10

Chaos, Nonlinearity, Complexity: A Unified Perspective

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Summary. In this paper we employ the topological-multifunctional mathematical language and techniques of non-injective illposedness developed in [30] to formulate a notion of *ChaNoXity* — Chaos, Nonlinearity, Complexity — in describing the specifically nonlinear dynamical evolutionary processes of Nature. Non-bijective illposedness is the natural mode of expression for chanoxity that aims to focus on the nonlinear interactions generating dynamical evolution of real irreversible processes. The basic dynamics is considered to take place in a matter-negmatter (regulating matter, defined below) *kitchen space* $X \times \mathfrak{X}$ of Nature that is inaccessible to both the matter (X) and negmatter ($\mathfrak{X}$) components. These component spaces are distinguished by opposing evolutionary directional arrows and satisfy the defining property

$$(\forall A \subseteq X,\ \exists \mathfrak{A} \subseteq \mathfrak{X})\,\text{s.t.}\,(A \cup \mathfrak{A} = \emptyset).$$

Dynamical equilibrium is considered to be represented by such *competitively collaborating* homeostatic states of the matter-negmatter constituents of Nature.

> *The reductionist approach to science today remains largely the dominant model. It fosters the detailed study of limited domains in individual subdisciplines within the vast tree of science. However, over the past 30 years or so, an alternative conceptual picture has emerged for the study of large areas of science which have been found to share many common conceptual features, regardless of the subdiscipline, be it physics, chemistry or biology. Self-organization and complexity are the watchwords for this new way of thinking about the collective behaviour of many basic but interacting units. In colloquial terms, we are talking about systems in which 'the whole is greater that the sum of parts'.*
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> *Complexity is the study of the behaviour of large collection of such simple, interacting units, endowed with the potential to evolve with time. The complex phenomena that emerge from the dynamical behaviour of these interacting units are referred to as self-organizing. More technically, self-organization is the spontaneous emergence of non-equilibrium structural reorganizations on a macroscopic level, due to the collective interactions*

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between a large number of (usually simple) microscopic objects. Such structural organizations may be of a spatial, temporal or spatio-temporal nature, and is thus an emergent property.

For self-organization to arise, a system needs to exhibit two properties: it must be both dissipative and nonlinear. Self-organization and complexity are essential scientific concepts for understanding integrated systems whether in physics, biology or engineering ··· with a much more 'holistic', yet equally rigorous, scientific perspective compared with the reductionist methods, and so provide new insights into many of the more intellectually challenging concepts, including the large-scale structure of the Universe, the origin and evolution of life on Earth (and more widely in the cosmos), consciousness, intelligence and language.

There is, therefore, a general and conceptual framework for the description of self-organizing phenomena, of a theoretical and essentially mathematical nature. This more or less boils down to the theory of nonlinear dissipative dynamical systems.

Coveney [8]

10.1 Introduction

A dissipative structure is an open, out-of-equilibrium, unstable system that maintains its form and structure by interacting with its environment through the exchange of energy, matter, and entropy, thereby inducing spontaneous evolutionary convergence to a complex, and possibly chaotic, equilibrated state. These systems maintain or increase their organization through exergy destruction in a locally reduced entropy state by increasing the entropy of the "global" environment of which they are a part. This paper applies the mathematical language and techniques of non-bijective, and in particular non-injective, ill-posedness and multifunctions introduced and developed in [30] to formulate an integrated approach to dissipative systems involving chaos, nonlinearity and complexity (ChaNoXity), where a complex system is understood to imply

- ▶ an assembly of many *interdependent parts*
- ▶ interacting with each other through *competitive nonlinear collaboration*
- ▶ leading to *self-organized, emergent* behaviour.[1]

[1] Competitive collaboration — as opposed to reductionism — in the context of this characterization is to be understood as follows: The interdependent parts retain their individual identities, with each contributing to the whole in its own characteristic fashion *within a framework of dynamically emerging global properties of the whole.* A comparison with reductionism as summarized in Fig. 10.10*c*, shows that although the properties of the whole are generated by the parts, the individual units acting independently on their own cannot account for the emergent global behaviour of the total.

We will show how each of these defining characteristics of complexity can be described and structured within the mathematical framework of our multi-functional graphical convergence of a net of functions (f_α). In this programme, convergence in topological spaces continues to be our principal tool, and the particular topologies of significance that emerge are the topology of saturated sets and the exclusion topology. We will demonstrate that a complex system can be described as an association of independent expert groups, each entrusted with a specific specialized task by a top-level coordinating command, that consolidates and regulates the inputs received from its different constituent units each working independently of the others within the global framework of the coordinating authority, by harmonizing and combining them into an emerging whole; thus the complexity of a system, broadly speaking, is the amount of information needed to describe it. In this task, and depending on the evolving complexity of the dynamics, the coordinating unit delegates its authority to subordinate units that report back to it the data collected at its own level of authority.

Recall that

(i) a multifunction — which constitutes one of the foundational notions of our work — and the non-injective function are related by

$$\begin{aligned} f \text{ is a non-injective function} &\Longleftrightarrow f^- \text{ is a multifunction} \qquad (10.1.1)\\ f \text{ is a multifunction} &\Longleftrightarrow f^- \text{ is a non-injective function.} \end{aligned}$$

and

(ii) the neighbourhood of a point $x \in (X, \mathcal{U})$ — which is a generalization of the familiar notion of distances of metric spaces — is a nonempty subset N of X containing an open set $U \in \mathcal{U}$; thus $N \subseteq X$ is a neighbourhood of x iff $x \in U \subseteq N$ for some open set U of X. The collection of all neighbourhoods of x

$$\mathcal{N}_x \stackrel{\text{def}}{=} \{N \subseteq X : x \in U \subseteq N \text{ for some } U \in \mathcal{U}\} \qquad (10.1.2)$$

is the neighbourhood system at x, and the subcollection U of $\mathcal{U}$ used in this expression constitutes a neighbourhood (local) base or basic neighbourhood system, at x. The properties

(N1) x belongs to every member N of $\mathcal{N}_x$,

(N2) The intersection of any two neighbourhoods of x is another neighbourhood of x: $N, M \in \mathcal{N}_x \Rightarrow N \cap M \in \mathcal{N}_x$,

(N3) Every superset of any neighbourhood of x is a neighbourhood of x: $(M \in \mathcal{N}_x) \wedge (M \subseteq N) \Rightarrow N \in \mathcal{N}_x$

characterize $\mathcal{N}_x$ completely and imply that a subset $G \subseteq (X, \mathcal{U})$ is open iff it is a neighbourhood of each of its points. Accordingly if $\mathcal{N}_x$ is an arbitrary collection of subsets of X associated with each $x \in X$ satisfying (N1) – (N3), then the special class of neighbourhoods G

$$\mathcal{U} = \{G \in \mathcal{N}_x : x \in B \subseteq G \text{ for some } B \in \mathcal{N}_x \text{ and each } x \in G\} \qquad (10.1.3)$$

defines a unique topology on X containing a basic neighbourhood B at each of its points x for which the neighbourhood system is the prescribed collection $\mathcal{N}_x$. Among the three properties (N1) – (N3), the first two now re-expressed as

(NB1) x belongs to each member B of $\mathcal{B}_x$.

(NB2) The intersection of any two members of $\mathcal{B}_x$ contains another member of $\mathcal{B}_x$: $B_1, B_2 \in \mathcal{B}_x \Rightarrow (\exists B \in \mathcal{B}_x : B \subseteq B_1 \cap B_2)$.

are fundamental in the sense that the resulting subcollection $\mathcal{B}_x$ of $\mathcal{N}_x$ generates the full system by appealing to (N3). This basic neighbourhood system, or *local base*, at x in $(X,\mathcal{U})$ satisfies

$$\mathcal{B}_x \stackrel{\text{def}}{=} \{B \in \mathcal{N}_x : x \in B \subseteq N \text{ for each } N \in \mathcal{N}_x\} \tag{10.1.4}$$

which reciprocally determines the full neighbourhood system

$$\mathcal{N}_x = \{N \subseteq X : x \in B \subseteq N \text{ for some } B \in \mathcal{B}_x\} \tag{10.1.5}$$

as all the supersets of these basic elements.

The topology of saturated sets is defined in terms of equivalence classes $[x]_\sim = \{y \in X : y \sim x \in X\}$ generated by a relation $\sim$ on a set X; the neighbourhood system $\mathcal{N}_x$ of x in this topology consists of all supersets of the equivalence class $[x]_\sim \in X/\sim$. In the x-exclusion topology of all subsets of X that exclude x (plus X, of course), the neighbourhood system of x is just $\{X\}$. While the first topology provides, as in [30], the motive force for an evolutionary direction in time, the second will define a complementary *negative* space $\mathfrak{X}$ of (associated with, generated by) X, with an oppositely directed evolutionary arrow. With *dynamic equilibrium* representing a state of *homeostasis*[2] between the associated opposing motives of evolution, *equilibrium* will be taken to mark the end of a directional evolutionary process represented by convergence of the associated sequence to an adherence set.

[2] *Homeostasis* (Greek, *homoio-*: same, similar; *stasis*: a condition of balance among various forces, literally means "resistance to change") is the property of an open system to maintain its structure and functions by means of a multiplicity of dynamical equilibria rigorously controlled by interdependent regulation mechanisms. Homeostatic systems by opposing changes to maintain internal balance — with failure to do so eventually leading to its death and destruction — represent the action of negative feedbacks in sustaining a constant state of equilibrium by adjusting its physiological processes.

Examples: (a) Homeostasis is the fundamental defining character of a healthy living organism that allows it to function more efficiently by maintaining its internal environment within acceptable limits in competitive collaboration with its environment: the internal processes are regulated according to need. With respect to a parameter, an organism may maintain it at a constant level regardless of the environment, while others can allow the environment to determine its parameter through behavioral adaptations. It is the second type that is relevant for homeostasy. (b) The gravitational collapse of a cloud of interstellar matter raises its temperature until the nuclear fuel at the center ignites halting the collapse. The

Let $f : X \to Y$ be a function and $f^- : Y \multimap X$ its multi-inverse: hence $ff^-f = f$ and $f^-ff^- = f^-$ although $f^-f \neq \mathbf{1}_X$ and $ff^- \neq \mathbf{1}_Y$ necessarily. Some useful identities for subsets $A \subseteq X$ and $B \subseteq Y$ are shown in Table 10.1, where the complement of a subset $A \subseteq X$ is denoted by $A^c = \{x : (x \in X - A) \wedge (x \notin A)\}$. Let the *f-saturation* of A and the *f-component* of B on the image of f

$$\begin{aligned}\mathcal{S}_f(A) &= f^-f(A)\\ \mathcal{C}_f(B) &= ff^-(B) = B\bigcap f(X)\end{aligned}$$

define generalizations of injective and surjective mappings in the sense that any f behaves one-one and onto on its saturated and component sets respectively; in particular f is injective iff $\mathcal{S}_f(A) = A$ for all subsets $A \subseteq X$ and surjective iff $B = \mathcal{C}_f(B)$ for all $B \subseteq Y$. It is possible therefore to replace each of the relevant assertions of Table 10.1 with the more direct injectivity and surjectivity conditions on f. Indeed

$$\begin{aligned}f(x) = y &\Longrightarrow f(f^-f(x)) = y = ff^-(y)\\ &\Longrightarrow f(\mathcal{S}_f(x)) = \mathcal{C}_f(y) &\quad (10.1.6a)\\ x = f^-(y) &\Longrightarrow f^-f(x) = [x] = f^-(ff^-(y))\\ &\Longrightarrow \mathcal{S}_f(x) = f^-(\mathcal{C}_f(y)) &\quad (10.1.6b)\end{aligned}$$

demonstrate the bijectivity of $f : \mathcal{S}_f(x) \to \mathcal{C}_f(y)$; hence in the bijective inverse notation the corresponding functional equation takes the form

$$f(\mathcal{S}_f(A)) = \mathcal{C}_f(B) \Longleftrightarrow \mathcal{S}_f(A) = f^{-1}(\mathcal{C}_f(B)). \qquad (10.1.7)$$

This significant generalization of bijectivity of functions is noteworthy because our notion of chaos and complexity is based on ill-posedness of non-bijective functional equations, and one of the principal objectives of this work is to demonstrate that *the natural law of entropy increase is caused by the urge of the system $f(x) = y$ to impose an effective state of uniformity throughout X by the generation of saturated and component open sets.*

All statements of the first column of the table for saturated sets $A = \mathcal{S}_f(A)$ apply to the quotient map q; observe that $q(A^c) = (q(A))^c$. Moreover combining the respective entries of both the columns, it is easy to verify the following results for the saturation map $\mathcal{S}_f$ on saturated sets $A = \mathcal{S}_f(A)$.

(a) $\mathcal{S}_f(\cup A_i) = \cup \mathcal{S}_f(A_i)$: The union of saturated sets is saturated.

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consequent thermal pressure gradient of expansion inhibits the dominant gravitational force of compression resulting in the birth of a star that is a state of dynamical equilibrium between these opposing forces.

Homeostasis, as the ability or tendency of an organism or cell to maintain internal equilibrium by adjusting its physiological processes, will be used in this work to denote a state of dynamical equilibrium among various forces acting on the system.

	$f\colon X \to Y$	$f^- : Y \multimap X$
1	$A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$ $\Leftarrow$ iff $A = \mathcal{S}_f(A)$	$B_1 \subseteq B_2 \Rightarrow f^-(B_1) \subseteq f^-(B_2)$ $\Leftarrow$ iff $B = \mathcal{C}_f(B)$
2	$f(A) \subseteq B \Leftrightarrow A \subseteq f^-(B)$ $B \subseteq f(A) \Rightarrow f^-(B) \subseteq A$ iff $A = \mathcal{S}_f(A)$	$f(A) \subseteq B \Leftrightarrow A \subseteq f^-(B)$ $B \subseteq f(A) \Leftarrow f^-(B) \subseteq A$ iff $B = \mathcal{C}_f(B)$
3	$A = \emptyset \Leftrightarrow f(A) = \emptyset$	$f^-(\emptyset) = \emptyset$ $f^-(B) = \emptyset \Rightarrow B = \emptyset$ iff $B = \mathcal{C}_f(B)$
4	$f(A_1) \cap f(A_2) = \emptyset \Rightarrow$ $A_1 \cap A_2 = \emptyset \Leftarrow$ iff $A = \mathcal{S}_f(A)$	$f^-(B_1) \cap f^-(B_2) = \emptyset \Leftarrow$ $B_1 \cap B_2 = \emptyset \Rightarrow$ iff $B = \mathcal{C}_f(B)$
5	$f(\cup_\alpha A_\alpha) = \cup_\alpha f(A_\alpha)$	$f^-(\cup_\alpha B_\alpha) = \cup_\alpha f^-(B_\alpha)$
6	$f(\cap_\alpha A_\alpha) \subseteq \cap_\alpha f(A_\alpha)$, "=" iff $A = \mathcal{S}_f(A)$	$f^-(\cap_\alpha B_\alpha) = \cap_\alpha f^-(B_\alpha)$
7	$f(A^c) = (f(A))^c \cap f(X)$ iff $A = \mathcal{S}_f(A)$	$f^-(B^c) = ((f^-(B))^c$

Table 10.1. The role of saturated and component sets in a function and its inverse; here $A = \mathcal{S}_f(A)$ and $B = \mathcal{C}_f(B)$ are to be understood to hold for all subsets $A \subseteq X$ and $B \subseteq Y$, with the conditions ensuring that f is in fact injective and surjective respectively. Unlike f, f^- preserves the basic set operations in the sense of 5, 6, and 7. This makes f^- rather than f the ideal instrument for describing topological and measure theoretic properties like continuity and measurability of functions.

(b) $\mathcal{S}_f(\cap A_i) = \cap \mathcal{S}_f(A_i)$: The intersection of saturated sets is saturated.

(c) $X - \mathcal{S}_f(A) = \mathcal{S}_f(X - A)$: The complement of a saturated set is saturated.

(d) $A_1 \subseteq A_2 \Rightarrow \mathcal{S}_f(A_1) \subseteq \mathcal{S}_f(A_2)$

(e) $\mathcal{S}_f(\cap A_i) = \emptyset \Rightarrow \cap A_i = \emptyset$.

While properties (a) and (b) lead to the topology of saturated sets, the third makes it a complemented topology when the (closed) complement of an open set is also an open set. In this topology there are no boundaries between sets which are isolated in as far as a sequence eventually in one of them converging to points in the other is concerned.

As the guiding incentive for this work is an understanding of the precise role of irreversibility and nonlinearity in the dynamical evolution of (irreversible) real processes, we will propose an index of nonlinear irreversibility in the kitchen space $X \times \mathfrak{X}$ of Nature, wherein all the evolutionary dynamics are postulated to take place. The physical world X is only a projection of this multifaceted kitchen that is distinguished in having a complementary "negative" component $\mathfrak{X}$ interacting with X to generate the dynamical reality perceived in the later. This nonlinearity index, together with the dynamical synthesis of opposites between opposing directional arrows associated with X and its complementing negworld $\mathfrak{X}$, suggests a description of time's arrow that is specifically nonlinear with chaos and complexity being the prime manifestations of strongly nonlinear systems.

The entropy produced within a system due to irreversibilities within it [15] are generated by nonlinear dynamical interactions between the system and its negworld, and the objective of this paper is to clearly define this interaction and focus on its relevance in the dynamical evolution of Nature.

