MTH-418: Inference-I

2023-2024: II Semester
End Semester Examination

Time Allowed: 3 Hours **Maximum Marks: 100**

NOTE: (i) This question paper has 8 questions, to be attempted in 3 hours. Attempt all the 8 questions.
(ii) Start answer of every question on a new page. Moreover, attempt all the parts of a question at one place.
(iii) Answer each question legibly, clearly and concisely. Illegible answers will not be graded.
(iv) For each question, provide details of all the steps involved in arriving at the final answer(s)/conclusion(s) and box your final answer(s)/conclusion(s).

1. Let X_1, X_2 be a random sample from a population having a pdf

$$f_\theta(x) = \begin{cases} \frac{2x}{\theta^2}, & \text{if } 0 < x < \theta \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in [1, \infty) = \Theta$, say, is an unknown parameter (**Note:** The parameter space is restricted).

   (i) Show that $X_{(2)} = \max\{X_1, X_2\}$ is **NOT** a complete statistic;

   (ii) Find the UMVUE of θ. $\boxed{6+6=12 \text{ Marks}}$

2. Let $X_1, \ldots, X_5$ be a random sample from a $\text{Bin}(1, \theta)$ distribution, where $\theta \in (0, 1) = \Theta$. Let $\delta_0(\underline{X}) = X_1 X_2$, so that $E_\theta(\delta_0(\underline{X})) = \theta^2, \theta \in \Theta$.

   (i) Show that $T_1 = \sum_{i=1}^{5} X_i$ is a minimal sufficient statistic and hence show that $T_2 = (X_1 + X_2 + X_3, X_4 + X_5)$ is a sufficient statistic;

   (ii) Using the Rao-Blackwell Theorem on δ_0 with sufficient statistic T_2, find an unbiased estimator δ_1 of θ^2 that has uniformly smaller variance than the variance of the estimator δ_0. Is δ_1 the UMVUE?;

   (iii) Using the Rao-Blackwell Theorem on δ_1 (obtained in (i) above) with minimal sufficient statistic T_1, find an unbiased estimator δ_2 of θ^2 that has uniformly smaller variance than the variance of the estimator δ_1. Is δ_2 the UMVUE? $\boxed{4+5+5=14 \text{ Marks}}$

3. Let X_1, X_2, X_3 be a random sample from an exponential distribution having the p.d.f.

$$f_\theta(x) = \begin{cases} \theta e^{-\theta x}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, \infty) = \Theta$. $\boxed{\text{6+6=12 Marks}}$

(i) Find the UMVUEs δ_0, δ_1 and δ_2 of the estimands $\psi_1(\theta) = \frac{1}{\theta}, \psi_2(\theta) = \theta$ and $\psi_3(\theta) = e^{-\theta}$ (**Hint:** $\delta(\underline{X}) = I(X_1 > 1)$ is an unbiased estimator of $e^{-\theta}$);

(ii) Using the generalized Rao-Cramer bound, find sharp bounds on the variances of unbiased estimators of $\psi_1(\theta), \psi_2(\theta)$ and $\psi_3(\theta)$.

4. (a) Let X_1, X_2 be a random sample from a population having p.d.f.

$$f_\theta(x) = \begin{cases} \frac{2}{\theta^2}(\theta - x), & \text{if } 0 < x < \theta \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in (0, \infty) = \Theta$. Find the MME and the MLE of θ. $\boxed{\text{4+8=12 Marks}}$

5. Let X_1, X_2 be a random sample from $U(0, \theta)$, where $\theta \in \Theta = \{1, \frac{3}{2}, 2\}$. For testing $H_0 : \theta \in \{1, \frac{3}{2}\}$ against $H_1 : \theta = 2$, consider the test function

$$\phi(\underline{X}) = \begin{cases} 1, & \text{if } X_1 + X_2 > \frac{7}{4} \\ 0, & \text{otherwise} \end{cases}.$$

Find the size and the power of the test. $\boxed{\text{8+4=12 Marks}}$

6. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from an $U(\theta, \theta + 1)$ distribution, where $\theta \in \mathbb{R} = \Theta$. Find an UMP($\alpha$) test for testing $H_0 : \theta \leq 0$ against $H_1 : \theta > 0$; here $\alpha \in (0, 1)$. . $\boxed{\text{14 Marks}}$

7. Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$, where $\theta \in \mathbb{R}$ and $\sigma > 0$ are both unknown. For testing $H_0 : \theta = 0$ against $H_1 : \theta \neq 0$, find an UMPU(α) test ($\alpha \in (0, 1)$). $\boxed{\text{12 Marks}}$

8. Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from populations having p.d.f.s

$$f_{\theta_1}(x) = \begin{cases} \frac{1}{\theta_1} e^{-\frac{x}{\theta_1}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and } f_{\theta_2}(x) = \begin{cases} \frac{1}{\theta_2} e^{-\frac{x}{\theta_2}}, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases},$$

respectively, where $(\theta_1, \theta_2) \in (0, \infty) \times (0, \infty) = \Theta$. For testing $H_0 : \theta_1 = \theta_2$ against $\theta_1 \neq \theta_2$ at α $(\in (0, 1))$ of significance, find an UMPU(α) test. $\boxed{\text{12 Marks}}$

