MTH-418a: Inference-I

2023-2024: II Semester
Quiz II

Time Allowed: 45 Minutes **Maximum Marks: 25**

1. Let $X_1, \ldots, X_n$ be a random sample from a population having the Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} e^{-(x-\theta)}, & \text{if } x > \theta \\ 0, & \text{otherwise} \end{cases},$$

where $\theta \in \Theta = (-\infty, 0]$ is unknown. For testing $H_0 : \theta \geq 0$ vs $H_1 : \theta < 0$, at $\alpha \in (0,1)$ level of significance, show that the test

$$\phi^*(\underline{X}) = \begin{cases} \alpha, & \text{if } X_{(1)} > 0 \\ 1, & \text{if } X_{(1)} < 0 \end{cases},$$

is an UMP(α) test. $\boxed{\text{15 Marks}}$

2. Let $X_1, \ldots, X_n$ be a random sample from N$(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$.

   (i) Find the Rao-Cramer lower bound on variances of unbiased estimators of $g(\theta) = e^{-\theta}$;

   (ii) Find the UMVUE of $g(\theta) = e^{-\theta}$. Is the Rao-Cramer bound obtained in (i) above is attaned by the variance of the UMVUE? $\boxed{\text{5+5 = Marks}}$

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Model Solutions

Problem No. 1 First consider testing $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$, where $\theta_1 \in (-\infty, 0)$ is a fixed constant. We have

$$\frac{g_{\theta_1}(\underline{x})}{g_0(\underline{x})} = e^{n\theta_1}\,\frac{I(x_{(1)} > \theta_1)}{I(x_{(1)} > 0)} = \begin{cases} \infty, & \text{if } \theta_1 < x_{(1)} < 0 \\ e^{n\theta_1}, & \text{if } x_{(1)} > 0. \end{cases}$$

For $c = e^{n\theta_1}$, a UMP(α) test for testing $H_0: \theta = 0$ vs $H_1: \theta = \theta_1$ is

$$\phi^*(\underline{x}) = \begin{cases} \gamma, & \text{if } x_{(1)} > 0 \\ 1, & \text{if } x_{(1)} < 0, \end{cases}$$

where

$$E_0(\phi^*(X)) = \alpha, \quad \text{i.e.} \quad \gamma = \alpha.$$

Thus a UMP(α) test is

$$\phi^*(\underline{x}) = \begin{cases} 1, & \text{if } x_{(1)} < 0 \\ \alpha, & \text{if } x_{(1)} > 0 \end{cases}$$

$\boxed{\text{6 MARKS}}$

Note that the test ϕ^* does not depend on the choice of θ_1, as along as $\theta_1 < 0$. It follows that ϕ^* is also UMP(α) for testing $H_0: \theta = 0$ vs $H_1: \theta < 0$.

... $\boxed{\text{3 MARKS}}$

For $\theta \geq 0$ we have

$$E_\theta(\phi^*(\underline{X})) = P_\theta(X_{(1)} < 0) + \alpha P_\theta(X_{(1)} > 0)$$
$$= \alpha$$

$\Rightarrow \quad \sup_{\theta \geq 0} E_\theta(Q^*(X)) = \alpha$

$\Rightarrow \quad \phi^*$ has level α

$\Rightarrow \quad \phi^*$ is also UMP(α) for testing $H_0: \theta \geq 0$ vs

$H_1: \theta < 0$

... $\boxed{6 \text{ MARKS}}$

Problem No. 2

(i) The joint pdf of $\underline{X} = (X_1, \dots, X_n)$ is

$$g_\theta(\underline{x}) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} e^{-\frac{1}{2}\sum(x_i-\theta)^2}, \quad \underline{x} \in \mathbb{R}^n,\ \theta \in \mathbb{R}$$

$$\ln g_\theta(\underline{x}) = -\frac{n}{2}\ln 2\pi - \frac{1}{2}\sum(x_i-\theta)^2$$

$$\frac{\partial}{\partial\theta}\ln g_\theta(\underline{x}) = \sum(x_i-\theta) = n\bar{x} - n\theta$$

$$I(\theta) = V_\theta\left(\frac{\partial}{\partial\theta}\ln g_\theta(\underline{X})\right) = V_\theta(n\bar{X} - n\theta) = n$$

Rao-Cramer LB is

$$R(\theta) = \frac{(g'(\theta))^2}{I(\theta)} = \frac{e^{-2\theta}}{n}. \quad \dots \boxed{5 \text{ MARKS}}$$

(ii) $\bar{X}$ is a complete-sufficient statistic, $\bar{X} \sim N(\theta, \frac{1}{n})$,

$M_X(t) = E(e^{tX}) = e^{\mu t + \frac{1}{2}t^2\sigma^2}$

$M_{\bar{X}}(t) = E_\theta(e^{t\bar{X}}) = e^{\theta t + \frac{t^2}{2n}}, \quad \forall t \in \mathbb{R}$

$\Rightarrow \quad E_\theta\left(e^{-(\bar{X} + \frac{1}{2n})}\right) = e^{-\theta}, \quad \forall \theta \in \mathbb{R}$

$\Rightarrow \quad \delta_0(\underline{X}) = e^{-(\bar{X}+\frac{1}{2n})}$ is the UMVUE of $e^{-\theta}$. ... $\boxed{3 \text{ MARKS}}$

We know that bound in exponential family (to which normal distribution family belongs) is attained iff

$$g_\theta(\underline{x}) = e^{\eta(\theta) \delta_0(\underline{x}) - B(\theta)} h(\underline{x})$$

$$= e^{\eta(\theta) e^{-(\bar{x}+\frac{1}{2n})} - B(\theta)} h(\underline{x})$$

But

$$g_\theta(\underline{x}) = e^{n\bar{x}\theta - \frac{n\theta^2}{2}} \times \frac{e^{-\frac{\sum x_i^2}{2}}}{(2\pi)^{n/2}}$$

$\Rightarrow$ R-C bound is not attained ... $\boxed{\text{2 MARKS}}$

$\boxed{\text{Aliter}}$

$$V_\theta(\delta_0) = V_\theta\left(e^{-\frac{1}{2n}} e^{-\bar{x}}\right) = e^{-\frac{1}{n}} V_\theta(e^{-\bar{x}})$$

$$= e^{-\frac{1}{n}} \left[E_\theta(e^{-2\bar{x}}) - (E_\theta(e^{-\bar{x}}))^2 \right]$$

$$= e^{-\frac{1}{n}} \left[e^{-2\theta} e^{\frac{2}{n}} - \left(e^{-\theta} e^{\frac{1}{2n}}\right)^2 \right]$$

$$= (e^{\frac{1}{n}} - 1) e^{-2\theta} > \frac{e^{-2\theta}}{n}$$

... $\boxed{\text{2 MARKS}}$

$\Rightarrow$ R-C bound is not attained.