10.2 ChaNoXity: Chaos, Nonlinearity, Complexity

10.2.1 Entropy, Irreversibility, and Nonlinearity

Here we provide a summary of the "modern" approach to entropy — which is a measure of the molecular disorder of the system, generated as it does work: entropy relates the multiplicity associated with a state so that if one state can be achieved in more ways than another then it is more probable with a larger entropy — due to De Donder [15], incorporating explicitly irreversibility into the formalism of the Second Law of Thermodynamics[3] thereby making it unnecessary to consider ideal, non-physical, reversible processes for computing (changes in) entropy. This follows from the original Clausius inequality

$$dS \geq \frac{dQ}{T}$$

written as

$$dS = d\mathcal{S} + d\mathbb{S}, \tag{10.2.1}$$

where $d\mathcal{S}$ is the change in the entropy of the system due to heat exchanged by it with its exterior and $d\mathbb{S}$, the "uncompensated heat" of Clausius, represents the entropy generated within the system from real irreversible processes occurring in it. Although $d\mathcal{S} = dQ/T$ can be either positive or negative, $d\mathbb{S}$ must always be positive due to the irreversibilities produced in the system, implying that although entropy can either increase or decrease through energy transport across its boundary, the system can only generate and never

[3] The Second Law of Thermodynamics for non-equilibrium processes essentially requires the *exergy* (the maximum useful work that can be obtained from a system at a given state in a specified environment, Eq. 10.2.2) of *isolated* systems to be continuously degraded by diabatic *irreversible* processes that drive systems towards equilibrium by generating entropy, eventually leading to a dead equilibrium state of maximum entropy. Statistically, the equilibrium state is interpreted to represent the most probable state. For a closed system, entropy gives a quantitative measure of the amount of thermal energy not available to do work, that is of the amount of unavailable energy.

destroy entropy. In an isolated system since the energy exchange is zero, the entropy will continue to increase due to effective irreversibilities, and reach the maximum possible value leading to a steady state of dynamical equilibrium in which all (irreversible) processes must cease. When the system exchanges entropy with its surroundings, it is driven to an out-of-equilibrium state and entropy producing irreversibilities begin to operate leading to a more probable disordered state. The entropy flowing out of an *adiabatic* system must, by Eq. (10.2.1), be larger than that flowing into it with the difference being equal to the amount generated by the irreversibilities. The basic point, as will be elaborated in the following, is that dissipative systems in communion with its exterior utilize the exergy (or thermodynamic availability) to organize emerging structures within itself: for a system to be in a non-equilibrium steady state, $dS \leq 0$; hence $d\mathcal{S}$ must be negative of magnitude greater than or equal to $d\mathbb{S}$. The exergy

$$\mathcal{E} = (U - U_{\text{eq}}) + P_0(V - V_{\text{eq}}) - T_0(S - S_{\text{eq}}) - \sum_{j=1}^{J} \mu_{j,0}(N_j - N_{j,\text{eq}}) \quad (10.2.2)$$

of a system is a measure of its deviation from thermodynamic equilibrium with the environment, and represents the maximum capacity of energy to perform useful work as the system proceeds to equilibrium, with irreversibilities increasing its entropy at the expense of exergy; here $_{\text{eq}}$ marks the equilibrium state, and $_0$ represents the environment with which the system interacts.

In postulating the existence of an entropy function $S(U, V, N)$ of the extensive parameters U, V, and $\{N\}_{j=1}^{J}$ of the internal energy, volume, and mole numbers of the chemical constituents comprising a composite compound system that is defined for all equilibrium states, we follow Callen [4] in postulating that in the absence of internal constraints the extensive parameters assume such values that maximize S over all the constrained equilibrium states. The entropy of the composite system is additive over the constituent subsystems, and is continuous, differentiable, and increases monotonically with respect to the energy U. This last property implies that $S(U, V, N)$ can be inverted in $U(S, V, N)$; hence

$$\begin{aligned} dU(S, V, \{N_j\}) = \frac{\partial U(S, V, \{N_j\})}{\partial S}\, dS + \frac{\partial U(S, V, \{N_j\})}{\partial V}\, dV \\ + \sum_{j=1}^{J} \frac{\partial U(S, V, \{N_j\})}{\partial N_j}\, dN_j \end{aligned} \quad (10.2.3)$$

defines the *intensive parameters*

$$\frac{\partial U}{\partial S} \stackrel{\text{def}}{=} T(S, V, \{N_j\}_{j=1}^{J}), \qquad V,\ \{N_j\}\ \text{held const} \quad (10.2.4)$$

$$\frac{\partial U}{\partial V} \stackrel{\text{def}}{=} -P(S, V, \{N_j\}_{j=1}^{J}), \qquad S,\ \{N_j\}\ \text{held const} \quad (10.2.5)$$

$$\frac{\partial U}{\partial N_j} \stackrel{\text{def}}{=} \mu_j(S, V, \{N_j\}_{j=1}^J), \qquad S,\, V \text{ held const} \tag{10.2.6}$$

of absolute temperature T, pressure P, and chemical potential μ_j of the j^{th} component, from the macroscopic extensive ones. Inversion of (10.2.3) gives the differential *Gibbs entropy* definition

$$dS(U,V,\{N_j\}) = \frac{1}{T(U,V,\{N_j\})}\, dU + \frac{P(U,V,\{N_j\})}{T(U,V,\{N_j\})}\, dV - \sum_{j=1}^{J} \frac{\mu_j(U,V,\{N_j\})}{T(U,V,\{N_j\})}\, dN_j \tag{10.2.7}$$

that provides an equivalent correspondence of the partial derivatives

$$(\partial S/\partial U)_{V,N_j} = \frac{1}{T(U,V,\{N_j\})} \tag{10.2.8}$$

$$(\partial S/\partial V)_{U,N_j} = \frac{P(U,V,\{N_j\})}{T} \tag{10.2.9}$$

$$(\partial S/\partial N_j)_{U,V} = -\frac{\sum_{j=1}^{J} \mu(U,V,\{N_j\})}{T} \tag{10.2.10}$$

with the intensive variables of the system.

In the spirit of the Pffafian differential form, dependence of the intensive variables of the First Law

$$\begin{aligned} dU(S,V,\{N_j\}) &= dQ(S,V,\{N_j\}) + dW(S,V,\{N_j\}) + dM(S,V,\{N_j\}) \\ &= dQ(S,V,\{N_j\}) - P(S,V,\{N_j\})\, dV \\ &\quad + \sum_{j=1}^{J} \mu_j(S,V,\{N_j\})\, dN_j \end{aligned} \tag{10.2.11}$$

on the respective extensive macroscopic variables U, V, or N_j serves to decouple the (possibly nonlinear) bonds between them; this is necessary and sufficient for the resultant thermodynamics to be classified as *quasi-static* or *reversible*. These ideal states as pointed out in [4] are simply an ordered class of equilibrium states, neutral with respect to time-reversal and without any specific directional property, distinguished from natural physical processes of ordered *temporal successions* of equilibrium and non-equilibrium states: *a quasi-static reversible process is a directionless collection of elements of an ordered set.*[4] From the definition (10.2.4) of the absolute temperature T it follows that *under quasi-static conditions*,

[4] The most comprehensive view of *irreversibility* follows from the notion of a *time-(a)symmetric theory* that requires the (non)existence of a backward process $\mathcal{P}_r :=$

$$dQ \stackrel{\text{def}}{=} T(S)\,dS \tag{10.2.12}$$

reduces the heat transfer dQ to formally behave work-like, permitting (10.2.11) to be expressed in the *combined first and second law* form

$$dU(S, V, \{N_j\}) = T(S)\,dS - P(V)\,dV + \sum_{j=1}^{J} \mu(N_j)\,dN_j \tag{10.2.13a}$$

$$dS(U, V, \{N_j\}) = \frac{1}{T(U)}\,dU + \frac{P(V)}{T(U)}\,dV - \sum_{j=1}^{J} \frac{\mu(N_j)}{T(U)}\,dN_j \tag{10.2.13b}$$

which are just the integrable *quasi-static versions* of Eqs. (10.2.3) and (10.2.7). Note that the total energy input and the corresponding entropy transfer in this quasi-static case reduces to a simple sum of the constituent parts of the change. For non quasi-static real processes, this linear superposition of the solution into its individual components is not justified as the resulting Pfaffian equation solves as the arbitrary $U(S, V, \{N_j\}) = \text{const}$. For any natural non-cyclic real process therefore, the identification

$$dQ(S, V, \{N_j\}) \stackrel{\text{def}}{=} T(S, V, \{N_j\})\,dS \tag{10.2.14}$$

reduces (10.2.3) to the first law form (10.2.11) for real processes that no longer decomposes into individual, non-interacting component parts like its quasi-static counterpart (10.2.13a). Equation (10.2.14) is graphically expressed [15] in the spirit of (10.2.1) as

$$\begin{aligned} dS &= \frac{dQ(S, V, \{N_j\})}{T(S, V, \{N_j\})} \\ &= \frac{d\mathcal{Q}(S, V, \{N_j\})}{T(S, V, \{N_j\})} + \frac{d\mathscr{Q}(S, V, \{N_j\})}{T(S, V, \{N_j\})} \\ &= d\mathcal{S} + d\mathscr{S}, \end{aligned} \tag{10.2.15}$$

where the total entropy exchange is expressed as a sum of two parts: the first

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$\{r(-t) : -t_f \le t \le -t_i\}$ for every permissible forward process $\mathcal{P} := \{s(t) : t_i \le t \le t_f\}$ of the theory; here $r = Rs$ with $R^2 = \mathbf{1}$, is the time-reversal of state s. Although in contrast with mechanics thermodynamics has no equations of motion, the second law endows it with a time-asymmetric character and a thermodynamic process is irreversible iff its reverse $\mathcal{P}_r$ is not allowed by the theory, iff time-symmetry is broken in the sense that an irreversible process cannot be reversed without introducing some change in the surroundings, typically by work transforming to heat. Reversible processes are useful idealizations used to measure how well we are doing with real irreversible processes. Entropy change in the universe is a direct quantification of irreversibility indicating how far from ideal the system actually is: irreversibility is directly related to the lost opportunity of converting heat to work.

$$d\mathcal{S} = \frac{d\mathcal{Q}}{T} \gtrless 0 \tag{10.2.16}$$

may be positive, zero or negative depending on the specific nature of energy transfer $d\mathcal{Q}$ with the (infinite) exterior reservoir, but the second

$$d\mathbb{S} = \frac{d\mathbb{Q}}{T} \geq 0 \tag{10.2.17}$$

representing the entropy produced by irreversible nonlinear processes within the system is always positive. Equation (10.2.13*b*) for a composite body $C = A \cup B$ of two parts A and B, each interacting with its own infinite reservoir under the constraint $U = U_A + U_B$, $V = V_A + V_B$ and $N = N_A + N_B$, yields the Gibbs expression

$$\begin{aligned} d\mathcal{S}_C(U,V,N) = \left[\frac{1}{T_A}\,dU_A + \frac{1}{T_B}\,dU_B\right] + \left[\frac{p_A}{T_A}\,dV_A + \frac{p_B}{T_B}\,dV_B\right] \\ - \left[\frac{\mu_A}{T_A}\,dN_A + \frac{\mu_B}{T_B}\,dN_B\right] \end{aligned} \tag{10.2.18}$$

for the entropy exchanged by C in reaching a state of *static equilibrium* with its infinite environment; here T, P and μ are the parameters of the reservoirs that completely determine the internal state of C. This exchange of energy with the surroundings perturbs the system from its state of equilibrium and sets up internal irreversible nonlinear processes between the two subsystems, driving C towards a new state of *dynamic equilibrium* that can be represented ([12], [15]) in terms of flows of extensive quantities set up by forces generated by the intensive variables. Thus for a composite *dynamically* interacting *compound system* C consisting of two chambers A and B of volumes V_A and V_B filled with two nonidentical gases at distinct temperatures, pressures, and mole numbers, the entropy generated by nonlinear irreversible processes within the system on removal of the partition between them, can be expressed in Gibbs form as

$$\begin{aligned} d\mathbb{S}_C(U,V,N) = \left[\frac{1}{T_A} - \frac{1}{T_B}\right] dU_A + \left[\frac{p_A}{T_A} - \frac{p_B}{T_B}\right] dV_A - \left[\frac{\mu_A}{T_A} - \frac{\mu_B}{T_B}\right] dN_A \\ U = U_A + U_B,\ V = V_A + V_B,\ \text{and } N = N_A + N_B \text{ remain constants.} \end{aligned} \tag{10.2.19}$$

Each term on the right — a product of an intensive thermodynamic force and the corresponding extensive thermodynamic flow — contributing to the uncompensated heat [15] generated within the system from the nonlinear irreversible interactions between its subsystems, is responsible for the increase in entropy accompanying all natural processes leading to the eventual degradation of energy in the universe to a state of inert uniformity.

The interaction of two finite subsystems is to be compared with the static interaction between a finite system and an infinite reservoir. Contrasted with the later, for which the time evolution is unidirectional with the system unreservedly acquiring the properties of the reservoir which undergoes no perceptible changes resulting in the *static equilibrium* from *passive interaction* of the system with its reservoir, the system-system interaction is fundamentally different as it evolves bidirectionally such that the properties of the composite are not of either of the systems, but an average of the individual properties that defines an eventual state of *dynamic interactive equilibrium.* This distinction between passive and dynamical equilibria resulting respectively from the uni- and bi-directional interactions is clearly revealed in Eqs. (10.2.18) and (10.2.19), with bi-directionality of the later being displayed by the *difference form of the generalized forces.* Accordingly, subsystems A and B have two directional arrows imposed on them: the first due to the evolution of the system opposed by a reverse arrow arising from its interactive interaction with the other, see Fig. 10.4. Evolution requires all macroscopic extensive variables — and hence all the related microscopic intensive parameters — to be functions of time so that equilibrium, in the case of (10.2.19) for example, demands

$$\frac{d\mathbb{S}_C}{dt} = 0 \Longrightarrow \left(\frac{dU_A}{dt} = 0\right) \bigwedge \left(\frac{dV_A}{dt} = 0\right) \bigwedge \left(\frac{dN_A}{dt} = 0\right)$$
$$\Longleftarrow (T_A(t) = T_B(t)) \bigwedge (p_A(t) = p_B(t)) \bigwedge (\mu_A(t) = \mu_B(t)). \tag{10.2.20}$$

While we return to this topic subsequently using the tools of directed sets and convergence in topological spaces, for the present it suffices to note that for an emerging, self-organizing, complex evolving system far from stable equilibrium, the reductionist linear proportionality between cause and effect[5] that decouples the entropy change into two independent parts, one with the exterior and the other the consequent internal generation as given by (10.2.15), is open to question as these constitute a system of interdependent evolutionary interlinked processes, depending on each other for their sustenance and contribution to the whole. Thus, "life" forms in which $d\mathcal{S}$, arising from the energy exchanged as food and other sustaining modes with the exterior, depends on the capacity $d\mathbb{S}$ of the life to utilize these resources, which in turn is regulated by $d\mathcal{S}$. These interdependent, non-reductionist, contributions of constituent parts to the whole is a direct consequence of nonlinearity that effectively implies $f(\alpha x_1 + \beta x_2) \neq \alpha f(x_1) + \beta f(x_2)$ for the related processes. The other "non-life" example requires the change to be determined by such internal parameters as mass, specific heat and chemical concentration of the constituents parts. Thus, for example, in the adiabatic mixing of a hot and cold body A and B the equilibrium temperature, given in terms of the respective mole

[5] Which, we recall, allows breaking up of the system into its constituent parts, studying their micro-dynamics and putting them back together in a linear sum to generate the macro-dynamics, thereby presuming that the macroscopic behaviour of a system of a large number of interacting parts is directly proportional to the character of its microscopic constituents.

numbers N, specific heat c and temperature T, by

$$N_A c_A (T_A - T) = N_B c_B (T - T_B) \tag{10.2.21}$$

sets up a state of dynamical equilibrium in which the bi-directional evolutionary arrow prevents A from annihilating B with the equilibrium condition $T = T_A$, $P = P_A$, $\mu = \mu_A$. Putting the heat balance equation in the form

$$dQ_A + (-dQ_B) = 0, \quad dQ = N\,c\,dT$$

suggests that the heat transfer out of a body, considered as a negative real number, be treated as the additive inverse to the positive transfers into the system. This sets up a one-to-one correspondence between two opposing directional real process that evolves to a state of dynamic equilibrium.

The basic feature of this evolutionary thermodynamics — based entirely on (linear) differential calculus — is that it reduces the dynamics of Eqs. (10.2.3) and (10.2.7) to a separation of the governing macroscopic extensive variables, thereby raising the question of the validity of such decoupling of the motive forces of evolution in strongly nonlinear, self-organizing, complex dynamical systems of nature[6]. Such a separation of variables tacitly implies, as in the example considered above, that the total energy exchange taking place

[6] The following extracts from the remarkably explicit lecture MIT-CTP-3112 by Michel [2], delivered possibly in 2000/2001, are worth recalling . *Chaos is still not part of the American university's physics curriculum; most students get physics degrees without ever hearing about it. The most popular textbook in classical mechanics does not include chaos. Why is that? The answer is simple. Physicists did not have the time to learn chaos, because they were fascinated by something else. That something else was 20th century physics of relativity, quantum mechanics, and their myriad of consequences. Chaos was not only unfamiliar to them; it was slightly distasteful!*

In offering an explanation for this, Baranger argues that in discovering calculus, Newton and Leibnitz *provided the scientific world with the most powerful new tool since the discovery of numbers themselves. The idea of calculus is simplicity itself. Smoothness (of functions) is the key to the whole thing. There are functions that are not smooth* $\cdots$. The discovery of calculus led to that of analysis and *after many decades of unbroken success with analysis, theorists became imbued with the notion that all problems would eventually yield to it, given enough effort and enough computing power. If you go to the bottom of this belief you find the following.* Everything *can be reduced to little pieces, therefore everything can be known and understood, if we analyze it to a fine enough scale. The enormous success of calculus is in large part responsible for the decidedly reductionist attitude of most twentieth century science, the belief in absolute control arising from detailed knowledge.*

Nonetheless, *chaos is the anti-calculus revolution, it is the rediscovery that calculus does not have infinite power. Chaos is the collection of those mathematical truths that have nothing to do with calculus. Chaos theory solves a wide variety of scientific and engineering problems which do not respond to calculus.*

when the gases are allowed to mix completely is separable into independent parts arising from changes in temperature, volume, and diffusion mixing of the gases, *with none of them having any effect on the others.* Observing that the defining property of a complex system responsible for its "complexity" is the interdependence of its interacting parts leading to non-reductionism, this contrary implication of independence of the extensive parameters conflicts with the foundational tenets of chaos and complexity.