Course No. MTM 418 : Inference-I

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Model Solutions

Problem No. 1 (i) The pdf of $X_{(2)} = \max\{X_1, X_2\}$ is

$$g_\theta(x) = \frac{4x^3}{\theta^4}, \quad 0 < x < \theta$$

Let the function $h(\cdot)$ be such that

$$E_\theta(h(X_{(2)})) = 0 \quad \forall\ \theta \geq 1$$

$$\Leftrightarrow \int_0^\theta h(x)\, x^3\, dx = 0 \quad \forall\ \theta \geq 1$$

$$\Leftrightarrow \int_0^1 h(x)\, x^3\, dx = 0 \quad \text{and} \quad \int_0^\theta h(x)\, x^3\, dx = 0 \quad \forall\ \theta \geq 1$$

$$\Leftrightarrow \int_0^1 h(x)\, x^3\, dx = 0, \quad \text{and} \quad h(t) = 0 \quad \forall\ t \geq 1$$

... $\boxed{3 \text{ MARKS}}$

Consider

$$h(x) = \begin{cases} x - \frac{4}{5}, & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

Then

$$E_\theta(h(X_{(2)})) = 0 \quad \forall\ \theta \geq 1$$

But $P_\theta(h(X_{(2)}) = 0) = P_\theta(X_{(2)} \geq 1)$

$$= \int_1^\theta \frac{4x^3}{\theta^4}\, dx = \theta^4 \neq 1 \quad \forall\ \theta \geq 1$$

$\Rightarrow$ $X_{(2)}$ is NOT Complete

... $\boxed{3 \text{ MARKS}}$

(ii) $T = X_{(2)}$ is a minimal sufficient statistic and therefore it suffices to consider only those estimators that depend on X_1 and X_2 only through $T = X_{(2)}$.

$$U \in \mathcal{U} \iff \int_0^1 u(t)\, t^3\, dt = 0 \quad \text{and} \quad u(t) = 0 \ \forall\, t \geq 1 \quad (\text{By (i)}) \text{ --- } (*)$$

Let $h(\cdot)$ be such that $E_\theta(h(X_{(n)})) = \theta, \ \forall\, \theta > 1$

$$\iff \int_0^\theta h(t) \frac{4t^3}{\theta^4}\, dt = \theta, \quad \forall\, \theta \geq 1$$

$$\iff 4\int_0^1 h(t)\, t^3\, dt + 4\int_1^\theta h(t)\, t^3\, dt = \theta^5, \ \forall\, \theta \geq 1$$

$$\Rightarrow 4\int_0^1 h(t)\, t^3\, dt = 1 \quad \text{and} \quad 4h(t)t^3 = 5t^4, \ \forall\, t \geq 1.$$

$$\iff h(t) = \frac{5}{4}t, \ \forall\, t \geq 1$$

Then, for $U \in \mathcal{U}$, we must have

$$E_\theta(h(t)U(t)) = \int_0^1 h(t)u(t) \frac{4t^3}{\theta^4}\, dt + \int_1^\theta h(t)u(t) \frac{4t^3}{\theta^4}\, dt$$

$$= \frac{4}{\theta^4}\int_0^1 h(t)u(t)t^3\, dt = 0 \quad \ldots \quad \boxed{3 \text{ MARKS}}$$

$\iff$ This suggests taking

$$\delta(X_{(2)}) = \begin{cases} a, & X_{(2)} < 1 \\ \frac{5}{4}X_{(2)}, & X_{(2)} \geq 1 \end{cases}$$

$$4a\int_0^1 t^3\, dt = 1 \Rightarrow a = 1.$$

Thus the UMVUE of θ is

$$\delta(X_{(2)}) = \begin{cases} 1, & X_{(2)} < 1 \\ \frac{5}{4}X_{(2)}, & X_{(2)} \geq 1 \end{cases} \quad \ldots \quad \boxed{3 \text{ MARKS}}$$

$\boxed{2/16}$

Problem No. 2

(i) For $\underline{x}, \underline{y} \in \mathcal{X}$

$$\frac{f_\theta(\underline{x})}{f_\theta(\underline{y})} = \frac{(\theta/_{1-\theta})^{T_1(\underline{x})}}{(\theta/_{1-\theta})^{T_1(\underline{y})}} \text{ is independent of } \theta \in (0, 1)$$

$$\Leftrightarrow \quad \eta^{T_1(\underline{x}) - T_1(\underline{y})} \text{ is independent of } \eta \in (0, \infty)$$

$$\Leftrightarrow \quad T_1(\underline{x}) = T_1(\underline{y})$$

Thus $T_1 \equiv T_1(\underline{X}) = \sum_{i=1}^{5} X_i$ is a minimal sufficient statistic ... $\boxed{\text{2 MARKS}}$

Since the sufficient statistic (in fact minimal sufficient statistic) T_1 is a function of the statistic T_2, it follows that T_2 is also a sufficient statistic. ... $\boxed{\text{2 MARKS}}$

