MTH 515a: Inference-II
Assignment No. 1: Invariance

1. Let $X_1, \ldots, X_n$ $(n \geq 2)$ be a random sample from an $N(\theta, 1)$ distribution, where $\theta \in \Theta = \mathbb{R}$. Consider estimation of θ under the squared error loss function. For any $\underline{x} \in \chi$, consider the randomized decision rule $\delta_0((-\infty, t]|\underline{x}) = \Phi(t - x_1)$.

   (i) Find a randomized estimator (say δ_1) that is a function of a minimal-sufficient statistic and has the same risk function as that of δ_0.

   (ii) Find a non-randomized estimator that dominates the estimators δ_0 and $\delta_2(\underline{X}) = X_1$.

2. Let $X \sim \text{Bin}(n, \theta)$, where θ $(\in \Theta = (0,1))$ is unknown and n is a fixed positive integer. For estimating θ, under the squared error loss function, consider the randomized estimator

$$\delta_0(a|x) = \begin{cases} \frac{1}{2}, & \text{if } a = \frac{1}{2} \\ \\ \frac{1}{2}, & \text{if } a = \frac{x}{n} \end{cases}, x = 0, 1, \ldots, n.$$

   Find a non-randomized estimator that has uniformly smaller risk than δ_0.

3. Let $X_1, \ldots, X_{2n+1}$ be i.i.d. $\text{N}(\theta, 1)$ random variables, where θ $(\in \Theta = \mathbb{R})$ is unknown. Consider estimation of θ under a loss function $L(\theta, a), \theta \in \Theta, a \in \mathcal{A} = \mathbb{R}$.

   (a) Find a randomized estimator based on the complete-sufficient statistic $\overline{X}$ which has the same risk function as the sample median $\widetilde{X}$.

   (b) Suppose that, for every fixed $\theta \in \Theta$, the loss function $L(\theta, a)$ is strictly convex in $a \in \mathcal{A}$. Find an estimator that dominates the sample median $\widetilde{X}$.

4. For an invariant estimation problem $(\mathcal{P}, \mathcal{A}, L)$, show that $\overline{\mathcal{G}} = \{\overline{g} : g \in \mathcal{G}\}$ is a group of transformations of $\mathcal{P}$ into itself.

5. For an invariant estimation problem $(\mathcal{P}, \mathcal{A}, L)$, show that $\tilde{\mathcal{G}} = \{\tilde{g} : g \in \mathcal{G}\}$ is a group of transformations of $\mathcal{A}$ into itself.

6. Let $\mathcal{P} = \{F_\theta : \theta \in \Theta\}$, where $\Theta = \mathbb{R}$ and F_θ is the distribution function of $X \sim \text{N}(\theta, 1)$, $\theta \in \Theta$. Let $\mathcal{A} = \Theta$ and consider estimating θ under the loss function

$$L(\theta, a) = W(|a - \theta|), \ a \in \mathcal{A}, \theta \in \Theta,$$

   where W is some non-negative function defined on $\mathbb{R}_+ = [0, \infty)$. Let $\chi = \mathbb{R}$.

   (a) Show that the estimation/decision problem is invariant under the additive group $\mathcal{G} = \{g_c : c \in \mathbb{R}\}$, where $g_c(x) = x + c, x \in \chi, c \in \mathbb{R}$;

(b) Show that a decision rule δ is invariant under $\mathcal{G}$ if, and only if, $\delta(A|x) = \delta(A+c|x+c), \forall A \in \mathcal{F}_{\mathcal{A}}, x \in \chi, c \in \mathbb{R}$, where $A + c = \{a + c : a \in A\}$ (or $Y_x + c \stackrel{d}{=} Y_{x+c}, \forall x \in \chi, c \in \mathbb{R}$, where, for each $x \in \chi$, Y_x is a random variable corresponding to probability measure $\delta(\cdot|x)$);

(c) Show that a non-randomized decision rule δ is invariant if, and only if, $\delta(x) = x + c, x \in \chi$, for some $c \in \mathbb{R}$;

(d) For any invariant decision rule δ, show that the risk function $R_\delta(\theta)$ is constant (does not depend on $\theta \in \Theta$).

(**Note:** The loss function $L(\theta, a) = (a-\theta)^2,\ a \in \mathcal{A}, \theta \in \Theta$, corresponding to the choice $W(x) = x^2, x \in \mathbb{R}_+$, is called the squared error loss function.)

7. Let $\mathcal{P} = \{F_\theta : \theta \in \Theta\}$, where $\Theta = (0,1)$ and F_θ is the distribution function of $X \sim \text{Bin}(n,\theta), \theta \in \Theta$; here n is a known positive integer. Let $\mathcal{A} = \Theta$, $\chi = \{0, 1, \ldots, n\}$ and consider the problem of estimating θ under loss function

$$L(\theta, a) = W(|a - \theta|),\ a \in \mathcal{A}, \theta \in \Theta,$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is some function.