Nonetheless, it should be clear from the above considerations that a non-isolated, "non-equilibrium" system can maintain a steady state of low entropy not only by discarding its excess entropy to the surroundings, but more importantly by utilizing [15] a part of this generation by the nonlinearities within itself to enhance its own state of organization consistent with the irreversibilities. Thus when the heated earth at a high level of non-equilibrium instability radiates heat to the cooler atmosphere through evaporation, the earth-atmosphere system is not scorched to the earth's temperature but instead stabilizes itself by "attracting, as it were, a stream of negative entropy upon itself" [29], through condensation of the water vapour back to the earth that essentially opposes this attempt to move the earth-atmosphere system away from its stable equilibrium by acting a gradient dissipator of the temperature difference. As the temperature difference increases, so does the opposition making it more and more difficult for the system to be away from equilibrium. The Second Law of Thermodynamics for non-equilibrium systems — recall footnote 3 — can accordingly be reformulated [27] to require that as the system is forced away from thermodynamic equilibrium it utilizes every possible avenue in "sucking orderliness from its environment" [29], to counter the applied gradients, with its ability to oppose continued displacement increasing with the gradient itself. For such systems the Second Law becomes a law of continuity for the entropy transferred in and out of the system.

The objective of this paper is to propose an explicitly nonlinear, topological formulation of dynamical evolution from an integrated chanoxity perspective that focuses on nonlinearly generated self-organization, adaption, and emergence of systems far from thermodynamic equilibrium. In this perspective, the following observations of Bertuglia and Vaio [3] are worth noting.

Linear approximations become increasingly unacceptable the further away we get from a condition of stable equilibrium. The world of classical science has shown a great deal of interest in linear differential equations for a very simple reason: apart from some exceptions, these are the only equations of an order above the first that can be solved analytically. The simplicity of linearization and the success that it has at times enjoyed have imposed the perspective from which scientists observed reality, encouraging scientific investigation to concentrate on linearity in its description of dynamic processes. On one hand this led to the idea that the elements that can be treated with techniques of linear mathematics prevail over nonlinear ones, and on the other hand it ended up giving rise to the idea that linearity is *intrinsically "elegant" because it is expressed in simple, concise formulae, and that a linear model is aesthetically more "attractive" than a nonlinear one. The practice of considering linearity as elegant encouraged a sort of self-promotion and gave rise to a real scientific prejudice: mainly linear aspects were studied. The success that was at times undeniably achieved in this ambit increasingly convinced scholars that linearization was the right way forward for other phenomena that adapted badly to linearization.*

However, an arbitrary forced aesthetic sense led them to think (and at times still leads us to think) that finding an equation acknowledged as elegant was, in a certain sense, a guarantee that nature itself behaved in a way that adapted well to an abstract vision of such mathematics.

Linear systems cannot generate dynamics that is sensitive to initial conditions with non-repeating orbits that remain confined in a bounded region of space. This defining character of chaos can be generated only by nonlinear interactions leading to increasing unpredictability of the system's future with increasing time: nonlinearity produces unexpected outcomes, linearity does not. Newtonian classical mechanics is reductionist and the solution of the equations of motion are uniquely determined by the initial conditions for all times.

10.2.2 Maximal Noninjectivity is Chaos

Chaos was defined in [30] as representing maximal non-injective ill-posedness in the temporal evolution of a dynamical system and was based on the purely set theoretic arguments of Zorn's Lemma and Hausdorff Maximal Chain Theorem. It was, however, necessary to link this with topologies because evolutionary directions are naturally represented by adherence and convergence of the associated nets and filters, which require topologies for describing their eventual and frequenting behaviour. For this we found the topology of saturated sets generated by the increasingly non-injective evolving maps to provide the motivation for maximally non-injective, degenerate ill-posedness leading to the concept of the *ininality of topologies* generated by a function $f : X \to Y$ that is simultaneously image and preimage continuous. In this case, the topologies on the range $\mathcal{R}(f)$ and domain $\mathcal{D}(f)$ of f are locked with respect to each other as far as further temporal evolution of f is concerned by having the respective topologies defined as the *f-images in Y of f^--saturated open sets of X*. Thus Eqs. (10.1.6a, b), and (10.1.7) taken with the definitions[7]

[7] If $(f_\alpha : X \to (Y_\alpha, \mathcal{V}_\alpha))_{\alpha\in\mathbb{D}}$ is a family of functions into topological spaces $(Y_\alpha, \mathcal{V}_\alpha)$, then the topology generated by the subbasis $\{f_\alpha^-(V_\alpha) : V_\alpha \in \mathcal{V}_\alpha\}_{\alpha\in\mathbb{D}}$ is the *initial topology* of X induced by the family $(f_\alpha)_{\alpha\in\mathbb{D}}$. Reciprocally, if $(f_\alpha : (X_\alpha, \mathcal{U}_\alpha) \to Y)_{\alpha\in\mathbb{D}}$ is a family of functions from topological spaces $(X_\alpha, \mathcal{U}_\alpha)$, then the collection $\{G \subseteq Y : f_\alpha^-(G) \in \mathcal{U}_\alpha\}_{\alpha\in\mathbb{D}}$ is the *final topology* of Y of the family $(f_\alpha)_{\alpha\in\mathbb{D}}$. A topology that is both initial and final is *ininal.*

$$\mathrm{IT}\{e;\mathcal{V}\} \stackrel{\text{def}}{=} \{U \subseteq X : U = e^{-}(V),\ V \in \mathcal{V}\} \qquad (10.2.22a)$$

and

$$\mathrm{FT}\{\mathcal{U};q\} \stackrel{\text{def}}{=} \{V \subseteq Y : q^{-}(V) = U,\ U \in \mathcal{U}\} \qquad (10.2.22b)$$

of initial and final topologies — that denote the coarsest (smallest) and finest (largest) topologies in X and Y respectively making f continuous — implies for open sets $V \in \mathcal{V}$ of Y and $G \subseteq U \in \mathcal{U}$ of X satisfying $U = \mathcal{S}_q(G)$ so that q acts only on saturated open sets,

$$f(\mathcal{S}_f(A)) = \mathcal{C}_f(B) \begin{cases} \xrightarrow{\text{IT}} (\mathcal{S}_e(U) = U)\,(e(U) = \mathcal{C}_e(V)) \\ \xrightarrow{\text{FT}} (q(\mathcal{S}_q(G) = V)\,(V = \mathcal{C}_q(V)); \end{cases} \qquad (10.2.23)$$

see also column 2, row 1 of Table 10.1. As these equations show, preimage and image continuous functions need not be open functions: a preimage continuous function is open iff $e(U)$ is an open set in Y and an image continuous function is open iff the q-saturation of every open set of X is also an open set. The generation of new topologies on the domain and range of a function — which will generally be quite different from the original topologies the spaces might have possessed — by the evolving dynamics of increasingly nonlinear maps is a basic property of the evolutionary process that constitutes the motive for such dynamical changes. Putting implications (10.2.23) together yields

$$U,\ V \in \mathrm{IFT}\{\mathcal{U};f;\mathcal{V}\} \Longleftrightarrow (\mathcal{U} = f^{-}(\mathcal{V}))\,(f(\mathcal{U}) = \mathcal{V}) \qquad (10.2.24a)$$

that effectively renders both e and q open functions, and reduces to

$$U,\ V \in \mathrm{HOM}\{\mathcal{U};f;\mathcal{V}\} \Longleftrightarrow (\mathcal{U} = f^{-1}(\mathcal{V}))\,(f(\mathcal{U}) = \mathcal{V}) \qquad (10.2.24b)$$

for a bijection satisfying both $\mathcal{S}_f(A) = A,\ \forall A \subseteq X$ and $\mathcal{C}_f(B) = B,\ \forall B \subseteq Y$; observe that the only difference between Eqs. (10.2.24a) and (10.2.24b) is in the bijectivity of f.

There are two defining components, temporal and spatial, in any natural evolutionary processes. However, these are equivalent in the sense that both can be represented as pre-ordered sets with the additional directional property of a *directed set* $(\mathbb{D}, \preceq)$ which satisfies

(DS1) $\alpha \in \mathbb{D} \Rightarrow \alpha \preceq \alpha$ ($\preceq$ is reflexive)
(DS2) $\alpha, \beta, \gamma \in \mathbb{D}$ such that $(\alpha \preceq \beta \wedge \beta \preceq \gamma)$ implies $\alpha \preceq \gamma$ ($\preceq$ is transitive)
(DS3) For all $\alpha, \beta \in \mathbb{D}$, there exists a $\gamma \in \mathbb{D}$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$

with respect to the *direction* $\preceq$. While the first two properties are obvious and constitutes the preordering of $\mathbb{D}$, the third replaces antisymmetry of an order with the condition that every pair of elements of $\mathbb{D}$, whether ordered or not, always has a successor. This directional property of $\mathbb{D}$, that imparts to the static pre-order a sequential arrow by allowing it to choose a forward path between possible alternatives when non-comparable elements bifurcate at the arrow, will be used to model evolutionary processes in space and time.

Besides the obvious examples $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}$, or $\mathbb{Z}$ of totally ordered sets, more exotic instances of directed sets imparting directions to neighbourhood systems in X tailored to the specific needs of convergence theory are summarized in Table 10.2, where $\beta \in \mathbb{D}$ is the directional index. Although the neighbourhood

Directed set $\mathbb{D}$	Direction $\preceq$ induced by $\mathbb{D}$
${}_{\mathbb{D}}N = \{N : N \in \mathcal{N}_x\}$	$M \preceq N \Leftrightarrow N \subseteq M$
${}_{\mathbb{D}}N_t = \{(N,t) : (N \in \mathcal{N}_x)(t \in N)\}$	$(M,s) \preceq (N,t) \Leftrightarrow N \subseteq M$
${}_{\mathbb{D}}N_\beta = \{(N,\beta) : (N \in \mathcal{N}_x)(x_\beta \in N)\}$	$(M,\alpha) \leq (N,\beta) \Leftrightarrow (\alpha \preceq \beta) \wedge (N \subseteq M)$

Table 10.2. Natural directions of decreasing subsets in $(X,\mathcal{U})$ induced by some useful directed sets of convergence theory. Significant examples of directed sets that are only partially ordered are $(\mathcal{P}(X),\subseteq)$, $(\mathcal{P}(X),\supseteq)$; $(\mathcal{F}(X),\supseteq)$; $(\mathcal{N}_x,\subseteq)$, $(\mathcal{N}_x,\supseteq)$ for a set X, We take $\mathcal{N}_x$, suitably redefined if necessary, to be always a system of nested subsets of X.

system ${}_{\mathbb{D}}N$ at a point $x \in X$ with the *reverse-inclusion* direction $\preceq$ is the basic example of natural direction of the neighbourhood system $\mathcal{N}_x$ of x, the directed sets ${}_{\mathbb{D}}N_t$ and ${}_{\mathbb{D}}N_\beta$ are more useful in convergence theory because unlike the first, these do not require a simultaneous application of the Axiom of Choice to every $N \in \mathcal{N}_x$.

Chaos as manifest in the limiting adhering attractors is a direct consequence of the increasing nonlinearity of the map under increasing iterations and with the right conditions, appears to be the natural outcome of the characteristic difference between a function f and its multi-inverse f^-. Equivalence classes of fixed points stable and unstable, as generated by the saturation operator $\mathcal{S}_f = f^- f$, determine the ultimate behaviour of an evolving dynamical system, and since the eventual (as also frequent) nature of a filter or net is dictated by topology on the set, chaoticity on a set X leads to a reformulation of the open sets of X to equivalence classes generated by the evolving map f. In the limit of infinite iterational evolution in time resulting in the multifunction Φ, the generated open sets constitute a basis for a topology on $\mathcal{D}(f)$ and the basis for the topology of $\mathcal{R}(f)$ are the corresponding Φ-images of these equivalent classes. It follows that the motivation behind evolution leading to chaos is the drive toward a state of the dynamical system that supports ininality of the limit multi Φ[8]. In this case therefore, the open sets of

[8] For the logistic map $f_\lambda(x) = \lambda x(1-x)$ with chaos setting in at $\lambda = \lambda_* = 3.5699456$, this drive in ininality implies an evolution toward values of the spatial parameter $\lambda \geq \lambda_*$; this is taken to be a spatial parameter as it determines the degree of surjectivity of f_λ. Together with the temporal evolution in increasing noninjectivity for any λ, this comprises the full evolutionary dynamics of the

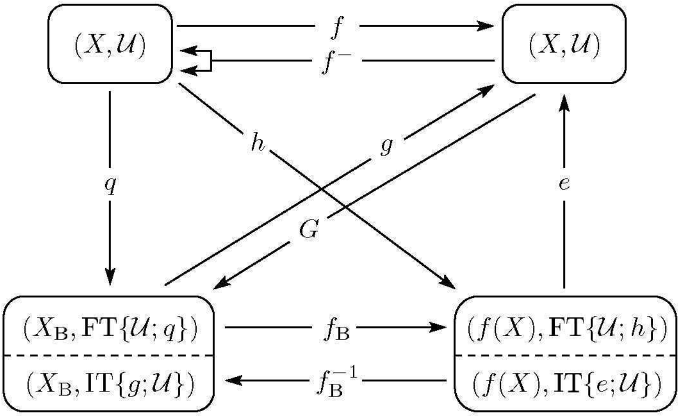

Fig. 10.1. Generation of a multifunctional inverse $x = f^{-}(y)$ of the functional equation $f(x) = y$ for $f : X \to X$; here $G : Y \to X_{\mathrm{B}}$ is a generalized inverse of f because $fGf = f$ and $GfG = G$ that follows from the commutativity of the diagrams. g and h are the injective and surjective restrictions of f; these will be topologically denoted by their generic notations e and q respectively.

the range $\mathcal{R}(f) \subseteq X$ are the multi images that graphical convergence generates at each of these inverse-stable fixed points. As readily verified from Fig. 10.1, X has two topologies imposed on it by the dynamics of f: the first of equivalence classes generated by the limit multi Φ in the domain of f and the second as Φ-images of these classes in the range of f. Hence while subdiagrams $X - (X_{\mathrm{B}}, \mathrm{FT}\{\mathcal{U}; q\}) - (f(X), \mathcal{U}_2)$ and $(X_{\mathrm{B}}, \mathcal{U}_1) - (f(X), \mathrm{IT}\{e; \mathcal{U}\}) - X$ apply to the final and initial topologies of X_{B} and $f(X)$ respectively, their superposition $X - (X_{\mathrm{B}}, \mathrm{FT}\{\mathcal{U}; q\}) - (f(X), \mathrm{IT}\{e; \mathcal{U}\}) - X$ *under the additional requirement of a homeomorphic* f_{B} leads to the conditions $\mathcal{U}_1 = \mathrm{IT}\{g; \mathcal{U}\}$ and $\mathcal{U}_2 = \mathrm{FT}\{\mathcal{U}; h\}$ that X_{B} and $f(X)$ must possess. For this to be possible,

$$\mathrm{FT}\{\mathcal{U}; q\} = \mathrm{IT}\{g; \mathcal{U}\}$$
$$\mathrm{IT}\{e; \mathcal{U}\} = \mathrm{FT}\{\mathcal{U}; h\}$$

requires the image continuous q and the preimage continuous e to be also be open functions which translates to the ininality of f on $(X, \mathcal{U})$, and hence for the topology of X to be simultaneously the direct and inverse images of itself under f; compare Eq. (10.2.24a). Since the map f and the topology $\mathcal{U}$ of X are already provided, this is interpreted to mean that the increasing nonlinear

logistic map. These two distinct dynamical mechanisms of increasing surjectivity and decreasing injectivity are not independent, however. Thus λ — which may be taken to represent the energy exchanges of all possible types that the system can have with the surroundings — determines the nature of the internal forward-backward stasis that leads to the eventual equilibrium of the system with its environment.

ill-posedness of the time-iterates of f is driven by ininality of the maximally "degenerate" ill-posed limit relation Φ on X^2. In this case Φ acts as a non-bijective open and continuous relation that forces the sequence of evolving functions (f^n) on X to eventually behave, by (10.1.7), homeomorphically on the saturated open sets of equivalence classes and their f^n-images in X. A homeomorphism, by establishing an equivalence between spaces $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ — algebraically through bijectivity and topologically by setting up a one-to-one correspondence between the respective open sets — renders the spaces as "essentially the same", with the non-bijective ininal function acting as an effective bijection $f\colon \mathcal{S}_f(A) \to \mathcal{C}_f(B)$ for all subsets A and B between X and Y. For a function defined *on* a space X, this means that, under ininality, the domain and range spaces are "effectively the same" thereby precluding any further interaction between them, which corresponds to a condition of equilibrium entropic death. We define the resulting ininal topology on X to be the *chaotic topology on X associated with f*. Neighbourhoods of points in this topology cannot be arbitrarily small as they consist of all members of the equivalence class to which any element belongs; hence a sequence converging to any of these elements necessarily converges to all, and the eventual objective of chaotic dynamics is to generate a topology in X (irrespective of the original $\mathcal{U}$) with respect to which elements of the space are grouped together in large equivalence classes such that if a net converges simultaneously to points $x \neq y \in X$ then $x \sim y$: x is of course equivalent to itself while x, y, z are equivalent to each other iff they are simultaneously in every open set where the net may eventually be in. This signature of chaos eradicates existing separation properties of the space: it makes X uniformly homogeneous and flat, devoid of any interaction inducing inducement among its parts, signifying thereby "death".