(ii) The desired estimator is

$$\delta_1(\underline{T_2}) = \delta_1(Z_1, Z_2) \quad \ldots \boxed{\text{2 MARKS}}$$

$$= E_\theta[\delta_0(\underline{X}) \mid (Z_1, Z_2)],$$

where $Z_1 = X_1 + X_2 + X_3$, ~~and~~ $Z_2 = X_4 + X_5$ and $\underline{T_2} = (Z_1, Z_2)$

For fixed $(z_1, z_2) \in \{0, 1, 2, 3\} \times \{0, 1, 2\}$,

$$\delta_1(\underline{t_2}) = P_\theta(X_1 X_2 = 1 \mid X_1 + X_2 + X_3 = z_1, X_4 + X_5 = z_2)$$

$$= \frac{P(X_1 = 1, X_2 = 1, X_3 = z_1 - 2, X_4 + X_5 = z_2)}{P_\theta(X_1 + X_2 + X_3 = z_1, X_4 + X_4 = z_2)}$$

$$\cancel{= \begin{cases} 0, & z_1 = 0, 1 \\ \dfrac{\cdots \binom{2}{z_2}\theta^{z_2}(1-\theta)^{2-z_2}}{\binom{3}{z_1}\theta^{z_1}(1-\theta)^{3-z_1}\binom{2}{z_2}\theta^{z_2}(1-\theta)^{2-z_2}}, & z_1 = 2, 3, 4, 5, \; z_2 \in 0, 1, 2 \end{cases}}$$

$$
= \begin{cases} 0, & z_1 = 0, 1 \\ \dfrac{\theta^2(1-\theta)}{3\theta^2(1-\theta)} = \dfrac{1}{3} & z_1 = 2 \\ \dfrac{\theta^3}{\theta^3} = 1 & z_1 = 3 \end{cases}
$$

Thus the desired estimator is

$$
\delta_1(T_2) = \frac{Z_1(Z_1-1)}{6}, \quad \text{where } Z_1 = X_1 + X_2 + X_3
$$

. The estimator δ_1 can be further improved by using Rao-Blackwel theorem on the minimal sufficient statistic T_1.
So δ_1 is NOT the UMVUE

3 MARKS

(iii) The desired estimator is

$$
\delta_2(T_4) : E_\theta[\delta_1(T_1) \mid T_4]
$$

For $t_2 \in \{0, 1, \ldots, 5\}$

$$
\delta_2(t_4) = \sum_{t_1=0}^{3} \delta_1(t_1) P_\theta(\delta_1(T_2) = \delta_1(t_2) \mid T_4 = t_4)
$$

$$
= \frac{1}{3} P_\theta\left(X_1 + X_2 + X_3 = 2 \,\middle|\, \sum_{i=1}^{5} x_i = t_4\right)
$$

$$
+ P_\theta\left(X_1 + X_2 + X_3 = 3 \,\middle|\, \sum_{i=1}^{5} x_i = t_4\right)
$$

$$
= \frac{1}{3} \frac{P_\theta(X_1 + X_2 + X_3 = 2) P_\theta(X_4 + X_5 = t_4 - 2)}{P_\theta\left(\sum_{i=1}^{5} x_i = t_4\right)}
$$

$$
+ \frac{P_\theta(X_1 + X_2 + X_3 = 3) P_\theta(X_4 + X_4 = t_4 - 3)}{P_\theta\left(\sum_{i=1}^{5} x_i = t_4\right)} \ldots
$$

2 MARKS

$$
\frac{1}{3} \frac{\binom{3}{2}\theta^2(1-\theta)(1-\theta)}{\binom{5}{2}\theta^2(1-\theta)^3}, \quad t_4 = 2
$$

$$
= \begin{cases} 0, & t_2 = 0, 1 \\ \dfrac{1}{3} \dfrac{\binom{3}{2}\theta^2(1-\theta)\binom{2}{t_4-2}\theta^{t_4-2}(1-\theta)^{4-t_4}}{\binom{5}{t_2}\theta^{t_4}(1-\theta)^{5-t_4}} + \dfrac{\theta^3\binom{2}{t_4-3}\theta^{t_4-3}(1-\theta)^{5-t_4}}{\binom{5}{t_2}\theta^{t_4}(1-\theta)^{5-t_4}}, & t_4 = 3, 4 \\ \dfrac{\theta^3 \times \theta}{\theta^5}, & t_4 = 5 \end{cases}
$$

$$= \begin{cases} 0, & t_2 = 0, 1 \\ \frac{1}{10}, & t_2 = 2 \\ \frac{\binom{2}{t_2-2}}{\binom{5}{t_2}} + \frac{\binom{2}{t_2-3}}{\binom{5}{t_2}}, & t_2 = 3, 4 \\ 1, & t_2 = 5 \end{cases}$$

$$= \begin{cases} 0, & t_2 = 0, 1 \\ \frac{1}{10}, & t_2 = 2 \\ \frac{t_2(t_2-1)(5-t_2)}{60} + \frac{t_2(t_2-1)(t_2-2)}{60}, & t_2 = 3, 4 \\ 1, & t_2 = 5 \end{cases}$$

$$= \begin{cases} 0, & t_2 = 0, 1 \\ \frac{1}{10}, & t_2 = 2 \\ \frac{t_2(t_2-1)}{20}, & t_2 = 3, 4 \\ 1, & t_2 = 5 \end{cases}$$

$\boxed{\text{3 MARKS}}$

$$\Rightarrow \quad \delta_2(T_2) = \frac{T_2(T_2-1)}{20}$$

δ_2 is the unique ~~un~~ unique UMVUE based on the complete sufficient statistic T_1 (Lehmann-Scheffe Thm).

