MTH 515a: Inference-II
Assignment No. 2: Location and Scale Invariant Estimation

1. Let $X_1, \ldots, X_n$ be a random sample from $\mathrm{N}(\theta, \sigma_0^2)$ distribution, where $\theta \in \mathbb{R} = \Theta$ is unknown and σ_0 is a known positive constant. Consider estimation of θ under the loss function
$$L(\theta, a) = W(a-\theta),\ a \in \mathcal{A} = \Theta, \theta \in \Theta,$$
where $W : \mathbb{R} \to \mathbb{R}$ is some convex function with $W(t) = W(-t), \forall t \in \mathbb{R}$. Show that $\delta_0(\underline{X}) = \overline{X}$ is the MRIE under additive group of transformations.

2. Let $X_1, \ldots, X_n$ be a random sample from a population with Lebesgue p.d.f.
$$f_\mu(x) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{1}{2}(x-\theta)^2}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta \end{cases},$$
where $\theta \in \mathbb{R} = \Theta$ is unknown. Find the MRIE of θ under the additive group of transformations and the squared error loss function.

3. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{E}(\theta, \sigma_0)$ random variables with common p.d.f.
$$f_\theta(x) = \begin{cases} \frac{1}{\sigma_0} e^{-\frac{x-\theta}{\sigma_0}}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta \end{cases},$$
where $\theta \in \mathbb{R} = \Theta$ is unknown and $\sigma_0 > 0$ is known. Consider estimation of θ under the loss function
$$L(\theta, a) = I_{(t,\infty)}(|\theta - a|), \theta, a \in \Theta,$$
where $t > 0$ is a given constant. Find the MRIE of θ under the additive group of transformations.

4. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{U}(\theta - \frac{1}{2}, \theta + \frac{1}{2})$ random variables, where $\theta \in \mathbb{R} = \Theta$ is unknown. Find the MRIE of θ under the additive group of transformations and the loss function $L(\theta, a) = W(a - \theta),\ a, \theta \in \Theta$, where $W(\cdot)$ is convex and even.

5. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{DE}(\theta, \sigma_0)$ (Double Exponential) random variables with common p.d.f.
$$f_\theta(x) = \frac{1}{\sigma_0} e^{-\frac{|x-\theta|}{\sigma_0}},\ -\infty < x < \infty,$$
where $\theta \in \mathbb{R} = \Theta$ is unknown and $\sigma_0 > 0$ is known. Under the squared error loss function and additive group of transformations, find the MRIE of θ.

6. Let $X_1, \ldots, X_n$ be i.i.d. $\mathrm{N}(\theta, \sigma_0^2)$ random variables, where $\theta \in \mathbb{R} = \Theta$ is unknown and $\sigma_0 > 0$ is known. Let the loss function be
$$L(\theta, a) = \begin{cases} \alpha(a - \theta), & \text{if } \theta < a \\ \beta(\theta - a), & \text{if } \theta \geq a \end{cases},$$
where α and β are positive constants. Find the MRIE of θ under the additive group of transformations.

7. Let δ_0 be a location invariant estimator of θ. Under the squared error loss function show that δ_0 is MRIE iff δ_0 is unbiased and $E_\theta(\delta_0(\underline{X})U(\underline{X})) = 0, \forall\theta \in \mathbb{R} = \Theta$, for any function $U(\cdot)$ satisfying $U(x_1 + c, \ldots, x_n + c) = U(x_1, \ldots, x_n)$, $\forall c \in \mathbb{R}, \underline{x} \in \mathbb{R}^n$, $\text{Var}_\theta(U) < \infty$ and $E_\theta(U(\underline{X})) = 0, \forall\theta \in \Theta$.

8. Let $X_1, \ldots, X_n$ be a random sample from $\text{N}(0, \theta^2)$ distribution, where $\theta \in \Theta = \mathbb{R}_{++}$ is unknown. Consider estimation of $\theta^r, r = 1, 2$, under the scaled squared error loss function $L(\theta, a) = (\frac{a}{\theta^r} - 1)^2,\ a, \theta \in \Theta$. Find the MRIE of θ under the multiplicative group of transformation.

9. Let $X_1, \ldots, X_n$ be i.i.d. $\text{E}(0, \theta)$ random variables with unknown scale parameter $\theta \in \mathbb{R}_{++} = (0, \infty)0$. Consider the scale goup of transformations.

   (a) Find the MRIE of θ under the loss function $L(\theta, a) = |\frac{a}{\theta} - 1|,\ \theta > 0, a > 0$;

   (b) Find the MRIE of θ under the loss function $L(\theta, a) = (\frac{a}{\theta} - 1)^2,\ \theta > 0, a > 0$;

   (c) Find the MRIE of θ^2 under the loss function $L(\theta, a) = (\frac{a}{\theta^2} - 1)^2,\ \theta > 0, a > 0$.

10. Let $X_1, \ldots, X_n$ be a random sample from $\text{U}(0, \theta)$ distribution with unknown $\theta \in \Theta = \mathbb{R}_{++}$. Find the MRIE of θ under the multiplicative group of transformations and the scaled absolute error loss function $L(\theta, a) = |\frac{a}{\theta} - 1|,\ a, \theta \in \Theta$.

11. Let $X_1, \ldots, X_n$ be a random sample from the Pareto distribution with the Lebesgue p.d.f.

$$f_\theta(x) = \begin{cases} \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, & \text{if } x \geq \theta \\ 0, & \text{if } x < \theta \end{cases},$$

where $\theta \in \mathbb{R}_{++}$ is unknown and $\alpha > 2$ is known. Find the MRIE of θ under the multiplicative group of transformations and the scaled squared error loss function $L(\theta, a) = (\frac{a}{\theta} - 1)^2,\ \theta, a \in \Theta$.

12. Let $X_1, \ldots, X_m$ be a random sample from $\text{N}(\theta_1, 1)$ distribution and let $Y_1, \ldots, Y_n$ be a random sample from $\text{N}(\theta_2, 1)$ distribution, where the two sample are mutually independent and $\underline{\theta} = (\theta_1, \theta_2) \in \mathbb{R}^2 = \Theta$ is unknown. Consider the problem of estimating $\eta = \theta_1 - \theta_2$ under the loss function $L(\underline{\theta}, a) = W(a - \eta),\ a \in \mathbb{R} = \mathcal{A},\ \underline{\theta} \in \Theta$, where $W(t)$ is, even, convex and non-monotone function. Find a suitable group of transformations under which the given Estimation/decision problem is invariant. Also find the best invariant decision rule.

13. Let $X_1, \ldots, X_m$ be a random sample from $\text{E}(0, \theta_1)$ distribution (exponential distribution with mean θ_1) and let $Y_1, \ldots, Y_n$ be a random sample from $\text{E}(0, \theta_2)$ distribution, where the two sample are mutually independent and $\underline{\theta} = (\theta_1, \theta_2) \in \mathbb{R}_+^2 = (0, \infty) \times (0, \infty) = \Theta$ is unknown. Consider the problem of estimating $\eta = \frac{\theta_1}{\theta_2}$ under the loss function $L(\underline{\theta}, a) = (\frac{a}{\theta} - 1)^2,\ a \in \mathbb{R}_+ = \mathcal{A},\ \underline{\theta} \in \Theta$. Find a suitable group of transformations under which the given estimation problem is invariant. Also find the best invariant estimator.