The generation of a new topology on X by the dynamics of f on X is a consequence of the topology of pointwise biconvergence $\mathcal{T}$ defined on the set of relations $\mathrm{Multi}((X,\mathcal{U}),(Y,\mathcal{V}))$, [30]. This generalization of the topology of pointwise convergence defines neighbourhoods of f in $\mathrm{Multi}((X,\mathcal{U}),(Y,\mathcal{V}))$ to consist of those functions in $(\mathrm{Multi}((X,\mathcal{U}),(Y,\mathcal{V})),\mathcal{T})$ whose images at any point $x \in X$ lie not only close enough to $f(x) \in Y$ (this gives the usual pointwise convergence) but additionally whose inverse images at $y = f(x)$ contain points arbitrarily close to x. Thus the graph of f apart from being sufficiently close to $f(x)$ at x in $V \in \mathcal{V}$, but must also be constrained such that $f^-(y)$ has at least one branch in the open set $U \in \mathcal{U}$ about x. This requires all members of a neighbourhood $\mathcal{N}_f$ of f to "cling to" f as the number of points on the graph of f increases with the result that unlike for simple pointwise convergence, no gaps in the graph of the limit relation is possible not only on the domain of f but on its range too.

For a given integer $I \geq 1$, the open sets of $(\mathrm{Multi}(X,Y),\mathcal{T})$ are

$$B((x_i),(V_i);(y_i),(U_i)) = \{g \in \mathrm{Map}(X,Y)\colon (g(x_i) \in V_i) \bigwedge (g^-(y_i) \bigcap U_i \neq \emptyset)\ , i = 1,2,\cdots,I\}, \quad (10.2.25)$$

where $(x_i)_{i=1}^{I} \in X$, $(y_i)_{i=1}^{I} \in Y$, $(U_i)_{i=1}^{I} \in \mathcal{U}$, $(V_i)_{i=1}^{I} \in \mathcal{V}$ are chosen arbitrarily with reference to $(x_i, f(x_i))$. A local base at f, for $(x_i, y_i) \in \text{Graph}(f)$, is the set of functions of (10.2.25) with $y_i = f(x_i)$, and the collection of all local bases $B_\alpha = B((x_i)_{i=1}^{I_\alpha}, (V_i)_{i=1}^{I_\alpha}; (y_i)_{i=1}^{I_\alpha}, (U_i)_{i=1}^{I_\alpha})$, for every choice of $\alpha \in \mathbb{D}$, is a base ${}_{\mathrm{T}}\mathcal{B}$ of $(\text{Multi}(X,Y), \mathcal{T})$; note that in this topology $(\text{Map}(X,Y), \mathcal{T})$ is a subspace of $(\text{Multi}(X,Y), \mathcal{T})$.

The basic technical tool needed for describing the adhering limit relation in $(\text{Multi}(X,Y), \mathcal{T})$ is a generalization of the topological concept of neighbourhoods to the algebraic concept of a filter which is a collection of subsets of X satisfying

(F1) The empty set $\emptyset$ does not belong to $\mathcal{F}$,

(F2) The intersection of any two members of a filter is another member of the filter: $F_1, F_2 \in \mathcal{F} \Rightarrow F_1 \cap F_2 \in \mathcal{F}$,

(F3) Every superset of a member of a filter belongs to the filter: $(F \in \mathcal{F}) \wedge (F \subseteq G) \Rightarrow G \in \mathcal{F}$; in particular $X \in \mathcal{F}$,

and is generated by a subfamily $(B_\alpha)_{\alpha \in \mathbb{D}} = {}_{\mathrm{F}}\mathcal{B} \subseteq \mathcal{F}$ of itself, known as the filter-base, characterized by

(FB1) There are no empty sets in the collection ${}_{\mathrm{F}}\mathcal{B}$: $(\forall \alpha \in \mathbb{D})(B_\alpha \neq \emptyset)$

(FB2) The intersection of any two members of ${}_{\mathrm{F}}\mathcal{B}$ contains another member of ${}_{\mathrm{F}}\mathcal{B}$: $B_\alpha, B_\beta \in {}_{\mathrm{F}}\mathcal{B} \Rightarrow (\exists B \in {}_{\mathrm{F}}\mathcal{B}\colon B \subseteq B_\alpha \cap B_\beta)$.

Hence any family of subsets of X that does not contain the empty set and is closed under finite intersections is a base for a unique filter on X, and the filter-base

$$ {}_{\mathrm{F}}\mathcal{B} \stackrel{\text{def}}{=} \{B \in \mathcal{F}\colon B \subseteq F \text{ for each } F \in \mathcal{F}\} \tag{10.2.26} $$

determines the filter

$$ \mathcal{F} = \{F \subseteq X : B \subseteq F \text{ for some } B \in {}_{\mathrm{F}}\mathcal{B}\} \tag{10.2.27} $$

as all its supersets. Since filters are purely algebraic without any topological content, to use it as a tool of convergence, a comparison of (F1)-(F3) and (FB1)-(FB2) with (N1)-(N3) and (NB1)-(NB2) of Sec. 10.1 show that the neighbourhood system $\mathcal{N}_x$ at x is the *neighbourhood filter at* x and any local base at x is a filter-base for $\mathcal{N}_x$ and generally for any subset A of X, $\{N \subseteq X : A \subseteq \text{Int}(N)\}$ is a filter on X at A. All subsets of X containing a point $p \in X$ is the *principal filter* ${}_{\mathrm{F}}\mathcal{P}(p)$ on X at p, and the collection of all supersets of a nonempty subset A of X is the principal filter ${}_{\mathrm{F}}\mathcal{P}(A)$ at A. The singleton sets $\{\{x\}\}$ and $\{A\}$ are particularly simple examples of filter-bases that generate the principal filters at $\{x\}$ and A; other useful examples that we require subsequently are the set of all residuals

$$ \text{Res}(\mathbb{D}) = \{\mathbb{R}_\alpha \colon \mathbb{R}_\alpha = \{\beta \in \mathbb{D}\colon \alpha \preceq \beta\}\} $$

of a directed set $\mathbb{D}$, and the neighbourhood systems $\mathcal{B}_x$ and $\mathcal{N}_x$. By adjoining the empty set to the principal filters yields the p-inclusion and A-inclusion topologies on X respectively[9].

The utility of filters in describing convergence in topological spaces is because a filter $\mathcal{F}$ on X can always be associated with the net $\chi_{\mathcal{F}}\colon {}_{\mathbb{D}}F_x \to X$ defined by

$$\chi_{\mathcal{F}}(F,x) \stackrel{\text{def}}{=} x \tag{10.2.28}$$

where ${}_{\mathbb{D}}F_x = \{(F,x) : (F \in \mathcal{F})(x \in F)\}$ is the directed set with direction $(F,x) \preceq (G,y) \Rightarrow (G \subseteq F)$; reciprocally a net $\chi\colon \mathbb{D} \to X$ corresponds to the filter-base

$${}_{\mathrm{F}}\mathcal{B}_{\chi} \stackrel{\text{def}}{=} \{\chi(\mathbb{R}_{\alpha})\colon \mathrm{Res}(\mathbb{D}) \to X \text{ for all } \alpha \in \mathbb{D}\}, \tag{10.2.29}$$

with the corresponding filter $\mathcal{F}_{\chi}$ being obtained by taking all supersets of the elements of ${}_{\mathrm{F}}\mathcal{B}_{\chi}$. Filters and their bases are extremely powerful tools for maximal, non-injective, degenerate ill-posedness in the context of the algebraic Hausdorff Maximal Principle and Zorn's Lemma, that is now summarized below[10].

Let f be a noninjective function in Multi(X) and $\mathfrak{I}(f)$ be the number of injective branches of f and let

$$F = \{f \in \mathrm{Multi}(X)\colon f \text{ is a noninjective function on } X\} \in \mathcal{P}(\mathrm{Multi}(X))$$

be the collection of all noninjective functions on X satisfying the properties

(a) For every $\alpha \in \mathbb{D}$, F has the extension property

$$(\text{For any } f_{\alpha} \in F)(\exists f_{\beta} \in F)\colon \mathfrak{I}(f_{\alpha}) \leq \mathfrak{I}(f_{\beta}).$$

Define a partial order $\preceq$ on Multi(X) as

$$\mathfrak{I}(f_{\alpha}) \leq \mathfrak{I}(f_{\beta}) \Longleftrightarrow f_{\alpha} \preceq f_{\beta}, \tag{10.2.30}$$

with $\mathfrak{I}(f) := 1$ for the smallest f. This is actually a preorder on Multi(X) in which all function with the same number of injective branches are equivalent

---

[9] A filter is almost a topology: both are closed under finite intersections and arbitrary unions, and both contain the base set X. It is only the empty set that must always be in the topology but never in a filter; adding it to a filter makes it a special type of topology that might be termed a filtered topology. Whereas any arbitrary family of sets can generate a topology as its subbase through finite intersections followed by arbitrary unions, the family must satisfy the finite intersection property before qualifying as a filter subbase; hence, every filter subbase is a topological subbase but not conversely.

[10] **Hausdorff Maximal Principle** (HMP)**:** Every partially ordered set has a maximal chain.

A partially ordered set X is said to be *inductive* if every chain of X has an upper bound in X.

Zorn's Lemma: Every inductive set has at least one maximal element.

to each other. Note that Multi(X) has two orders imposed on it: the first $\preceq$ between its elements f, and the second the usual $\subseteq$ that orders subsets of these functional elements.

(b) Let

$$\mathcal{X} = \{C \in \mathcal{P}(F) \colon C \text{ is a chain in } (\text{Multi}(X), \preceq)\} \in \mathcal{P}^2(\text{Multi}(X)) \quad (10.2.31)$$

be a collection of chains in Multi(X) with respect to the order (10.2.30) where

$$C_\nu = \{f_\alpha \in \text{Multi}(X) \colon f_\alpha \preceq f_\nu\} \in \mathcal{P}(\text{Multi}(X)), \qquad \nu \in \mathbb{D}, \qquad (10.2.32)$$

are the chains of non-injective functions where $f_\alpha \in F$ is to be identified with the iterates f^i, the number of injective branches $\mathcal{I}(f)$ depending on i. The chains are to be built from the smallest $C_0 = \mathcal{D}$ the domain of f, by application of a choice function g_c that generates the immediate successor

$$C_j := g(C_i) = C_i \bigcup g_c(\mathcal{G}(C_i) - C_i) \in \mathcal{X}$$

of C_i by picking one from the many

$$\mathcal{G}(C_i) = \{f \in F - C_i \colon \{f\} \bigcup C_i \in \mathcal{X}\}$$

that C_i may possibly possess. Application of g to C_0 n-times generates the chain $C_n = \{\mathcal{D}, f(\mathcal{D}), \cdots, f^n(\mathcal{D})\}$, and the smallest common chain

$$\begin{aligned} \mathcal{C} &= \{C_j \in \mathcal{P}(\text{Multi}(X)) \colon C_i \subseteq C_k \text{ for } i \le k\} \subseteq \mathcal{X} \\ &= \{\mathcal{D}, \{\mathcal{D}, f(\mathcal{D})\}, \{\mathcal{D}, f(\mathcal{D}), f^2(\mathcal{D})\}, \cdots\} \qquad \mathcal{D} := C_0 \end{aligned} \quad (10.2.33)$$

of all the possible g-towered chains $\{C_i\}_{i=0,1,2,\cdots}$ of Multi(X) constitutes a principal filter of totally ordered subsets of (Multi(X), $\subseteq$) at C_0. Notice that while $\mathcal{X} \in \mathcal{P}^2(\text{Multi}(X))$ is a set of sets, $C \in \mathcal{P}(\text{Multi}(X))$ is relatively simpler as a set of elements of $f \in \text{Multi}(X)$, which at the base level of the tree of interdependent structures of Multi(X), is canonically the simplest.

To continue further with the application of Hausdorff Maximal Principle to the partially ordered set $(\mathcal{X}, \preceq)$ of sets, it is necessary to postulate that

(i) There exists a smallest element C_0 in $\mathcal{X}$ with no predecessor,

(ii) Every element C of $\mathcal{X}$ has an immediate successor $g(C)$ in $\mathcal{X}$; hence there is no element of $\mathcal{X}$ lying strictly between C and $g(C)$, and

(iii) $\mathcal{X}$ is an *inductive set* so that every chain $\mathcal{C}$ of $(\mathcal{X}, \preceq)$ has a supremum $\sup_{\mathcal{X}}(\mathcal{C}) = \cup_{C \in \mathcal{C}} C$ in $\mathcal{X}$.

Any subset $\mathcal{T}$ of $\mathcal{X}$ satisfying these properties is known as a *tower*; $\mathcal{X}$ is of course a tower by definition. The intersection of all possible towers of $\mathcal{X}$ is the towered chain $\mathcal{C}$ of $\mathcal{X}$, Eq. (10.2.33). Criterion (iii) is especially crucial as it effectively disqualifies $(F, \preceq)$ as a likely candidate for HMP: the supremum of the chains of increasingly non-injective functions need not be a *function,* but is more likely to be a multifunction. Hence $\mathcal{X}$ in the conditions above is the

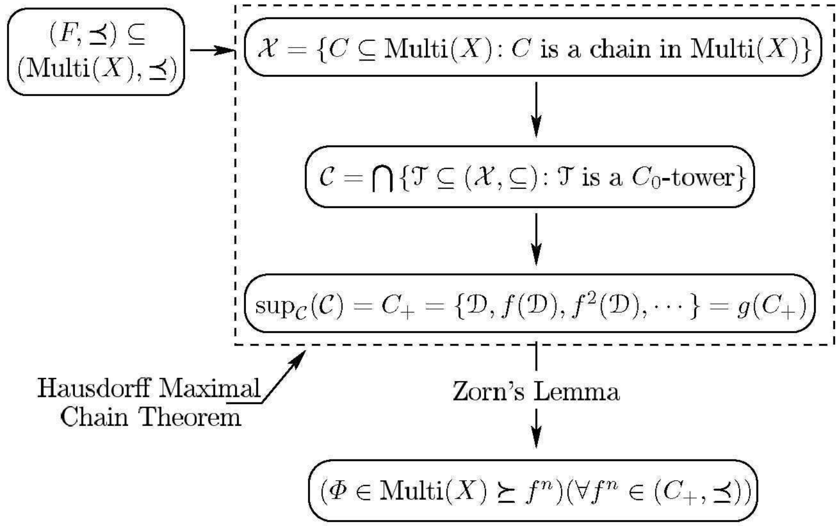

Fig. 10.2. Application of Zorn's Lemma to a partially ordered set F of non-injective functions f in Multi(X). $\mathcal{C} = \{\mathcal{D}, \{\mathcal{D}, f(\mathcal{D})\}, \{\mathcal{D}, f(\mathcal{D}), f^2(\mathcal{D})\}, \cdots\}$ is a chain of towered chains of *functions* in Multi(X) with $C_0 = \mathcal{D}$, the domain of f. Notice that to obtain a maximal Φ at the base level Multi(X), it is necessary to go two levels higher: $\mathcal{X} \in \mathcal{P}^2(\text{Multi}(X)) \to C \in \mathcal{P}(\text{Multi}(X)) \to \Phi \in \text{Multi}(X)$ is a three-tiered structure with the two-tiered HMP feeding to the third of Zorn's Lemma.

space of relations, and it is necessary to consider C of (10.2.32) as a subset of this Multi(X) rather than of F. The careful reader cannot fail to note that the requirement of inductivity of $\mathcal{X}$ effectively leads to an "extension" of Map(X) to Multi(X) where the supremum of the chain of non-injective functions can possibly lie. However since this is purely in an algebraic setting without topologies on the sets, the supremum constitutes only a static cap on the family of equilibrium ordered states: the chains being only ordered and not directed are devoid of any dynamical evolutionary property.

(c) The Hausdorff Maximal Principle applied to $(\mathcal{X}, \subseteq)$ now yields

$$\begin{aligned}\sup_{\mathcal{C}}(\mathcal{C}) &= C_+ = \{f_\alpha, f_\beta, f_\gamma, \cdots\} \\ &= \{\mathcal{D}, f(\mathcal{D}), f^2(\mathcal{D}), \cdots\} = g(C_+) \in \mathcal{C} \qquad (10.2.34)\end{aligned}$$

as the supremum of $\mathcal{C}$ in $\mathcal{C}$, defined as a fixed-point of the tower generator g, without any immediate successor. Identification of this fixed-point supremum as one of the many possible maximal elements of $(\mathcal{X}, \subseteq)$ completes the application of Hausdorff Principle, yielding C_+ as the required maximal chain of $(\mathcal{X}, \subseteq)$.

The technique of HMP is of interest because it presents a graphic step-wise algorithmic rule leading to an equivalent filter description and the algebraic notion of a *chained tower*. Not possessing any of the topological directional properties associated with a net or sequence, the tower comprises an ideal mathematical vocabulary for an ordered succession of equilibrium states of a quasi-static, reversible, process. The directional attributes of convergence and adherence must be externally imposed on towered filters like $\mathcal{C}$ by introducing the neighbourhood system: a filter $\mathcal{F}$ converges to $x \in (X,\mathcal{U})$ iff $\mathcal{N}_x \subseteq \mathcal{F}$.

(d) Returning to the partially ordered set $(\text{Multi}(X), \preceq)$, Zorn's Lemma applied to the maximal chained element C_+ of the inductive set $\mathcal{X}$ finally yields the required maximal element $\Phi \in \text{Multi}(X)$ as an upper bound of the maximal chain $(C_+, \preceq)$. Because this limit need not in general be a function, the supremum does not belong to the towered chain having it as a fixed point, and may be considered as a contribution of the inverse functional relations (f_α^-) in the following sense. From Eq. (10.1.1), the net of increasingly non-injective functions of Eq. (10.2.30) implies a corresponding net of decreasingly multivalued functions ordered inversely by the relation $f_\alpha \preceq f_\beta \Leftrightarrow f_\beta^- \preceq f_\alpha^-$. Thus the inverse relations which are as much an integral part of graphical convergence as are the direct relations, have a smallest element belonging to the multifunctional class. Clearly, this smallest element as the required supremum of the increasingly non-injective tower of functions defined by (10.2.30), serves to complete the significance of the tower by capping it with a "boundary" element that can be taken to bridge the classes of functional and non-functional relations on X.