Problem No. 3 (i) $T = X_1 + X_2 + X_3 \sim \text{Gamma}(3, \frac{1}{\theta})$

$= 3\bar{X}$

T is a complete sufficient statistic.

$$E_\theta(T^r) = \frac{\theta^3}{\Gamma 3} \int_0^\infty t^r e^{-\theta t} t^2 \, dt$$

$$= \frac{\Gamma(r+3)}{2} \theta^{-r}, \quad r > -3$$

$$E_\theta(T) = \frac{\Gamma 4}{2} \psi_1(\theta) = 3\psi_1(\theta), \quad \forall \theta > 0$$

$$\Rightarrow E_\theta\left(\frac{T}{3}\right) = \psi_1(\theta), \quad \forall \theta > 0$$

$\Rightarrow \bar{X}$ is UMVUE of $\psi_1(\theta)$ 1-MARK

$$E_\theta(T^{-1}) = \frac{\Gamma 2}{2} \cdot \frac{1}{\theta} \Rightarrow E_\theta\left(\frac{2}{3\bar{X}}\right) = \psi_2(\theta) \quad \forall \theta > 0$$

$\Rightarrow \frac{2}{3\bar{X}}$ is the UMVUE of $\psi_2(\theta)$ 1-MARK

The UMVUE of $\psi_3(\theta)$ is

$$\delta_0(T) = E_\theta(\delta \mid T)$$

$$= P_\theta(X_1 > 1 \mid T)$$

For $t > 0$

$$\delta_0(t) = P_\theta(X_1 > 1 \mid T = t)$$

$$= P_\theta\left(\frac{X_1}{X_1 + X_2 + X_3} > \frac{1}{t} \,\Big|\, T = t\right)$$

$$= P_\theta\left(\frac{X_1}{X_1 + X_2 + X_3} > \frac{1}{t}\right) \quad (\text{Basu's Thm})$$

$$= P_\theta\left(\text{Beta}(1, 2) > \frac{1}{t}\right) = \begin{cases} 2\int_{1/t}^{1} (1-\lambda)\, d\lambda & 0 < \frac{1}{t} < 1 \\ 0 & \frac{1}{t} > 1 \end{cases}$$

$$= \begin{cases} \left(1-\frac{1}{t}\right)^2, & \text{if } t > 1 \\ 0 & \text{if } t < 1 \end{cases}$$

$$\Rightarrow \quad \delta_0(T) = \begin{cases} \left(1-\frac{1}{3\bar{x}}\right)^2, & \text{if } \bar{x} > \frac{1}{3} \\ 0 & \text{if } \bar{x} < \frac{1}{3} \end{cases}$$ ... 4 MARKS

(ii) $$J_{3\times 1} = \begin{pmatrix} \varphi_1'(\theta) \\ \varphi_2'(\theta) \\ \varphi_3'(\theta) \end{pmatrix} = \begin{pmatrix} -\frac{1}{\theta^2} \\ 1 \\ -e^{-\theta} \end{pmatrix}$$

$$I(\theta) = \frac{3}{V_\theta(X_1)} = \frac{3}{\theta^2}.$$

For unbiased estimator δ_i of $\varphi_i(\theta)$ .. 3 MARKS

$V_\theta(\underline{\delta}) - JI^{-1}J'$ is p.d.

$$V_\theta(\delta) - \frac{\theta^2}{3}\begin{pmatrix} -\frac{1}{\theta^2} \\ 1 \\ -e^{-\theta} \end{pmatrix}\left(-\frac{1}{\theta^2}, 1, -e^{-\theta}\right) \text{ is p.d.}$$

$$\Leftrightarrow \quad V_\theta(\delta) - \frac{\theta^2}{3}\begin{pmatrix} \frac{1}{\theta^4} & -\frac{1}{\theta^2}, & \frac{e^{-\theta}}{\theta^2} \\ -\frac{1}{\theta^2} & 1, & -e^{-\theta} \\ \frac{e^{-\theta}}{\theta^2} & -e^{-\theta} & e^{-2\theta} \end{pmatrix}$$

$$\Rightarrow \quad V_\theta(\delta_1) \geq \frac{1}{3\theta^2}, \qquad V_\theta(\delta_2) \geq \frac{\theta^2}{3}$$

$$V_\theta(\delta_3) \geq \frac{\theta^2 e^{-2\theta}}{3}$$ ... 3 MARKS

$\boxed{\text{Problem No. 4}}$ $E_\theta(X) = \frac{2}{\theta^2}\int_0^\theta x(\theta - x)\,dx$

$$= \frac{\theta}{3}$$

$\Rightarrow \quad \hat{\theta}_{MME} = 3\bar{X}$ ... $\boxed{\text{4 MARKS}}$

For any fixed $\underline{x} = (x_1, x_2) \in (0, \infty)^2$

$$L_{\underline{x}}(\theta) = \frac{4(\theta - x_{(1)})(\theta - x_{(2)})}{\theta^4}, \quad \theta \geq x_{(2)}$$

