigma^2}(n_1\bar{x}_1+n_2\bar{x}_2) - \frac{n_1\mu_1}{\sigma^2} - \frac{n_2\mu_2}{\sigma^2} - \frac{n_1+n_2}{2}\ln(2\pi) - \frac{n_1+n_2}{2}\ln\sigma^2 \right\}$$

$\theta = \dfrac{\mu_2-\mu_1}{\sigma^2\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$, $Y = \bar{X}_2 - \bar{X}_1$, $\underline{\psi} = \left(\dfrac{n_1\mu_1+n_2\mu_2}{(n_1+n_2)\sigma^2}, -\dfrac{1}{2\sigma^2} \right)$

$\underline{U} = \left(n_1\bar{X}_1 + n_2\bar{X}_2, \sum_{i=1}^{2}\sum_{j=1}^{n_i} X_{ij}^2 \right)$

$$V(\underline{Y}, \underline{U}) = \frac{(\bar{X}_2-\bar{X}_1)\Big/\sqrt{\frac{1}{n_1}+\frac{1}{n_2}}}{\sqrt{\dfrac{\sum_{i=1}^{2}\sum_{j=1}^{n_i}\left(X_{ij} - \frac{n_1\bar{X}_1+n_2\bar{X}_2}{n_1+n_2}\right)^2}{n_1+n_2-2}}} \quad \uparrow Y$$

$\sim T_{n_1+n_2-1}$ (Student's t distribution with n_1+n_2-1 d.f.)

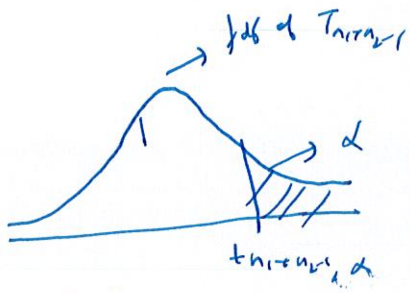
under H_0. (or $\mu_1=\mu_2$)

Also, $\mu_1=\mu_2$, $V(\underline{Y}, U)$ and $\underline{U}$ are independent by Basu's theorem.

Case II A: $H_0: \mu_1 \geq \mu_2$ vs $H_1: \mu_1 < \mu_2$ $\iff$ $H_0: \theta \leq 0$ vs $H_1: \theta > 0$

UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & V > c \\ 0, & \text{o.w.} \end{cases}$$



Where $E_{\theta=0}(\phi^*(\underline{X})) = \alpha \Rightarrow c = t_{n_1+n_2-1,\,\alpha}$

Case II B: $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 \neq \mu_2$ $\iff$ $H_0: \theta = 0$ vs $H_1: \theta \neq 0$

UMPU(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & V < c_1 \text{ or } V > c_2 \\ 0, & \text{o.w.} \end{cases}$$

Where $E_{\mu_1=\mu_2}(\phi^*(\underline{X})) = \alpha$, & $E_{\mu_1=\mu_2}(\phi^*(\underline{X})V) = \alpha\, E_{\mu_1=\mu_2}(V)$

$\Rightarrow E_{\mu_1=\mu_2}(1-\phi^*(\underline{X})) = 1-\alpha$, & $E_{\mu_1=\mu_2}((1-\phi^*(\underline{X}))V) = (1-\alpha)\, E_{\mu_1=\mu_2}(V)$

As before we will have $c_2 = -c_1 = t_{n_1+n_2-1,\,1-\alpha/2}$

Testing for Independence in the Bivariate Normal Model

Let $\underset{\sim}{X_j} = \begin{pmatrix} x_{1j} \\ x_{2j} \end{pmatrix}$, $j=1,2,\dots,n$ be a random sample from

$$N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$$

Where $\underline{\theta} = (\mu_1, \mu_2, \sigma_1, \sigma_2, \rho) \in \Theta = \mathbb{R}^2 \times (0,\infty)^2 \times (-1,1)$ is a vector of of unknown parameters.

Consider testing

$H_0: \rho \le 0$ vs $H_1: \rho > 0$ or $H_0: \rho = 0$ vs $H_1: \rho \neq 0$

We have

$$f_{\theta, \underline{\psi}}(\underline{x}) = \left(2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}\right)^{-n} \times$$

$$\exp\left[-\frac{1}{2(1-\rho^2)}\left[\frac{\sum_{j=1}^{n}(x_{1j}-\mu_1)^2}{\sigma_1^2} + \frac{\sum_{j=1}^{n}(x_{2j}-\mu_2)^2}{\sigma_2^2} - \frac{2\rho}{\sigma_1\sigma_2}\sum_{j=1}^{n}(x_{1j}-\mu_1)(x_{2j}-\mu_2)\right]\right],$$

$$\theta = \frac{\rho}{\sigma_1\sigma_2(1-\rho^2)}, \quad Y = \sum_{j=1}^{n} x_{1j} x_{2j}, \quad \underline{U} = \left(\sum_{j=1}^{n} x_{1j}^2, \sum_{j=1}^{n} x_{2j}^2, \sum_{j=1}^{n} x_{1j}, \sum_{j=1}^{n} x_{2j}\right)$$

Consider

$$V(Y, \underline{U}) = R = \frac{\sum_{j=1}^{n}(x_{1j}-\bar{x}_1)(x_{2j}-\bar{x}_2)}{\sqrt{\left\{\sum_{j=1}^{n}(x_{1j}-\bar{x}_1)^2\right\}\left\{\sum_{j=1}^{n}(x_{2j}-\bar{x}_2)^2\right\}}}$$

Under H_0, V and $\underline{U}$ are independent and

$$T = \frac{\sqrt{n-2}\, R}{\sqrt{1-R^2}} \sim T_{n-2}$$ (Student t-distribution with $n-2$ d.f.)

Thus one can construct a t test based on statistic T.