Having been assured of the existence of a largest element $\Phi \in \text{Multi}(X)$, we now proceed to construct it topologically. Let $(\chi_i := f^i(A))_{i\in\mathbb{N}}$ for a subset $A \subseteq X$ that we may take to be the domain of f, correspond to the ordered sequence (10.2.30). Using the notation of (10.2.29), let the sequences $\chi(\mathbb{R}_i) = \cup_{j\geq i} f^j(A)$ for each $i \in \mathbb{N}$ generate the decreasingly nested filter-base

$$
\begin{aligned}
{}_{\mathrm{F}}\mathcal{B}_\chi &= \left\{ \bigcup_{j\geq i} f^j(A) \right\}_{i\in\mathbb{N}} \\
&= \left\{ \bigcup_{j\geq i} f^j(x) \right\}_{i\in\mathbb{N}}, \qquad \forall x \in A, \qquad (10.2.35)
\end{aligned}
$$

corresponding to the sequence of functional iterates $(f^j)_{j\geq i\in\mathbb{N}}$. The existence of a maximal chain with a maximal element guaranteed by the Hausdorff Maximal Principle and Zorn's Lemma respectively implies a nonempty core of ${}_{\mathrm{F}}\mathcal{B}_\chi$. We now identify this filterbase with the neighbourhood base at Φ and thereby define

$$
\begin{aligned}
\Phi(A) &\overset{\text{def}}{=} \text{adh}({}_{\mathrm{F}}\mathcal{B}_\chi) \\
&= \bigcap_{i\geq 0} \text{Cl}(A_i), \qquad A_i = \{f^i(A), f^{i+1}(A), \cdots\} \qquad (10.2.36)
\end{aligned}
$$

as the attractor of A, where the closure is with respect to the topology of pointwise bi-convergence induced by the neighbourhood filter base ${}_{\mathrm{F}}\mathcal{B}_\chi$. Clearly the attractor as defined here is the graphical limit of the sequence of functions $(f^i)_{i\in\mathbb{N}}$ with respect to the directed sets of Table 10.2. This attractor represents, in the product space $X \times X$, the converged limit of the bi-directional evolutionary dynamics occurring in the kitchen $X \times \mathfrak{X}$ that induces the image $\Phi(A)$ in X. The exclusion space $\mathfrak{X}$ is not directly observable, being composed of complementary *negelements* $\mathfrak{x}$ that correspond in an unique, one-to-one fashion to the corresponding defining observables $x \in X$, just as the negative reals — which are not physically directly observable either — are attached in a one-to-one fashion with their corresponding defining positive counterparts by

$$r + (-r) = 0, \qquad r \in \mathbb{R}_+. \tag{10.2.37}$$

The exclusion space $(\mathfrak{X}, \mathfrak{U})$ introduced next is necessary for the understanding of bi-directional evolutionary process responsible for a synthesis of opposites of two sub-systems competitively collaborating with each other. The basic example of an exclusion space is the negative reals with a *forward* arrow of the *decreasing* negatives resulting from an *exclusion topology* $\mathcal{U}_-$ generated by the topology $\mathcal{U}_+$ of the observable positive reals R_+. This generalization of the additive inverse of the real number system to sets follows.

The Negative Exclusion Space of a Topological Space

Postulate NEG-1. The Negative $\mathfrak{X}$ of a set X.[11] Let X be a set and suppose that for every $x \in X$ there exists a *negative element* $\mathfrak{x} \in \mathfrak{X}$ with the property that

$$\mathfrak{X} \stackrel{\text{def}}{=} \{\mathfrak{x} \colon \{x\} \bigcup \{\mathfrak{x}\} = \emptyset\} \tag{10.2.38a}$$

defines the *negative,* or *exclusion, set* of X. This means that for every subset A of X there is a complementary neg(ative)set $\mathfrak{A} \subseteq \mathfrak{X}$ *associated with* (*generated by*) it such that

$$A \bigcup \mathfrak{G} \stackrel{\text{def}}{=} A - G, \qquad G \longleftrightarrow \mathfrak{G}, \tag{10.2.38b}$$

implies $A \cup \mathfrak{A} = \emptyset$. Hence a neg-set and its generating set act as relative *discipliners* of each other in restoring a measure of order in the evolving confusion, disquiet and tension, with the intuition of the set-negset pair "undoing", "controlling", or "stabilizing" each other. The complementing neg-element is an unitive inverse of its generating element, with $\emptyset$ the corresponding *identity* and G the *physical manifestation of* $\mathfrak{G}$. Thus for $r > s \in \mathbb{R}_+$, the physical manifestation of any $-s \in \mathfrak{R}_+ (\equiv \mathbb{R}_-)$ is the smaller element $(r - s) \in \mathbb{R}_+$.

As compared with the directed set $(\mathcal{P}(X), \subseteq)$ that induces the natural direction of *decreasing subsets* of Table 10.2, the direction of *increasing supersets* induced by $(\mathcal{P}(X), \supseteq)$ — which understandably finds no ready application in convergence theory — proves useful in generating a co-topology $\mathcal{U}_-$ on $(X, \mathcal{U}_+)$ as follows. Let $(x_0, x_1, x_2, \cdots)$ be a sequence in X converging to $x_* \in X$ with

[11] These quantities will be denoted by 𝔣𝔯𝔞𝔨𝔱𝔲𝔯 letters.

reference to any of the reverse-inclusion directions of decreasing neighbourhoods of Table 10.2[12], and consider the backward arrow induced at x_* by the directed set $(\mathcal{P}(X), \supseteq)$ of increasing supersets at x_*. As the reverse sequence $(x_*, \cdots, x_{i+1}, x_i, x_{i-1}, \cdots)$ does not converge to x_0 unless it is eventually in every neighbourhood of this initial point, we employ the closed-open subsets

$$N_i - N_j = \begin{cases} (N_i - N_j) \bigcap N_i, & \text{(open)} \\ (N_i - N_j) \bigcap (X - N_j) & \text{(closed)} \end{cases} \tag{10.2.39}$$

$(j > i)$ in the *inclusion* topology $\mathcal{U}_+$ of X with $x_i \in N_i - N_{i+1}$, $N_i \in \mathcal{N}_{x_*}$, to define an additional *exclusion topology* $\mathcal{U}_-$ on $(X, \mathcal{U}_+)$ as follows. First recall that whereas the *x-inclusion topology* $\mathcal{U}_+$ of X comprises, together with $\emptyset$, all subsets of X that *include* x with the neighbourhood system $\mathcal{N}_x$ being just these non-empty subsets of X, the *x-exclusion topology* is, along with X, all the subsets $\mathcal{P}(X - \{x\})$ that *exclude* x. The $A \subseteq X$ exclusion topology $\{\mathcal{P}(X - A), X\}$ therefore consists of all subsets of X that do not intersect A and the $(X - A)$-exclusion topology $\{\mathcal{P}(A), X\}$ comprises, with X, only the subsets of A. Since $\mathcal{N}_x = \{X\}$ and $\mathcal{N}_{y \neq x} = \{\{y\}\}$ are the neighbourhood systems at x and any $y \neq x$ in the x-exclusion topology, it follows that while every net must converge to the defining point of its own topology, only the eventually constant net $\{y, y, y, \cdots\}$ converges to any $y \neq x$[13]. The exclusion topology of x therefore has the remarkable property of compelling every other element of X to either submit to the dictum of x by being in its sphere of influence, or else to effectively isolate any other member of X from establishing its own sphere of influence. All directions with respect to x are consequently rendered equivalent; hence the directions of $\{1/n\}_{n=1}^{\infty}$ and $\{n\}_{n=1}^{\infty}$ are equivalent in $\mathbb{R}_+$ as they converge to 0 in its exclusion topology, and this basic property of the exclusion topology induces an opposing direction in X.

It is now possible to postulate with respect to the directed set $_{\mathbb{D}}N_i = \{(N_i, i) : (N_i \in \mathcal{N}_{x_*})(x_i \in N_i)\}$ of Table 10.2 and a sequence $(x_i)_{i \geq 0}$ in $(X, \mathcal{U}_+)$ converging to $x_* = \cap_{i \geq 0} \mathrm{Cl}(N_i) \in X$, that

Postulate NEG-2. The $\mathbf{x_0}$-exclusion topology $\mathcal{U}_-$ of $(\mathbf{X}, \mathcal{U}_+)$. There exists an increasing sequence of negelements $(\mathfrak{x}_i)_{i \geq 0}$ of $\mathfrak{X}$ that converges to $\mathfrak{x}_*$ in the $\mathfrak{x}_*$-inclusion topology $\mathfrak{U}$ of $\mathfrak{X}$ generated by the $\mathfrak{X}$-images of the neighbourhood system $\mathcal{N}_{x_*}$ of $(X, \mathcal{U}_+)$. Since the only manifestation of neg-sets in the observable world is their regulating property, the $\mathfrak{X}$-increasing sequence $(\mathfrak{x}_i)_{i \geq 0}$ converges to $\mathfrak{x}_*$ in $(\mathfrak{X}, \mathfrak{U})$ if and only if the sequence $(x_0, x_1, x_2, \cdots)$ converges to x_* in $(X, \mathcal{U}_+)$. Affinely translated to X, this means that the $\mathfrak{x}_*$-inclusion

---

[12] We henceforth adopt the convention that the arrow induced by the inclusion topology of the real world is the forward arrow of the system, and the exclusion neg-matter manifests in this real world as its backward arrow. The forward arrow therefore corresponds to the increasing direction of an appropriate pre-ordering of the real physical world.

[13] I thank Joseph T. H. Lo for his clarifications on the subtleties of the exclusion topology, Private Communication, May 2004.

arrow in $(\mathfrak{X},\mathfrak{U})$ transforms to an x_0-exclusion arrow in $(X,\mathcal{U}_+)$ generating an additional topology $\mathcal{U}_-$ in X that opposes the arrow converging to x_*. This *direction of increasing supersets of* $\{x_*\}$ *excluding* x_0 associated with $\mathcal{U}_-$ of Table 10.3, is to be compared with the *natural direction of decreasing subsets containing* x_* in $(X,\mathcal{U}_+)$, Table 10.2. We take the reference natural direction in $X \cup \mathfrak{X}$ to be that of X pulling the inclusion sequence $(x_0, x_1, x_2, \cdots)$ to x_*; hence the decreasing subset direction in $\mathfrak{X}$ of the inclusion sequence $(\mathfrak{x}_0, \mathfrak{x}_1, \mathfrak{x}_2, \cdots, \mathfrak{x}_*)$ appears in X as an exclusion sequence converging to x_0 because any sequence in an exclusion space must necessarily converge to the defining element in its own topology. In this perspective, the left side of Eq. (10.2.38b), read in the more familiar form $a+(-b) = a-b$ with $a, b \in \mathbb{R}_+$ and $-b := \mathfrak{b} \in \mathfrak{R}_+$, represents "+" evolution in the base kitchen of Nature, which is then served in its bi-directional physical-world manifestation on the dining-table of the right side supporting retraction along the "−" direction. At the risk of an apparent "abuse of language", $(\mathfrak{X},\mathfrak{U})$ will be termed the *exclusion space* of $(X,\mathcal{U}_+)$.

Directed set $\mathbb{D}$	Direction $\preceq$ induced by $\mathbb{D}$
${}_{\mathbb{D}}\mathfrak{N} = \{\mathfrak{N} \colon \mathfrak{N} \in \mathcal{N}_{\mathfrak{x}}\}$	$\mathfrak{M} \preceq \mathfrak{N} \Leftrightarrow \mathfrak{M} \subseteq \mathfrak{N}$
${}_{\mathbb{D}}\mathfrak{N}_{\mathfrak{t}} = \{(\mathfrak{N},\mathfrak{t}) \colon (\mathfrak{N} \in \mathcal{N}_{\mathfrak{x}})(\mathfrak{t} \in \mathfrak{N})\}$	$(\mathfrak{M},\mathfrak{s}) \preceq (\mathfrak{N},\mathfrak{t}) \Leftrightarrow \mathfrak{M} \subseteq \mathfrak{N}$
${}_{\mathbb{D}}\mathfrak{N}_{\beta} = \{(\mathfrak{N},\beta) \colon (\mathfrak{N} \in \mathcal{N}_{\mathfrak{x}})(\mathfrak{x}_\beta \in \mathfrak{N})\}$	$(\mathfrak{M},\alpha) \leq (\mathfrak{N},\beta) \Leftrightarrow (\alpha \preceq \beta) \wedge (\mathfrak{M} \subseteq \mathfrak{N})$

Table 10.3. Natural directions of increasing supersets in $(\mathfrak{X},\mathfrak{U})$ is to be compared with Table 10.2 of the natural reverse directions in $(X,\mathcal{U})$. The direction of coevents in $\mathfrak{X}$ is opposite to that of X in the sense that the temporal sequence of images of events in X opposes that in $\mathfrak{X}$ and the order of occurrence of events induced by the coworld appear to be reversed to the physical observer stationed in X.

Although the backward sequence $(x_j)_{j=\cdots,i+1,i,i-1,\cdots}$ in $(X,\mathcal{U}_+)$ does not converge, the effect of $(\mathfrak{x}_i)_{i\geq 0}$ of $\mathfrak{X}$ on X is to regulate the evolution of the forward arrow $(x_i)_{i\geq 0}$ to an effective state of stasis of dynamical equilibrium, that becomes self-evident on considering for X and $\mathfrak{X}$ the sets of positive and negative reals, and for x_*, $\mathfrak{x}_*$ a positive number r and its negative inverse image $-r$. The existence of a negelement $x \leftrightarrow \mathfrak{x}$ in $\mathfrak{X}$ for every $x \in X$ requires all forward arrows in X to have a matching forward arrow in $\mathfrak{X}$ that actually *appears backward when viewed from* X. It is this opposing complimentary effect of the apparently backward-$\mathfrak{X}$ sequences on X — responsible by (10.2.38b) for moderating the normal uni-directional evolution in X — that is useful in establishing a stasis of dynamical balance between the opposing forces generated in the composite of a compound system with its environment. Obviously, the evolutionary process ceases when the opposing influences in X due to it-

	$(X,\mathcal{U}_+)$	$(X,\mathcal{U}_-)$
T_0	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x : N \cap \{y\} = \emptyset) \vee (\exists\, M \in \mathcal{N}_y : M \cap \{x\} = \emptyset)$	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x : N \cap \{y\} \neq \emptyset) \vee (\exists\, M \in \mathcal{N}_y : M \cap \{x\} \neq \emptyset)$
T_1	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x : N \cap \{y\} = \emptyset) \wedge (\exists\, M \in \mathcal{N}_y : M \cap \{x\} = \emptyset)$	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x : N \cap \{y\} \neq \emptyset) \wedge (\exists\, M \in \mathcal{N}_y : M \cap \{x\} \neq \emptyset)$
T_2	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x \wedge M \in \mathcal{N}_y) : (M \cap N = \emptyset)$	$(\forall x \neq y \in X)\,(\exists\, N \in \mathcal{N}_x \wedge M \in \mathcal{N}_y) : (M \cap N \neq \emptyset)$

Table 10.4. Comparison of the separation properties of $(X,\mathcal{U}_+)$ and its inhibitor $(X,\mathcal{U}_-)$.

self and that of its moderator $\mathfrak{X}$ balance out marking a state of dynamic equilibrium.

It should be noted that the moderating image $\mathfrak{X}$ of X needs to be endowed with inverse inhibiting properties if Eq. (10.2.38*b*) is to be meaningful which leads to the separation properties of the conjugate spaces $(X,\mathcal{U}_+)$ and $(\mathfrak{X},\mathfrak{U})$ as shown in Table 10.4. Significantly, the exclusion space is topologically distinguished in having its sequences converge with respect to the only neighbourhood X of the limit point, a property that leads as already pointed out earlier to the existence of a multiplicity of equivalent limits in large neighbourhoods of x_0 to which the backward sequences in X converges, even when $(X,\mathcal{U})$ is Hausdorff. In the context of iterational evolution of functions that concerns us here, that the function-multifunction asymmetry of (10.1.1) introduced by the non-injectivity of the iterates is directly responsible for the difference in the separation properties of $\mathcal{U}_+$ and $\mathcal{U}_-$, which in turn prohibits the system from annihilating B mentioned earlier and forces it to adopt the forward-backward stasis of opposites. Recalling that non-injectivity of one-dimensional maps translate to pairs of injective branches with *positive* and *negative* slopes, we argue with reference to Fig. 10.3 that whereas branches with positive slope represent matter, those with negative slope correspond to reg(ulating)-matter by Eq. (10.2.38*b*) and the disjoint union of these components represents the compound system of forward-backward opposites. Taking $T_A > T_B$, $p_A > p_B$ and $\mu_A > \mu_B$, the dynamical evolution represented by the shaded boxes would, in the absence of the backward arrow induced by the exclusion space, eventually spread uniformly over the full domain, and equilibrium would be characterized completely by T_A, p_A, μ_A, at the exclusion of B. Denoting matter by 1 and (the effect of) negmatter by 0, the progressively refined partition of $\mathcal{D}(t)$ induced by the evolving map is indicated in (ii), (iii) and (iv).