For $\theta \geq x_{(2)}$

$\frac{\partial}{\partial \theta} L_{\underline{x}}(\theta) \; > (<) \; 0$

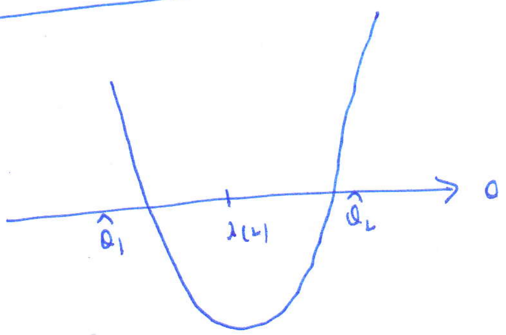
$\Leftrightarrow g(\theta) = 2\theta^2 - 3(x_{(1)} + x_{(2)})\theta + 4x_{(1)}x_{(2)} < (>) 0.$ ... $\boxed{\text{4 MARKS}}$

The quadratic equation $2\theta^2 - 3(x_{(1)} + x_{(2)})\theta + 4x_{(1)}x_{(2)} = 0$ has 2 real roots (discriminant $\Delta = 9x_{(1)}^2 + 9x_{(2)}^2 - 14x_{(1)}x_{(2)} \uparrow$ in $x_{(2)}$ so $\Delta \geq 9x_{(1)}^2 + 9x_{(1)}^2 - 14x_{(1)}^2 = 4x_{(1)}^2 > 0$).

$$\hat{\theta}_1 = \frac{3(x_{(1)} + x_{(2)}) - \sqrt{9(x_{(1)} + x_{(2)})^2 - 32x_{(1)}x_{(2)}}}{4}$$

and $$\hat{\theta}_2 = \frac{3(x_{(1)} + x_{(2)}) + \sqrt{9(x_{(1)} + x_{(2)})^2 - 32x_{(1)}x_{(2)}}}{4}$$

with $\hat{\theta}_1 < x_{(2)} < \hat{\theta}_2$



Thus $L_{\underline{x}}(\theta) \uparrow$ in $\theta \in (x_{(2)}, \hat{\theta}_2)$ and $\downarrow$ in $\theta \in (\hat{\theta}_2, \infty)$

$$\Rightarrow \quad \hat{\theta}_{MLE} = \hat{\theta}_2 = \frac{3(X_{(1)} + X_{(2)}) + \sqrt{9(X_{(1)} + X_{(2)})^2 - 32X_{(1)}X_{(2)}}}{4}$$

... $\boxed{\text{4 MARKS}}$

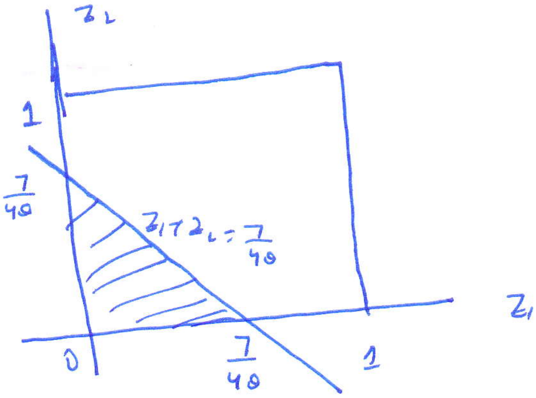
Problem No. 5

For $\theta \in \Theta = \{1, \frac{3}{2}, 2\}$

$$E_\theta(\phi(\underline{X})) = P_\theta\left(X_1 + X_2 > \frac{7}{4}\right)$$

$$= P_\theta\left(Z_1 + Z_2 > \frac{7}{4\theta}\right),$$

where Z_1 and Z_2 are iid $U(0, 1)$



For $\frac{7}{4\theta} \in (0, 1)$, i.e. $\theta > 7/4$

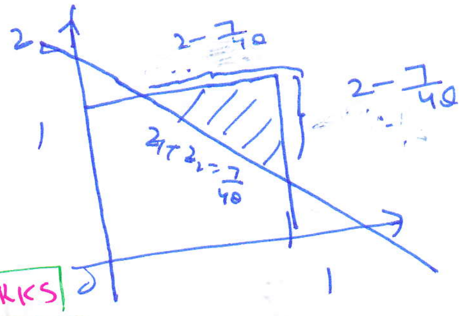
$$E_\theta(\phi(X)) = 1 - \frac{1}{2}\left(\frac{7}{4\theta}\right)^2$$