(a) Find a suitable group of transformations $\mathcal{G}$ under which the problem of estimating θ is invariant;

(b) Show that a decision rule δ is invariant under $\mathcal{G}$ if, and only if, $\delta(A|x) = \delta(1-A|n-x), \forall A \in \mathcal{F}_{\mathcal{A}}, x \in \chi$, where $1 - A = \{1 - a : a \in A\}$ (or $1 - Y_x \stackrel{d}{=} Y_{n-x}, \forall x \in \chi$, where, for each $x \in \chi$, Y_x is a random variable corresponding to probability measure $\delta(\cdot|x)$);

(c) Show that a non-randomized decision rule δ is invariant if, and only if, $1 - \delta(x) = \delta(n-x), x \in \chi$;

(d) For any invariant decision rule δ, show that $R_\delta(\theta) = R_\delta(1-\theta), \forall\theta \in \Theta$;

(e) Does $\overline{\mathcal{G}}$ acts transitively on $\mathcal{P}$?

8. Let $\mathcal{P} = \{F_\theta : \theta \in \Theta\}$, where $\Theta = \mathbb{R}_{++} = (0,\infty)$ and $F_\theta(x) = F(\frac{x}{\theta}),\ x \in \chi = \mathbb{R}_+ = [0,\infty), \theta \in \mathbb{R}_{++}$, for some distribution function F. Consider the problem of estimating θ under the loss function

$$L(\theta, a) = W(|\frac{a}{\theta} - 1|),\ a \in \mathcal{A} = \Theta, \theta \in \Theta,$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is some function.

(a) Show that the estimation/decision problem is invariant under the scale (multiplicative) group $\mathcal{G} = \{g_c : c \in \mathbb{R}\}$, where $g_c(x) = cx, x \in \chi, c \in \mathbb{R}_{++}$;

(b) Show that a decision rule δ is invariant under $\mathcal{G}$ if, and only if, $\delta(A|x) = \delta(cA|cx), \forall A \in \mathcal{F}_{\mathcal{A}}, x \in \chi, c \in \mathbb{R}_{++}$, where $cA = \{ca : a \in A\}$ (or $cY_x \stackrel{d}{=} Y_{cx}, \forall x \in \chi, c \in \mathbb{R}_+$, where, for each $x \in \chi$, Y_x is a random variable corresponding to probability measure $\delta(\cdot|x)$);

(c) Show that a non-randomized decision rule δ is invariant if, and only if, $\delta(x) = cx, x \in \chi$, for some $c \in \mathbb{R}$;

(d) For any invariant decision rule δ, show that the risk function $R_\delta(\theta)$ is constant (does not depend on $\theta \in \Theta$).

(**Note:** The loss function $L(\theta, a) = (\frac{a}{\theta} - 1)^2,\ a \in \mathcal{A}, \theta \in \Theta$, corresponding to the choice $W(x) = x^2, x \in \mathbb{R}_+$, is called the scaled squared error loss function.)

9. Let $\mathcal{P} = \{F_\theta : \theta \in \Theta\}$, where $\Theta = \mathbb{R}$ and $F_\theta(x) = F(x - \theta),\ x \in \chi = \mathbb{R}, \theta \in \Theta$, for some distribution function F. Consider the problem of estimating θ under the loss function

$$L(\theta, a) = W(|a - \theta|),\ a \in \mathcal{A} = \Theta, \theta \in \Theta,$$

where $W : \mathbb{R}_+ \to \mathbb{R}_+$ is some function.

(a) Show that the estimation/decision problem is invariant under the additive group $\mathcal{G} = \{g_c : c \in \mathbb{R}\}$, where $g_c(x) = x + c, x \in \chi, c \in \mathbb{R}$;

(b) Show that a decision rule δ is invariant under $\mathcal{G}$ if, and only if, $\delta(A|x) = \delta(A + c|x + c), \forall A \in \mathcal{F}_{\mathcal{A}}, x \in \chi, c \in \mathbb{R}$, where $A + c = \{a + c : a \in A\}$ (or $Y_x + c \stackrel{d}{=} Y_{x+c}, \forall x \in \chi, c \in \mathbb{R}$, where, for each $x \in \chi$, Y_x is a random variable corresponding to probability measure $\delta(\cdot|x)$);

(c) Show that a non-randomized decision rule δ is invariant if, and only if, $\delta(x) = x + c, x \in \chi$, for some $c \in \mathbb{R}$;

(d) For any invariant decision rule δ, show that the risk function $R_\delta(\theta)$ is constant (does not depend on $\theta \in \Theta$).