As an example, we return to Eqs. (10.2.18) and (10.2.19) for the entropy change due to exchange of resources and its non-linear, irreversible, internal

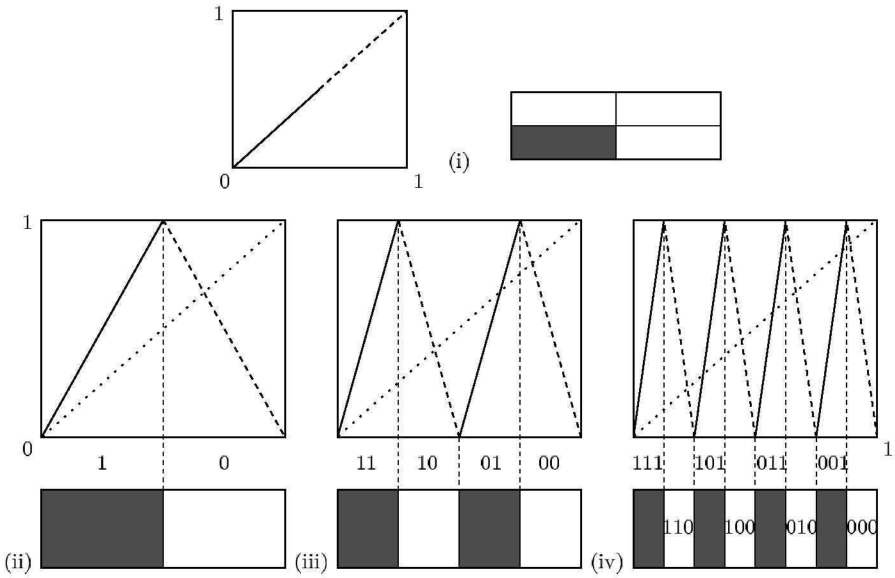

Fig. 10.3. Matter-negmatter synthesis of an evolving system $C = A \cup B$ under the tent interaction. A and B are represented by the solid and dashed lines as injective branches with positive and negative slopes respectively.

generation respectively. The external exchange with the environment leads to a change in the internal state of the system which is then utilized in performing irreversible useful work relative to the environment, conveniently displayed in terms of the *neutral-neutral* convergence mode of a net of Fig. 10.4 adapted from Fig. 22 of [30], which illustrates the irreversible internal generation of entropy in a universe $C = A \cup B$, where A and B are two disjoint components of a system prepared at different initial conditions shown in the figure, with B the physical manifestation of a compatible space $\mathfrak{B}$ endowed with an exclusion topology and a direction opposing that of A. In the real interval $[0, 1]$, notable examples of A and B are $f(x)$ and $f(1-x)$ with B the physical manifestation of $\mathfrak{A}$. This allows us to make the

Definition 10.1 (Interaction Between Two Spaces). *A space* $(A, \mathcal{U})$ *will be said to* interact *with a* disjoint *space* $(B, \mathcal{V})$ *if there exists a function* f *on the compound* disjoint sum $(C = A \cup B, \mathcal{W})$ *where*

$$\begin{aligned}\mathcal{W} &= \{W := U \bigcup V : (U \in \mathcal{U}) \bigwedge (V \in \mathcal{V})\} \\ &= \{W \subseteq C : (W \bigcap A \,\text{is open in}\, A) \bigwedge (W \bigcap B \,\text{is open in}\, B)\},\end{aligned}$$

which evolves graphically to a well defined limit in the topology of pointwise biconvergence on $(C, \mathcal{W})$. *The function* f *will be said to be a* bidirectional interaction *between the subsystems* A *and* B *of* C.[14]

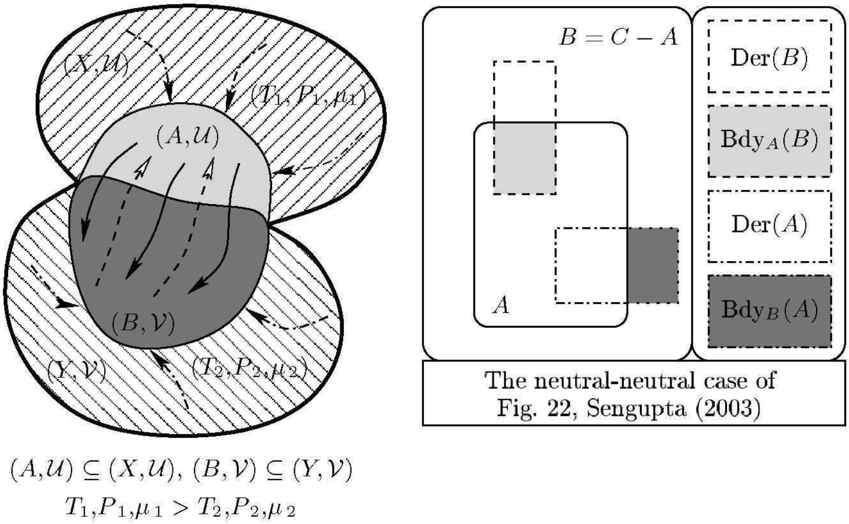

Fig. 10.4. Schematic representation of irreversible entropy generation in $C = A \cup B$ with respect to the universe $X \cup Y$. We will identify the solid arrows in C from the hot body to the cold with inverse limit, neg-entropic self-organization, and the dashed arrows from the cold to hot as direct limit, second law entropic emergence, see Fig. 10.5*b*.

While A and B by themselves need not display any notable features (see Fig. 10.10*c*), the evolution of A in the disjoint compound $(C, \mathcal{W})$, motivated by the inducement of an ininal topology on C, is effectively opposed by the influence of the exclusion topology of B, with the equivalence classes generated in C being responsible for the multi-inverses of the evolving f characterizing the nonlinear state of C following the internal preparation of the system. This irreversible process, indicated in Fig. 10.4 by the nets of full arrows from $(A, \mathcal{U})$ to $(B, \mathcal{V})$ representing transfer of energy, volume, or mass driven

[14] If A and B are not disjoint, then this construction of the compound sum may not work because A and B will generally induce distinct topologies on C; in this case $\mathcal{W}$ is obtained as follows. Endow the disjoint copies $A_1 := A \times \{1\}$ and $B_2 := B \times \{2\}$ of A and B with topologies $\mathcal{U}_1 = \{U \times \{1\} : U \in \mathcal{U}\}$ and $\mathcal{V}_2 = \{V \times \{2\} : V \in \mathcal{V}\}$, which are homeomorphic with their originals with $a \mapsto (a, 1)$ and $b \mapsto (b, 2)$ being the respective homeomorphisms. Then $C = A_1 \cup B_2$ is the *disjoint union (sum)* of A_1 and B_2 with the topology $\mathcal{W} = \{W \subseteq C : W = (U \times \{1\}) \cup (V \times \{2\}) : (U \in \mathcal{U}) \wedge (V \in \mathcal{V})\}$ that induces the subspaces $(A_1, \mathcal{U}_1)$ and $(B_2, \mathcal{V}_2)$.

by an appropriate evolutionary directed set of a thermodynamic force (for instance due to a temperature gradient $T_A > T_B$ inducing the energy transfer), provides the forward impetus for directional transport motivated by ininality. The dashed open arrows show the reverse evolution in C due to its inhibitor $\mathfrak{C}$. The dash-dot arrows stand for the uni-directional transfer of energy from a reservoir that continues till the respective parts of C acquire the characteristics of their reservoirs.

Since physical evolution powered by changes in the internal intensive parameters is represented by convergence of appropriate sequences and nets, it is postulated in keeping with the role of ininality, that equilibrium in uni-directional temporal evolutions like $X \to A \subseteq X$ or $Y \to B \subseteq Y$ sets up A and B as subspaces of X and Y respectively. For bi-directional processes like $A \leftrightarrow B$, the open headed dashed arrows of Fig. 10.4 from B to A represent the backward influence of $(B, \mathcal{V})$ on $(C, \mathcal{W})$. The assumptions

▶ Both the subsets A and B of the compound C are perfect in the sense that $A = \mathrm{Der}(A)$ and $B = \mathrm{Der}(B)$ so that there are no isolated points in A and B with all points of each of the sets accessible by sequences eventually in them, and

▶ $\mathrm{Bdy}_B(A) = B$ and $\mathrm{Bdy}_A(B) = A$ which enables all points of A and B to be directly accessed as limits by sequences in B and A,

imply that any exchange of resource from the environment $E = X \cup Y$ to system C will be evenly dispersed throughout by the irreversible, internal evolution of the system, once C attains equilibrium with E and is allowed to evolve unperturbed thereafter. This *global homogenizing principle of detailed balance*, applicable to evolutionary processes at the micro-level provides a rationale for equilibration in nature that requires every forward arrow to be balanced by a backward, leading to the global equilibrium of thermodynamics. If these influences exactly balance each other resulting in a complete restoration of all the intermediate stages, then the resulting *reversible process* is actually quasi-static with no effective changes; hence nontrivial dynamical equilibrium cannot be generated by reversible processes.

For unimodal maps like the logistic $f_\lambda = \lambda x(1 - x)$ that are defined with respect to the forward-backward, positive-negative slope characteristic, which for a particular λ can be taken to represent the subspace $C \subseteq E$ at equilibrium with its environment E, evolutionary changes in the effective available resources λ induce changes in the internal intensive thermodynamic parameters that follow uni-directional exchanges of C with E. This perturbs the equilibrium between components A and B resulting in further evolutionary iterational interaction between them. The iterational evolution of f_λ is relatively moderated by the reverse effect of the evolution of f_λ^- which suppresses the continual increase of noninjectivity of f_λ that would otherwise lead to a state of maximum noninjective ill-posedness for this λ: note the negative branches of f appear positive to its inverse, and conversely. Measurable global dynamic equilibrium represents a balance between the opposing induced local forces that are determined by, and which in turn determine, the degree of resource exchange λ. The eventual ininality at $\lambda = 4$ represents continual resource utilization from E that is dissipated for the globalizing uniformity of Figs. 10.3 and 10.10a(iii). In the range $3 < \lambda \leq \lambda_* = 3.5699456$, the input is gainfully employed to generate the complex structures that are needed to sustain the process at that level of λ.

Recalling footnote 8, we now summarize the principal features of the non-linear evolutionary dynamics following interaction of a compound system C with its surroundings.

▶ If the state of dynamic equilibrium of a composite system $C = A \cup B$ with its surroundings, as represented by the logistic map is disturbed by an interaction between them, forces are set up between the components A and B so as to absorb the effect of this disturbance.

▶ Consumption of the effects of the exchange is motivated by a simultaneous, non-reductionist drive towards increasing surjectivity and decreasing injectivity of $(f_\lambda)_\lambda$ and its evolved iterated images, that eventually leads to a state of maximal non-injective degeneracy on the domain of f. Owing to the function-multifunction asymmetry of f, such a condition would signify static equilibrium and an end to all further evolutionary processes, a state of dissipative annihilation, burn-out and ininality.

▶ Since such eventual self-destruction cannot be the stated objective of Nature, this unrelenting thrust toward collapse is opposed by the negworld exclusion effects we have described earlier, generating a reversed sequential direction effectively inhibiting the drive towards self-destruction induced by the simultaneous increase of λ and the increased noninjectivity under iterations. The resulting state of dynamic equilibrium is the observed equilibrium of Nature. Like all others, nature's *kitchen* $C \times \mathfrak{C}$ where the actual dynamical evolution occurs is beyond direct observation; only its disciplining effect in $C \times C$ is perceived by the observer stationed in $\mathcal{D}(f) = C$.

As an example, consider an isolated system of two parts each locally in equilibrium with its environment as in (10.2.19) that can now be re-written as

$$\mathcal{S}_C(t) = \mathcal{S}_C(0) + \left[N_A c_A \ln\left(\frac{T}{T_A(t)}\right) + N_B c_B \ln\left(\frac{T}{T_B(t)}\right)\right] - R\left[N_A \ln\left(\frac{P_A}{p_A(t)}\right) + N_B \ln\left(\frac{P_B}{p_B(t)}\right)\right], \quad (10.2.40)$$

where we note with reference to Fig. 10.4 that $T_A = T_1$, $T_B = T_2$ are the temperatures of subsystems A and B, $V_A + V_B = V$ is the total volume of C, p_A, p_B are the pressures of A and B, $P_{A,B} := N_{A,B}RT_{A,B}/V$ are their partial pressures with $P = P_A + P_B$ the total pressure exerted by the gases in V, and T is the equilibrium temperature of (10.2.21).

Then

(i) If the parts containing nonidentical ideal gases at different temperatures are brought in contact with each other, the equilibrium state of stasis resulting from the flows of *heat* and *cold* (= negheat) between the bodies lead to the equality of temperature, $T_A = T = T_B$, and the vanishing of the first part of (10.2.40).

(ii) If the gas in the first half expands into the second then equilibrium is reached when the *gas outflow* is exactly balanced by the *vacuum inflow* into it if the second is evacuated, or if it is filled with a nonidentical gas then equalization of pressure of the chambers by outflow of the gases from their respective halves into the other, results in the vanishing of the second term of (10.2.40). In either case, competitive collaboration of the two opposites with unequal resources, rather than annihilation of the weaker by the more resourceful, leads to the state of mutual equilibrium.

In all these instances, the two disjoint opposing parts act in competitive non-reductionist collaboration to generate a moderated and inhibited stasis of the union: this is its only manifestation of the complementary neg-world on its observable physical partner. Thus *cold, vacuum* and *a nonidentical substance* are the negations of *heat* and *matter* — just as $-r \in \mathfrak{R}_+$ is the negation of $r \in \mathbb{R}_+$. These negations as elements of the negworld are no more observable than -5, for example, is to us in our physical world: we cannot collect -5 objects around us or measure the distance between two places to be -100 kilometers; more generally, the set of complex numbers can be considered to constitute the coreals, without which there would have been no zero, no starting initial point in any ordered set, and no "equilibrium" either. Nature, propelled by its unidirectional increasing entropic disorder, without the containing Schrodinger and de Broglie $\lambda = h/p$ waves, would have probably crashed out of existence long ago!

In summary, then, for an interaction $f : C \to C$ and the bijective map $\mathfrak{f}\colon C \to \mathfrak{C}$ corresponding to (10.2.38b), the hierarchal order

$$\begin{aligned}\text{Dynamics of } \mathfrak{f}f\colon C \to \mathfrak{C} \text{ in nature's } \textit{kitchen}\,(C,\mathcal{W}) \times (\mathfrak{C},\mathfrak{W})\\ \longrightarrow \text{Evolution of } f \text{ on } (C,\mathcal{W})^2\\ \longrightarrow \text{Experimental observation in } \mathfrak{D}(f) = C\end{aligned}$$

accompanied by

► Increasing iterates of f, driven by ininality of topology generated on C, constitutes the activating sense of the dynamics, that as we see subsequently, corresponds to the backward, entropy increasing, destabilizing direction of the evolutionary process. The function-multifunction asymmetry between f and f^- generates and sustains this unidirectional ininality, and

► Decreasing iterates of f corresponds to the forward, entropy decreasing, stabilizing direction in the evolving, competitive collaboration of interactions generated by f and f^-,

defines the state of equilibrated stasis schematized in Fig. 10.4. From the discussion in connection with Fig. 10.1 that ininality is an effective expression of a *non-bijective homeomorphism* when the sequence of evolutions (f^n) become progressively more bijective on the saturated open sets of equivalence classes and their respective images, Eq. (10.1.7), it can be argued that *the incentive towards the resulting effective simplicity of invertibility on the definite classes of sets associated with* (f^n) *is responsible for evolutionary dynamics on* C.

This account of "providing a mechanical (i.e., dynamical) explanation of why classical systems behave thermodynamically" [5] is to be compared with [10], see also [31]. The distinctive feature of the present approach is in its use of difference equations rather than the *microscopic* Hamilton differential equations that yield the Liouville equation of *macroscopic* mechanical systems. As so forcefully inquired by Baranger [2], can the emerging evolutionary properties of strongly nonlinear, emergent, self-organizing systems be described by linear (Hamiltonian) differential equations? By employing functional interactions as solutions to difference equations by the technique of graphical convergence of their iterates, we explicitly invoke the immediate past in determining its future and are thereby able to circumvent the issues of time reversal invariance and Poincare recurrence that are inherently associated with the microscopic dynamics of Hamilton's differential equations. This also enables us to avoid direct reference to statistical and probabilistic arguments except in so far as are inherently implied by the Axiom of Choice.

While our preference for unimodal, single-humped, logistic-like difference equations is based on the understanding that only an appropriate juxtaposition of the opposing directional effects — like that of $x-a$ and $b-x$ in the interval $a \leq x \leq b$ — can lead to meaningful emergence and self-organization, it is also well known [17, 20] that time evolution of a discrete model and its continuous counterpart can be so different as to have no apparent correlation with each other. Thus the logistic differential equation

$$\dot{x} = g(x) := (\lambda - 1)x\left(1 - \frac{\lambda}{\lambda - 1}\,x\right) \qquad (10.2.41a)$$

having the same equilibrium fixed points $x = 0$, $x_* = (\lambda - 1)/\lambda$ as the discrete version, has the harmless "trivial" solution

$$\begin{aligned} x(t) &= \frac{x_0\, x_* e^{(\lambda-1)t}}{x_* + x_0(e^{(\lambda-1)t} - 1)} \\ &\xrightarrow{t\to\infty} x_*. \end{aligned} \qquad (10.2.41b)$$

Compared with the structurally rich multifunctional graphical convergence leading to chaos and entropic drive of the discrete form, the tranquil differential variety can only produce a simple *monotonic* convergence to the basic fixed point x_* which is responsible for the complex dynamics of the former; in fact, linear systems can only admit stable or exponentially growing oscillatory or non-oscillatory solutions. This apparently surprising, though not unexpected, result arises from the fundamental difference in the bifurcation characteristics of these equations: *the availability of additional spatial dimensions allows the dynamical system a greater latitude in its evolution so that the complex hierarchal structure generated by iteration of one-dimensional maps are absent in flows under Hopf bifurcations.* In fact, for Eq. (10.2.41a), 0 is unstable and x_* stable for all values of $\lambda > 1$ because $g'(x) = (\lambda - 1) - 2\lambda x$ is positive at $x = 0$ and negative for $x = x_*$. In contrast with the bifurcation dominated rich and varied dynamics of maps, bifurcation-less evolution of vector fields on the real line — capable only of monotonically converging to fixed points or diverging to infinity without any oscillations or other dynamically interesting features — precludes any qualitative change in the evolution of solutions like (10.2.41b).

The alert reader would not have failed to notice that our use of the qualifiers "discipliner", "inhibitor", "stasis" signifying a condition of balance among various forces of the forward-backward opposites, can only provisional as the existence of a set of negatives for every positive as postulated in (10.2.38b) does not necessarily imply that their natural directions interact to generate a smaller positive. This crucial dynamical manifestation of matter-negmatter is provided by the second law of thermodynamics which is formalized through the concept of inverse and direct limits that incorporates directional arrows in their definitions.