$$= 1 - \frac{49}{32\theta^2}$$

Power $= E_{\theta=2}(\phi(X))$

$$= 1 - \frac{49}{128} = \frac{79}{128}$$

4 MARKS



For $\frac{7}{4\theta} > 1$, i.e. $\theta < \frac{7}{4}$

$$E_\theta(\phi(X)) = \frac{1}{2}\left(2 - \frac{7}{4\theta}\right)^2$$

$$E_{\theta=1}(\phi(X)) = \frac{1}{2}\left(2 - \frac{7}{4}\right)^2 = \frac{1}{32}$$ 2 MARKS

$$E_{\theta=\frac{3}{2}}(\phi(X)) = \frac{1}{2} \times \frac{25}{36} = \frac{25}{72}$$ 2 MARKS

Size $= \max\{E_{\theta=1}(\phi(X)), E_{\theta=\frac{3}{2}}(\phi(X))\}$

$$= \max\left\{\frac{1}{32}, \frac{25}{72}\right\} = \frac{25}{72}$$ 4 MARKS

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Problem No. 6

First consider testing $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$, where $\theta_1 > 0$ is a fixed constant.

Then $P_\theta(X_{(1)} > 0 \text{ and } X_{(n)} < \theta_1 + 1) = 1$, $\forall \theta \in \{0, \theta_1\}$.

$$\frac{f_{\theta_1}(\underline{x})}{f_0(\underline{x})} = \begin{cases} 0 & x_{(1)} < \theta_1 \text{ and } x_{(n)} < 1 \\ 1 & x_{(1)} > \theta_1, \text{ and } x_{(n)} < 1 \\ \infty & x_{(1)} > \theta_1 \text{ and } x_{(n)} > 1. \end{cases}$$

For $k = 0$, a competing test is

$$\phi_0(\underline{x}) = \begin{cases} \gamma & \text{if } x_{(1)} < \theta_1 \text{ and } x_{(n)} < 1 \\ 1, & \text{if } x_{(1)} > \theta_1 \end{cases}$$

where $\gamma \in [0, 1]$ is s.t.

$E_0(\phi_0(X)) = \alpha$

$\Leftrightarrow \gamma P_0(X_{(1)} < \theta_1, X_{(n)} < 1) + P_0(X_{(1)} > \theta_1) = \alpha$

$\Leftrightarrow \gamma P_0(X_{(1)} < \theta_1) + P_0(X_{(1)} > \theta_1) = \alpha$

$$\Leftrightarrow \gamma = \frac{\alpha - P_0(X_{(1)} > \theta_1)}{1 - P_0(X_{(1)} > \theta_1)} = \begin{cases} \dfrac{\alpha - (1-\theta_1)^n}{1-(1-\theta_1)^n} & 0 < \theta_1 < 1 \\ \alpha & \theta_1 > 1 \end{cases}$$

For $\theta_1 \in (0, 1)$ and $\alpha \geq (1-\theta_1)^n$, UMP($\alpha$) test is

$$\phi_0(x) = \begin{cases} \dfrac{\alpha - (1-\theta_1)^n}{1-(1-\theta_1)^n}, & X_{(1)} < \theta_1, X_{(n)} < 1 \\ 1, & X_{(1)} > \theta_1 \end{cases}$$

With power $= E_{\theta_1}(\phi(\underline{X})) = P_\theta(X_{(1)} > \theta_1) = 1$ ... (A)

For $\theta_1 \geq 1$ and $\alpha \in (0, 1)$, the UMP(α) test is

$$\phi_0(\underline{x}) = \begin{cases} \alpha, & x_{(1)} < \theta_1, x_{(n)} < 1 \\ 1 & x_{(1)} > \theta_1 \end{cases}$$

With power $E_{\theta_1}(\phi(\underline{X})) = P_{\theta_1}(X_{(1)} > \theta_1) = 1$. ... (B)

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3 MARKS

For $k=1$, UMP(α) test is

$$\phi_1(x) = \begin{cases} 0 & x_{(1)} < \theta_1, \quad x_{(n)} < 1 \\ \gamma & x_{(1)} > \theta_1, \quad x_{(n)} < 1 \\ 1 & x_{(1)} > \theta_1, \quad x_{(n)} > 1 \end{cases}$$

Where $\gamma \in [0,1]$ is s.t.

$$E_0(\phi_1(\underline{x})) = \alpha$$

$$\Leftrightarrow \quad \gamma P_0(x_{(1)} > \theta_1) = \alpha$$

$$\Leftrightarrow \quad \gamma = \begin{cases} \dfrac{\alpha}{(1-\theta_1)^n} & 0 < \theta_1 < 1 \\ \infty & \theta_1 > 1 \end{cases}$$

Thus for $\theta_1 \in (0,1)$ and $0 < \alpha \le (1-\theta_1)^n$ UMP(α) test is

$$\phi_1(\underline{x}) = \begin{cases} 0 & x_{(1)} < \theta_1, \quad x_{(n)} < 1 \\ \dfrac{\alpha}{(1-\theta_1)^n}, & x_{(1)} > \theta_1, \quad x_{(n)} < 1 \\ 1 & x_{(1)} > \theta_1, \quad x_{(n)} > 1 \end{cases}$$

With power $E_{\theta_1}(\phi_1)$

$$= \frac{\alpha}{(1-\theta_1)^n} P_{\theta_1}(x_{(n)} < 1) + P_{\theta_1}(x_{(n)} > 1)$$

$$= \frac{\alpha}{(1-\theta_1)^n} (1-\theta_1)^n + 1 - (1-\theta_1)^n$$

$$= 1 + \alpha - (1-\theta_1)^n$$

.... (C)

... $\boxed{\text{3 MARKS}}$

Consider the test

$$\phi_2(\underline{x}) = \begin{cases} 0 & x_{(1)} < c, \quad x_{(n)} < 1 \\ 1 & x_{(1)} > c \text{ or } x_{(n)} > 1 \end{cases}$$

Where c is chosen so that

$$E_0(\phi_2(\underline{x})) = \alpha$$

$\Leftrightarrow \quad P_0 (X_{(1)} < c, \quad X_{(n)} < 1) = 1-\alpha$

$\Leftrightarrow \quad P_0 (X_{(1)} < c) = 1-\alpha$

$\Leftrightarrow \quad 1 - (1-c)^n = 1-\alpha$

$\Leftrightarrow \quad c = 1-\alpha^{1/n}$

The power of this test is

$E_{\theta_1} (Q_2(\underline{X})) = P_{\theta_1} (X_{(1)} > 1-\alpha^{1/n}$ or $X_{(n)} > 1)$

Clearly for $\theta_1 \geq 1$, $P_{\theta_1} (X_{(n)} \geq 1) = 1$ and therefore

power $= E_{\theta_1} (Q_2(\underline{X})) = 1$ ... (D)

For $\theta_1 \in (0, 1)$

power $= E_{\theta_1} (Q_2(\underline{X}))$

$= P_{\theta_1} (X_{(1)} > 1-\alpha^{1/n}$ or $X_{(n)} > 1)$

$= P_\theta (X_{(1)} > 1-\alpha^{1/n}, X_{(n)} < 1) + P_0 (X_{(n)} > 1)$

$$= \begin{cases} (1-\theta_1)^n + 1 - (1-\theta_1)^n = 1 & \theta_1 > 1-\alpha^{1/n} \\ \alpha + 1 - (1-\theta_1)^n & \theta_1 < 1-\alpha^{1/n} \end{cases} \quad \ldots \text{(E)}$$

Comparing (A) - (E), we conclude that UMP(α) test is

$$Q_2(\underline{X}) = \begin{cases} 0 & X_{(1)} < 1-\alpha^{1/n}, \quad X_{(n)} < 1 \\ 1 & X_{(1)} > 1-\alpha^{1/n} \text{ or } X_{(n)} > 1 \end{cases}$$

As Q_2 does not depend on θ_1 as long as $\theta_1 > 0$, Q_2 is also UMP(α) for testing $H_0: \theta = 0$ vs $H_1: \theta > 0$. ... $\boxed{\text{4 MARKS}}$

For $\theta \le 0$,

$$E_\theta(\phi_2(X)) = P_\theta(X_{(1)} > 1-\alpha^{1/n} \text{ or } X_{(n)} > 1)$$

$$= P_\theta(X_{(1)} > 1-\alpha^{1/n})$$

$$= \begin{cases} 0 & \theta \le -\alpha^{1/n} \\ (\theta + \alpha^{1/n})^n & -\alpha^{1/n} \le \theta \le 0 \end{cases}$$

and

$$\sup_{\theta \le 0} E_\theta(\phi_2(X)) = \alpha$$

$\Rightarrow$ ϕ_2 has size α (level α) completing that ϕ_2 is UMP(α) for testing $H_0: \theta \le 0$ vs $H_1: \theta > 0$

$\boxed{\text{2 MARKS}}$

Problem No. 7 The joint pdf of $\underset{\sim}{X} = (X_1, \ldots, X_n)$ is

$$f_\theta(\underset{\sim}{x}) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{n\theta\bar{x} - \frac{1}{2\sigma^2}\sum x_i^2 - \frac{n\theta^2}{2}}$$

Let $\eta = n\theta$. Then $H_0: \eta = 0$ and $H_1: \eta \neq 0$. We have

$$Y = \bar{X} \quad \text{and} \quad U = \sum_{i=1}^{n} X_i^2$$

$$T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{\frac{\sum X_i^2}{n}}} \uparrow \text{ in } \bar{X} \text{ for fixed } U.$$

Thus UMPU(α) test is

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1, & T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{\frac{\sum X_i^2}{n}}} < C_1(U) \text{ or } T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{\frac{\sum X_i^2}{n}}} > C_2(U) \\ 0 & \text{o.w} \end{cases}$$

...... $\boxed{\text{4 MARKS}}$

Where

$$E_0(\phi_0 | U) = \alpha$$

and $E_0(\phi_0 Y | U) = \alpha E_0(Y|U)$

Under H_0, $U = \sum X_i^2$ is C-S and T is ancillary. So by Basu's Thm, $\sum_{i=1}^{n} X_i^2$ and U are independent.