Direct Limit, Inverse Limit, Irreversibility

Otherwise put, every "it" — every particle, every field of force, even space-time continuum itself — derives its function, its meaning, its very existence, from answers to yes-no questions, binary choices, bits. "It from bit" symbolizes the idea that every item of the physical world has at bottom — at a very deep bottom, in most instances — an immaterial source and explanation, that which we call reality arises in the last analysis from the posing of yes-no questions; in short, that all things physical are information-theoretic in origin and this is a participatory universe.

J. A. Wheeler (1990)

These limits also known as colimit and limit respectively, with the confusing terminology arising possibly from the fact that the "natural" direction in convergence theory is a reverse direction where the counting index increases with decreasing size of the defining open sets, is summarized in Fig 10.5a.

Direct limit. The direct (or inductive) limit is a general method of taking limits of a "directed families of objects". Let $(\mathbb{D}, \preceq)$ be a directed partially ordered set, $\{X_\kappa\}_{\kappa\in\mathbb{D}}$ a family of spaces, and $\eta_{\alpha\beta} : X_\alpha \to X_\beta$ a family of continuous connecting maps *oriented along* $(\mathbb{D}, \preceq)$ satisfying the properties

$$\eta_{\alpha\alpha}(x) = x, \qquad \text{for all } x \in X_\alpha \tag{10.2.42a}$$

$$\eta_{\alpha\gamma} = \eta_{\beta\gamma} \circ \eta_{\alpha\beta}, \quad \text{for all } \alpha \preceq \beta \preceq \gamma. \tag{10.2.42b}$$

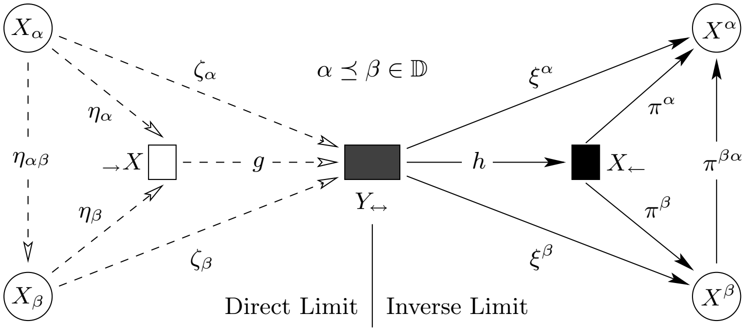

Fig. 10.5*a*. Direct and inverse limits of commutative diagrams. η_α, π^α are projections and $\eta_{\alpha\beta}$, $\pi^{\beta\alpha}$ are connecting maps.

Then the pair $(X_\alpha, \eta_{\alpha\beta})$ is called a *direct* (or *inductive*) *system over* $\mathbb{D}$. The image of a $x_\alpha \in X_\alpha$ under any connecting map is called the *successor of* x_α, and a direct system $(X_\alpha, \eta_{\alpha\beta})$ yields a direct limit space $_\rightarrow X$ as follows. Let $X = \uplus_\kappa X_\kappa$ be the disjoint union of $\{X_\kappa\}$ and let $x_\alpha \in X_\alpha$. The class of elements

$$[x_\alpha] = \{x_\beta \in X_\beta \colon \exists\, \gamma \succeq \alpha,\, \beta \text{ such that } \eta_{\alpha\gamma}(x_\alpha) = \eta_{\beta\gamma}(x_\beta)\} \qquad (10.2.43)$$

with a common successor in the union constitutes an equivalence class of x_α: while reflexivity and symmetry are obvious enough, transitivity of $\sim_{\mathrm{D}}$ follows from

$$\begin{aligned}[][x_\alpha \sim_{\mathrm{D}} x_\beta] \bigwedge [x_\beta \sim_{\mathrm{D}} x_\gamma] &\Longrightarrow \exists\, \delta, \epsilon \succeq \alpha,\, \beta,\, \gamma \text{ s.t. } [\eta_{\alpha\delta}(x_\alpha) = \eta_{\beta\delta}(x_\beta)] \\ &\qquad\qquad \bigwedge [\eta_{\beta\epsilon}(x_\beta) = \eta_{\gamma\epsilon}(x_\gamma)] \\ &\Longrightarrow \exists\, \zeta \succeq \delta,\, \epsilon \text{ s.t. } \eta_{\alpha\zeta}(x_\alpha) = \eta_{\gamma\zeta}(x_\gamma) = \eta_{\beta\zeta}(x_\beta),\end{aligned}$$

with two elements in the disjoint union being equivalent iff they are "eventually equal" in the direct system. Then the quotient space

$$_\rightarrow X \overset{\text{def}}{=} \biguplus_\kappa X_\kappa / \sim_{\mathrm{D}} \qquad (10.2.44a)$$

of the disjoint union of $\{X_\kappa\}$ modulo $\sim_{\mathrm{D}}$ is known as the *direct,* or *inductive, limit* of the system $(X_\alpha, \eta_{\alpha\beta})$. The pair $({}_\rightarrow X, \eta_\alpha)$ must be universal in the sense that *if there exists* any other such pair $(Y_\leftrightarrow, \zeta_\alpha)$ there is a unique morphism $g \colon {}_\rightarrow X \to Y_\leftrightarrow$ with the respective sub-diagrams commuting for all $\alpha \preceq \beta \in \mathbb{D}$. If

$$p \colon \biguplus_\kappa X_\kappa \to {}_\rightarrow X$$

is the projection, then its restriction

$$\eta_\kappa \colon X_\kappa \to {}_{\to}X$$

maps each element to its equivalence class, see Fig. 10.5a; hence

$$_{\to}X = \bigcup_\kappa \eta_\kappa(X_\kappa) \qquad (10.2.44b)$$

implies that ${}_{\to}X$ is not empty whenever at least one X_α is not empty and the algebraic operations on ${}_{\to}X$ are defined via these maps in an obvious manner. Clearly, $\eta_\alpha = \eta_\beta \eta_{\alpha\beta}$.

If the directed family is a family of disjoint sets $(X_\alpha)_\alpha$, with each X_α the domain of an injective branch of f that partitions $\mathcal{D}(f)$, then the direct limit ${}_{\to}X$ of (X_α) is isomorphic to the basic set X_{B} of Fig. 10.1, where $\eta_{\alpha\beta}(x_\alpha)$ is the element of $[x_\alpha]_f$ in X_β.

Inverse Limit. The inverse (or projective) limit is a construction that allows the "glueing together" of several related objects, the precise nature of the glueing being specified by morphisms between the objects. Let $(\mathbb{D}, \preceq)$ be a directed partially ordered set, $\{X^\alpha\}_{\alpha\in\mathbb{D}}$ a family of spaces, and $\pi^{\beta\alpha} \colon X^\beta \to X^\alpha$ a family of continuous connecting maps *oriented against* $(\mathbb{D}, \preceq)$ satisfying the properties

$$\pi^{\alpha\alpha}(x) = x, \qquad \text{for all } x \in X^\alpha \qquad (10.2.45a)$$

$$\pi^{\gamma\alpha} = \pi^{\beta\alpha} \circ \pi^{\gamma\beta}, \qquad \text{for all } \alpha \preceq \beta \preceq \gamma. \qquad (10.2.45b)$$

Then the pair $(X^\alpha, \pi^{\beta\alpha})$ is called an *inverse,* or *projective*, *system* over $\mathbb{D}$. The image of a $x^\beta \in X^\beta$ under any connecting map is the *predecessor of* x^β and the *inverse,* or *projective, limit*

$$X_{\leftarrow} \overset{\text{def}}{=} \{x \in \prod_\kappa X^\kappa \colon p^\alpha(x) = \pi^{\beta\alpha} \circ p^\beta(x) \text{ for all } \alpha \preceq \beta \in \mathbb{D}\}, \qquad (10.2.45c)$$

of $(X^\alpha, \pi^{\beta\alpha})$, where

$$p^\alpha \colon \prod_\kappa X^\kappa \to X^\alpha$$

is the projection of the product onto its components, is a subspace of $\prod X^\kappa$ with the property that a point $x = (x^\kappa) \in \prod X^\kappa$ is in $X_{\leftarrow}$ iff its coordinates satisfy $x_\alpha = \pi^{\beta\alpha}(x_\beta)$ for all $\alpha \preceq \beta \in \mathbb{D}$. Every element of $X_{\leftarrow}$ has a unique representation in each X_κ, but an element of X_κ may correspond to many points of the limit. As for direct limits, the pair $(X_{\leftarrow}, \pi^\alpha)$ must be universal such that the existence of any other such pair $(Y_{\leftrightarrow}, \xi^\alpha)$ implies the existence of a unique morphism $h \colon Y_{\leftrightarrow} \to X_{\leftarrow}$ with the respective sub-diagrams commuting for all $\alpha \preceq \beta \in \mathbb{D}$. The sets $(\pi^\alpha)^{-1}(U)$, $U \subseteq X^\alpha$ open, is a topological basis of $X_{\leftarrow}$, and all pairs of points of $X_{\leftarrow}$ obeying $x_\alpha = \pi^{\beta\alpha}(x_\beta)$ for $\alpha \preceq \beta$ is identical iff their images coincide for every α. The restrictions

$$\pi^\alpha \colon X_{\leftarrow} \to X^\alpha$$

of p^α is the continuous *canonical morphism* of $X_\leftarrow$ into X^α with two points of $X_\leftarrow$ being identical iff their images coincide for every α.

Straightforward examples of these limits are

(a) Let $\{X_k\}_{k\in\mathbb{Z}_+}$ be an increasing family of subsets of a set X, and let $\eta_{mn} : X_m \to X_n$ be the inclusion map for $m \leq n$. The direct limit of this system is

$$ {}_\rightarrow X = \bigcup_{k=1}^{\infty} X_k \qquad (10.2.46) $$

with the inclusion functions mapping from each X_k into this union. Generally, if $\mathbb{D}$ is any directed partially ordered set with a greatest element ω, then the direct limit of any corresponding direct system is isomorphic to X_ω and the canonical morphism $\eta_\omega : X_\omega \to {}_\rightarrow X$ is an isomorphism.

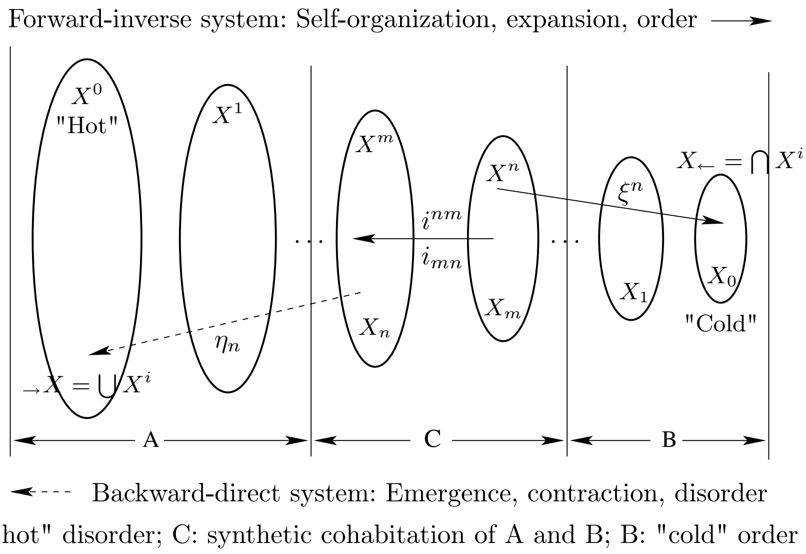

Fig. 10.5*b*. Direct-inverse limits for a family of nested subsets of a set X, with the direction of "order" $\mathbb{O}$ and "disorder" $\mathbb{D}$ — to be understood as implying smaller and larger multiplicities of the state — shown opposite so that all maps of the systems are now in the same direction. In the absence of a direct component, the inverse on its own would cause bottom-up, *self-organized* "cold death" to $X_\leftarrow$; if the inverse system were absent, *emergent*, "heat death" from the lone effect of the direct system would follow in ${}_\rightarrow X$, with each acting essentially as a gradient dissipator of the other. The nested decreasing subsets denote stability inspired *expansion* and self-organization as the system's response of utilizing "every possible avenue in sucking orderliness from its environment" to counter attempts to move it away from thermodynamic equilibrium, while the increasing supersets signify instability driven contraction and emergence. Compare Fig. 10.10*a*.

(b) Let $\{X^k\}_{k\in\mathbb{Z}_+}$ be a decreasing family of subsets of X, and let $\pi^{nm}: X^n \to X^m$ be the inclusion map for $m \leq n$. Since the inverse limit consists of only those points of the cartesian product whose "eventual" coordinate can be assigned independently,

$$X_{\leftarrow} \simeq \bigcap_{k=1}^{\infty} X^k \tag{10.2.47}$$

might be empty even though $X^k \neq \emptyset$ for each k, with $\pi^{kk} = \mathbf{1}$ being the identity map. What this result means is that the limit $X_{\leftarrow}$ must have all its components from the intersection only. Thus the inverse limit of $X_1 = [0,1]$, $X_2 = [0, 0.6]$ and $X_3 = [0, 0.2]$ is of the form $X_{\leftarrow} = \{(x_1, x_2, x_3) : x_1 \in X_3,\ x_2 \in X_3,\ x_3 \in X_3\}$ with the first and second coordinates considered to be elements of X_1 and X_2 respectively, by the inclusion map.

The consequence of these limit constructs in providing a dynamical basis to Postulates NEG-1 and NEG-2 of the exclusion space is contained in the following arguments. For a given resource λ, the inverse and direct limits $X_{\leftarrow}$ and ${}_{\rightarrow}X$, in competitive collaboration with each other, can be taken to represent respectively the *anabolic synthesis* of expansion, order, entropy-decrease and *catabolic analysis* of contraction, disorder, entropy-increase of the corresponding systems[15] leading to the dynamically equilibrated state $X_{\leftrightarrow}$: recall that everything else remaining the same, "hot" objects have higher entropy than "cold" ones, and when two bodies of different resources are brought in contact, entropy of the hot body *decreases* while that of the cold body *increases* such that the entropy decrease in the former is more than compensated by its increases in the later. This spontaneous flow of "heat" is associated with an overall entropy increase that continues till the combined entropy is a maximum. This is the essence of entropy production in the universe at the expense of exergy of the more resourceful constituent that in simple terms represents the opposition of a cold stable system to the urge of a hot unstable component to stabilize at its expense. The second law represents a straightforward stipulation that a part of the useful energy of a closed system must always be wasted as heat with the entropy being a quantitative measure of the amount of thermal energy not available for doing work, of the tendency for all matter and energy in the universe to evolve toward a dead state of inert uniformity. In the absence of the direct limit component, however, the inverse system would proceed to its logical destination of $X_{\leftarrow}$ leading to its

[15] *Metabolism* comprises the chemical processes taking place within a living cell or organism involving consumption and breakdown of complex compounds necessary for the maintenance of life, often accompanied by liberation of energy and waste products. It is the major process of living systems affecting all its chemical processes, consisting of a series of changes in an organism by means of which food is manufactured and utilized, and waste materials are eliminated. Metabolism is broadly subdivided into two opposing parts: *anabolic synthesis* of simple substances into complex materials is its constructive phase, and *catabolic analysis* of complex substances into simpler ones is the destructive.

minimum-entropy frozen "cold death", which translated to practical terms requires the whole system to acquire the unmoderated properties of the infinite colder reservoir. Inverse limits, however, demand the existence of connecting maps *opposing* $\mathbb{O}$; this manifests itself through generation of the reverse direction $\mathbb{D}$ of the direct limit which acting on its own would likewise lead to a maximum-entropy roasted "heat death" condition of ${}_{\rightarrow}X$. In communion with each other, $X_{\leftrightarrow}$ shares properties of both the opposites with the equilibrium representing some intermediate state $X^m = X_n$ of Fig. 10.5*b*. Physically this represents either (i) a hot body A^* interacting with another body B to yield the compound system $A^* (\equiv X_0) + B (\equiv X^0)$ which then evolves with time, or (ii) an infinite reservoir A^* that induces a temperature gradient in B; in this case the heat source remains external to the system. This reading of the dual limits, suggested by the directions of Fig. 10.5*b* representing converging sequences generated by points in the respective $\{X^\alpha\}$ and $\{X_\alpha\}$, can be viewed to be the basis of our postulate of an exclusion space leading to dynamical homeostasis, with the direction of the inverse limit being effectively inhibited by that of the direct limit. A second related interpretation is to consider, by the definition of footnote 7, the family of spaces and the restrictions of the associated projections to generate final and initial topologies on ${}_{\rightarrow}X$ and $X_{\leftarrow}$ respectively. The dynamically equilibrated steady state

$$X_{\leftrightarrow} = X^m = X_n \tag{10.2.48}$$

is therefore in an ininal state because all sub-diagrams of Fig. 10.5*a* must commute and the connecting sequences converge to the respective limits iff these carry the final and initial topologies of the direct and initial systems. Note that the dynamic equilibrium of (10.2.48) is effectively a saddle-node centre manifold, and is in fact the state $_{\text{eq}}$ of Eq. (10.2.2).

A thermodynamic analysis of the preceding heuristic rationale for the existence of a $X_{\leftrightarrow}$ will be given below that reduces the inverse-direct system to a coupled engine-pump dual with the natural inverse-limit engine $E : T_h \to T_c$ generating, under proper condition of irreversibility, a direct-limit pump $P : T_c \to T_h$ such that $X_{\leftrightarrow}$ is characterized by an equilibrium temperature $T \in [T_c, T_h]$.

In applying these considerations to the iterative evolution of maps, we take the domain of the interaction f to be a disjoint union C of a physical space A and an exclusion space B, when f generates bi-directional forward-backward arrows on C that are quite distinct from the catabolic-direct and anabolic-inverse limits. Accordingly two sets of arrows, the forward-inverse and backward-direct, are imposed on an evolving system and the character of the system depends on which of the two plays the role of an activating partner and which the restraining, representing a dynamical balance between the competitive collaboration of forward, self-organization and backward emergence, with new structures appearing only for the first few steps that is subsequently self-organized into a composite whole. This interpretation of the restoring effects implies that with appropriate interactions f, even extreme irreversibilities of non-injective ill-posedness can be effectively reversed with time, fully or partially depending on the nature of f, through internally generated regulating effects. Irreversibilities therefore need not be only wasteful: given adequate interactive support these can actually be utilized to induce higher-level order and discipline in the otherwise naturally occurring emerging entropic disorder, through a regulated process of adaption and self-organization. We employ this basic characteristic of the synthesis of matter and negative matter in formulating the definitions of complexity and "life" below.