$\Leftrightarrow$ $C_1(U) = C_1$ and $C_2(U) = C_2$, Under H_0

$$T = \frac{\sqrt{n}\,\bar{X}}{\sqrt{\frac{\sum X_i^2}{n}}} = \frac{N(0,1)}{\sqrt{\frac{\chi_n^2}{n}}} \Big\} \text{ independent}$$

$$\sim t\text{-dist}^n \text{ with } n \text{ d.f.}$$

$$E_0(\phi_0|U) = \alpha \Leftrightarrow P_0(T < C_1) + P_0(T > C_2) = \alpha \quad \ldots \text{(I)}$$

$$E_0(Y|U) - E_0((1-\phi_0)Y|U) = \alpha E_0(Y|U)$$

$$E_0(Y|U) = E_0\left(\frac{1}{n}\sqrt{U}\, T|U\right) = \frac{\sqrt{U}}{n} E_0(T|U)$$

$$= \frac{\sqrt{U}}{n} E_0(T) = 0$$

$\Rightarrow \quad E_0((1-\phi)Y|U) = 0$

$\Leftrightarrow \quad P_0(c_1 < T < c_2 | U) = 0 \Leftrightarrow P_0(c_1 < T < c_2) = 0$ ... 4 MARKS

$\Leftrightarrow \quad P_0(-c_2 < T < -c_1) = 0$

Thus we may take $c_1 = -c_2$. Using in (I) we get

$$P_0(T < -c_2) + P_0(T > c_2) = \alpha$$

$$2\left[1 - P_0(T \le c_2)\right] = \alpha$$

$$P_0(T \le c_2) = 1 - \alpha/2$$

$$P_0(T > c_2) = \alpha/2$$

$$\Rightarrow c_2 = t_{n,\alpha/2} = -c_1$$

Thus the UMPULα test is

$$\phi_0(\underline{x}) = \begin{cases} 1 & T < -t_{n,\alpha/2} \text{ or } T > t_{n,\alpha/2} \\ 0 & \text{o.w} \end{cases}$$

$$= \begin{cases} 1 & |T| > t_{n,\alpha/2} \\ 0 & \text{o.w.} \end{cases}$$ ... 4 MARKS

Problem No. 8 The joint pdf of $\underline{Z} = (X_1, \dots, X_m, Y_1, \dots, Y_n)$ is

$$h_{\theta_1,\theta_2}(\underline{x}, \underline{y}) = \frac{1}{\theta_1^m \theta_2^n} e^{-\frac{\sum x_i}{\theta_1} - \frac{\sum y_i}{\theta_2}}$$

$$= \frac{1}{\theta_1^m \theta_2^n} e^{\underbrace{\left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right)}_{\eta} \sum y_i - \frac{1}{\theta_1}\left(\sum x_i + \sum y_i\right)}$$

$H_0: \theta_1 = \theta_2 \iff H_0: \eta = 0$ $\quad Y = \sum Y_i, \quad U = \sum X_i + \sum Y_i$

$H_1: \theta_1 \neq \theta_2 \iff \eta \neq 0$

Under $\bar{\Theta}_{01} = \{(\theta_1, \theta_2): \theta_1 = \theta_2\}$ U is complete sufficient. For given Y, $T = \frac{Y}{U} \uparrow$ in Y. Thus UMPU(α) test is

$$\phi_0(\underline{x}) = \begin{cases} 1, & T < c_1 \text{ or } T > c_2 \\ 0 & \text{o.w.} \end{cases}$$

... 6 MARKS

Where

(*) $E_{\theta_1=\theta_2}(\phi_0(\underline{x})|U) = \alpha$

(**) $E_{\theta_1=\theta_2}(\phi_0(\underline{x}) Y | U) = \alpha E_{\theta_1=\theta_2}(Y|U)$

Under $\theta_1 = \theta_2$

$B = T \sim \text{Beta}(n, m)$ and U are independent

(*) $\iff P_0(c_1 < B(n, m) < c_2) = 1 - \alpha$

$$\iff \boxed{\int_{c_1}^{c_2} \frac{t^{n-1}(1-t)^{m-1}}{B(n, m)} dt = 1 - \alpha}$$

... 3 MARKS

(**) $\iff E_{\theta_1=\theta_2}((1 - \phi_0) Y | U) = (1-\alpha) E_{\theta_1=\theta_2}(Y|U)$

$\iff U \, P_{\theta_1=\theta_2}(I(c_1 < T < c_2) T) = (1-\alpha) \, U \, E_{\theta_1=\theta_2}(T|U)$

$$\iff \boxed{\int_{c_1}^{c_2} t \frac{t^{n-1}(1-t)^{m-1}}{B(n, m)} dt = (1-\alpha) \frac{n}{n+m+1}}$$

... 3 MARKS