The Lorenz Equation

To fully appreciate these observations and arrive at an understanding of the dynamics of difference equations vis-a-vis differential equations, we consider the Lorenz-Rayleigh-Benard model of two-dimensional convection of a horizontal layer of fluid heated from below involving three dynamical variables: x proportional to the circulatory convection velocity of the fluid that produces the flow pattern with positive x indicating clockwise circulation, y proportional to the temperature difference between the ascending warm and descending cold flows at a given height h, and z proportional to the nonlinear deviation of the vertical temperature profile from equilibrium linearity. The Lorenz equations

$$\dot{x} = \sigma(-x + y) \qquad (10.2.49a)$$

$$\dot{y} = Rx - y - xz \qquad (10.2.49b)$$

$$\dot{z} = xy - bz, \qquad (10.2.49c)$$

with σ the Prandtl number (ratio of the kinematic viscosity of the fluid to its thermal diffusivity), $R = r/r_{\mathrm{c}}$ the relative Rayleigh number (where $r := g\alpha d^3 \triangle T/(\kappa\nu)$ is the Rayleigh number — with g acceleration due to gravity, α, κ, ν coefficients of volume expansion, thermal diffusivity, kinematic viscosity, $\triangle T$ temperature difference between the upper and lower surfaces of the fluid separated by a distance d — and $r_{\mathrm{c}} := (a^2 + \pi^2)^3/a^2 = 27\pi^4/4$ is the critical value that defines $a = \pi/\sqrt{2}$ to give the lowest r at which convection starts), and b (ratio of the width to the height of the region in which convection is occurring), represents a state of competing collaboration between the downward stabilizing arrow of gravity and an upward buoyancy-driven instability of viscous friction and conductive heat losses. The equilibrium fixed point $\dot{\mathbf{x}} = 0$ of supercritical pitchfork bifurcation

$$\dot{\mathbf{x}} = 0 \iff x^3 - b(R-1)x = 0$$

has the roots

$$C_0 = (0,0,0), \qquad \text{all } R \qquad (10.2.50a)$$

$$C_\pm = (\pm\sqrt{b\rho}, \pm\sqrt{b\rho}, \rho), \qquad R > 1,\ \rho = R - 1. \tag{10.2.50b}$$

A linear stability analysis about C_0 requires the characteristic polynomial of the combined linearized equation

$$\dot{\mathbf{x}} = \begin{pmatrix} -\sigma & \sigma & 0 \\ R & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

to satisfy

$$f(\lambda) := (\lambda + b)[\lambda^2 + (1+\sigma)\lambda - \sigma(R-1)] = 0 \tag{10.2.51}$$

with the real eigenvalues

$$\begin{aligned} \lambda_z &= -b \\ \lambda_\pm &= -\frac{1+\sigma}{2} \pm \frac{1}{2}\sqrt{(1+\sigma)^2 + 4\sigma(R-1)} \end{aligned} \tag{10.2.52}$$

in which only $\lambda_\pm$ depends on the control parameter R. It can now be verified that for all positive R, σ and b:

(a) $\underline{R < 1}$: All the zeros λ_z, $-(1+\sigma) \le \lambda_- \le -1$ (upper and lower bounds occurring at $R = 1$ and $R = 0$), $-\sigma \le \lambda_+ \le 0$ (bounds occurring at $R = 0$ and $R = 1$), are negative which means that C_0 is a *stable node.*

(b) $\underline{R = 1}$: λ_z, $\lambda_- = -(1+\sigma)$ are negative and $\lambda_+ = 0$ with corresponding eigenvectors $\mathbf{u}_z = (0,0,1)^{\mathrm{T}}$, $\mathbf{u}_- = (-\sigma,1,0)^{\mathrm{T}}$, and $\mathbf{u}_+ = (1,1,0)^{\mathrm{T}}$; hence C_0 is *marginally* (neutrally) *stable*, leading to its *pitchfork bifurcation.* The three real equilibria for $R > 1$ as given in (c) below merge to the single stable node of $R < 1$ at $R = 1$.

(c) $\underline{R > 1}$: λ_z and λ_- are negative, λ_+ is positive; hence C_0 is an unstable fixed point. The flows along the eigenvectors of λ_z and λ_- are stable that become unstable along the of λ_+ direction. Hence C_0 undergoes a *saddle node* in three dimensions in this parameter range.

Linearization about the two other equilibrium points $C_\pm$ according to $x \mapsto x \mp \sqrt{b\rho}$, $y \mapsto y \mp \sqrt{b\rho}$, and $z \mapsto z - \rho$ leads to the eigenvalue equation

$$\begin{aligned} g(\mu) &:= \begin{vmatrix} \sigma+\mu & -\sigma & 0 \\ -1 & 1+\mu & \pm\sqrt{b\rho} \\ \mp\sqrt{b\rho} & \mp\sqrt{b\rho} & b+\mu \end{vmatrix} \\ &= \mu^3 + (1+b+\sigma)\mu^2 + b(\sigma+R)\mu + 2b\sigma(R-1) = 0 \end{aligned} \tag{10.2.53}$$

Since all its coefficients are positive and $g(0) > 0$ when $R > 1$, there is always a negative real root μ_z of Eq. (10.2.53). At $R = 1$, the three zeros of Eq. (10.2.53) are $\mu_z = -b$, $\mu_- = -(1+\sigma)$ and $\mu_+ = 0$, there are therefore two

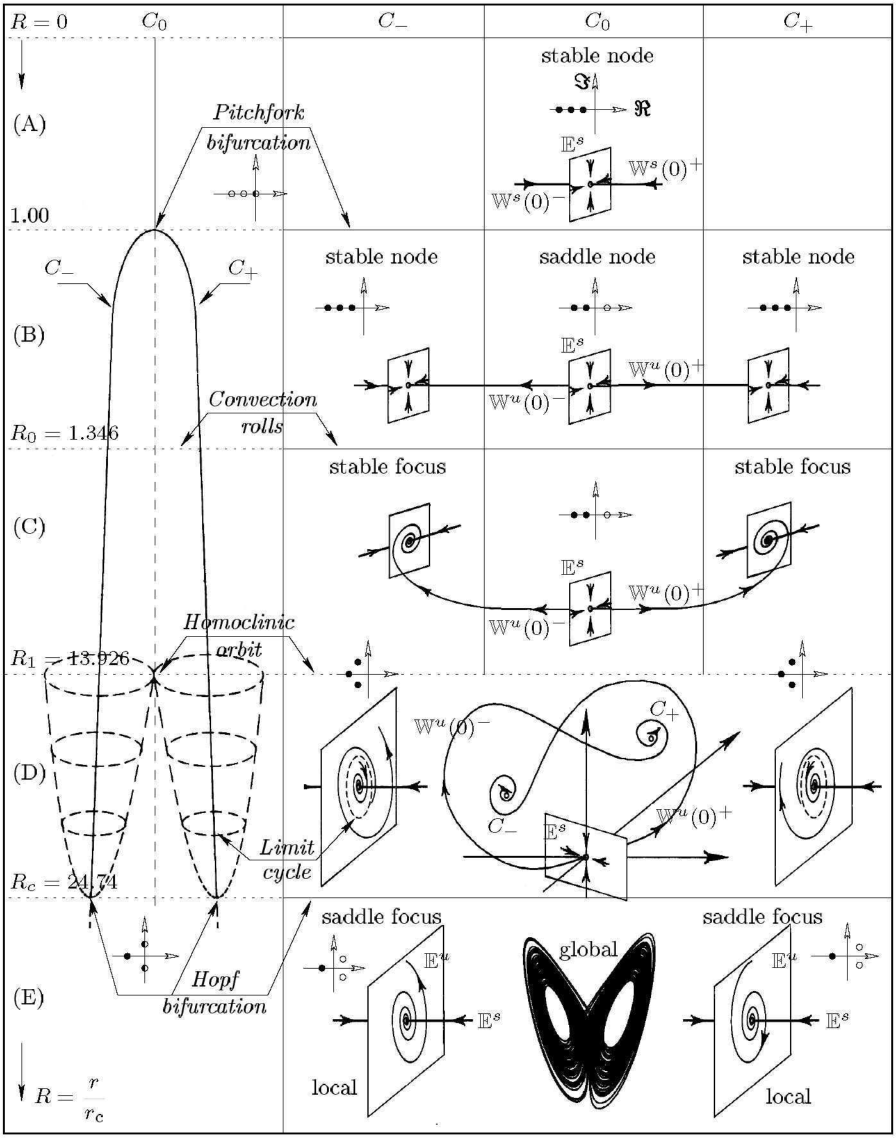

Fig. 10.6. Dynamics of the Lorenz equations. $\mathbb{E}$ and $\mathbb{W}$ are spanned by the respective eigenvectors of $\{\lambda_z, \lambda_-\}$ and λ_+ of Eq. (10.2.52). The local directions of the manifolds in panel (E) are determined by the eigenvectors of $C_\pm = (\pm\sqrt{b\rho}, \pm\sqrt{b\rho}, \rho)$, where $\rho = R - 1$ with R the relative Rayleigh number. Figure adapted from Argyris et al. [1]

stable (negative) roots and one marginally stable 0 root, in agreement with Eq. (10.2.51). From $\mu_z + \Re(\mu_-) + \Re(\mu_+) = -(1+b+\sigma) < 0$, it follows that the two complex roots cross over from negative to positive real parts, for $b = 8/3$ and $\sigma = 10$, when

$$\mu_z = -(1+b+\sigma) = -13.6667,$$
$$\mu_\pm = \pm i\sqrt{\frac{2\sigma b(\sigma+1)}{\sigma-b-1}} = \pm 9.62453\, i,$$

which leads, from $g(\mu_z) = 0$, to the critical magnitude

$$R_c = \sigma\left(\frac{\sigma+b+3}{\sigma-b-1}\right) = \frac{470}{19} \simeq 24.7368$$

of R marking the birth of a subcritical Hopf bifurcation.

The behaviour of the characteristic polynomials $f(\lambda)$ and $g(\mu)$ with variation of R in the range $R < 13.926$ is as follows, see Fig. 10.6. For $R < 1$, $\lambda_\pm$ repel each other but for $1 \le R < 1.346$ the $\mu_\pm$ attract as the graph of $g(\mu)$ moves up until at $R = 1.34561718$ the zeros merge, $g(\mu_-) = g(\mu_+) = 0$, and complex roots appear maintaining $\Re(\mu_-) = \Re(\mu_+) < 0$ which marks the initiation of convective rolls in the flow. At $R = 13.926$, homoclinic orbits starting at the origin along the unstable manifolds return to it as stable manifolds, the real parts thereafter increasing through 0 at $R = R_c$, with μ_z remaining negative along the z-direction for all $R > 1$. Hence

(d) $1.00 \le R < 1.3456 := R_0$, panel (B). The character of the equilibria $C_\pm$ change from nodes to spirals in the first appearance of oscillatory behaviour. This occurs when the graph of $g(\mu)$ becomes tangent to the μ-axis at its turning point for $\Re(\mu_+) = \Re(\mu_-) < 0$. At $R = R_0$, the molecular conduction of this region becomes unstable yielding place to convection rolls of highly structured coherent patterns. This increases the rate of heat transfer reducing the temperature gradient of the system, and constitutes "the system's response to attempts to move it away from equilibrium", [26].

(e) $R_0 \le R < 13.926 := R_1$, panel (C). The trajectory leaving C_0 along the local unstable manifold of λ_+ spirals into the nearer of the two stable manifolds C_- and C_+, tangent to the span of the respective eigenfunctions of μ_-, μ_+. These spirals of unstable manifolds on looping around C_- and C_+ increase in size with increasing R, until at

(f) $R = R_1$ they tend toward C_- and C_+ in wide arcs, eventually returning as *homoclinic orbits* to C_0 in the "infinite period limit" $t \to \pm\infty$. While no qualitative changes in the distribution of the zeros of Eq. (10.2.53) occur at this value of R, the emergence of homoclinic orbits can be attributed to the transformation of Eq. (10.2.53) to a monotonically increasing function of μ for all $R > R_1$. This is a significant event in the time evolution of the Lorenz equations that eventually leads to chaos at $R = R_c$. This mechanism to chaotic transition is common in systems modeled by differential equations and is not — unlike for maps — accompanied by any change in the character of fixed points but is due to interaction of the trajectory with various instabilities.

(g) $R_1 \le R < R_c \simeq 24.7368$, panel (D). As R increases beyond R_1, the monotonically increasing $g(\mu)$ results in the homoclinic orbits transforming to increasing *finite period* unstable orbits that eventually coalesce to disappear in a subcritical Hopf bifurcation at $R = R_c$. These increasingly oscillatory solutions of the *pre-chaotic* range $R_1 < R < 24.06$ travel back and forth between C_- and C_+ many times before finally spiraling into one of them: as R increases in this range, the generated unstable limit cycles repel $\mathbb{W}^u(0)$ so that the branch leaving C_0 in the octant of C_- converges to C_+ and that generated in the octant of C_+ ends up at C_-, with the number of crossings between C_- and C_+ increasing with R before eventually converging to one of them. The unstable limit cycles associated with C_-, C_+ shrink in size as R increases, passing over to a subcritical Hopf bifurcation at $R = R_c$. In the range $24.06 < R < R_c$ although the equilibria $C_\pm$ remain stable, some of the pre-chaotic orbits pass over into true chaos; hence in this region there is a chaotic attractor beside the two spiral attractors. At $R = R_c$, the stable spirals become unstable by absorbing the unstable spirals.

This dynamics of the Lorenz equation summarized in Fig. 10.6 allows us to draw the following correspondences with the logistic interaction $\{f_\lambda\}_{\lambda \in [0,4]}$.

- $0 \le R < 1.00 \Leftrightarrow 0 \le \lambda < 1$, panel (A). Heat is transferred from the hot bottom to the cold top by molecular thermal conduction. The tendency of the warm, lighter fluid to rise is inhibited by viscous damping and loss by conduction from the hot fluid to the surrounding cooler medium, and the temperature varies linearly with the height of separation between the plates. Recall that the only logistic fixed point $x_0 = 0$ is stable in this range, like the Lorenz C_0. See Fig. 10.8*a*
- $1.00 \le R < 1.3456 \Leftrightarrow 1 \le \lambda < 2$, panel (B). This λ-region of loss of stability of x_0 at $\lambda = 1$ and the simultaneous birth of a new stable fixed point marks the onset of a radial R-interaction between the now unstable C_0 and the new stable pair $C_\pm$, Fig. 10.8*a*.
- $1.3456 \le R < 13.926 \Leftrightarrow 2 \le \lambda < 3$, panel (C). Oscillations occur in the stable evolution of the logistic map, Fig. 10.8*a*(iv), corresponding to the appearance of the circular convective rolls in the Lorenz equations along the second angular θ-direction consequent of the appearance of complex roots of $g(\mu)$, Eq. (10.2.53).
- $13.926 \le R < 24.7368 \Leftrightarrow 3 \le \lambda < 1 + \sqrt{6} = 3.4495$, panel (D). This region of the initiation of period doubling of the one-dimensional map relates to the homoclinic orbit and the unstable limit cycles representing radial interaction between C_0 and $C_\pm$ that activates the third angular φ-direction at C_0. Note that as in the logistic interaction, this R-region is distinguished by the coexistence of the opposite directions due to the stable fixed points C_- and C_+ corresponding to the stable 2-cycle of the map of Fig. 10.8*b*.

The important point to note here is that *unlike for period doubling of the logistic map, the supercritical pitchfork bifurcation in a multidimensional space enables the unstable C_0 to interact with the stable $C_\pm$ by opening up new pathways along the angular coordinate directions.* In the one dimensional logistic case where the luxury of the new directions acting as additional tunable parameters are unavailable, *a tiered hierarchal communication system is established between the unstable and stable points* in order to utilize the additional λ-resource available to carry the evolutionary dynamics forward. In fact, compared to the sufficient conditions

$$f = 0, \quad \frac{\partial f}{\partial x} = 0, \quad (x, \mu) = (0, 0) \tag{10.2.54a}$$

and

$$\begin{aligned} \frac{\partial f}{\partial \mu} &= 0, & \frac{\partial^2 f}{\partial x^2} &= 0, \\ \frac{\partial^2 f}{\partial x \partial \mu} &\neq 0, & \frac{\partial^3 f}{\partial x^3} &\neq 0 \end{aligned} \tag{10.2.54b}$$

for non-hyperbolicity and pitchfork bifurcation respectively of a one-parameter, one-dimensional vector field $\dot{x} = f(x, \mu)$, a one-dimensional map $x \mapsto f(x, \mu)$ with non-hyperbolic fixed points

$$f = 0, \quad \frac{\partial f}{\partial x} = \pm 1, \quad (x, \mu) = (0, 0), \tag{10.2.55a}$$

not only undergoes pitchfork bifurcation at $\partial f / \partial x = 1$ for the same conditions as given by Eq. (10.2.54*b*), but more importantly a period doubling bifurcation appears whenever the non-hyperbolic slope $\partial f / \partial x = -1$ emerges and the second iterate of the map passes through a pitchfork

$$\begin{aligned} \frac{\partial f^2}{\partial x} = 1, \quad \frac{\partial f^2}{\partial \mu} &= 0, & \frac{\partial^2 f^2}{\partial x^2} &= 0, \\ \frac{\partial^2 f^2}{\partial x \partial \mu} &\neq 0, & \frac{\partial^3 f^2}{\partial x^3} &\neq 0 \end{aligned} \tag{10.2.55b}$$

at (x, μ). More generally, any increase in λ is gainfully employed by the logistic map through a series of period doublings such that a 2^N cycle is generated to effectively utilize the resource λ in N bifurcations, as can be verified from Figs. 10.8*b*, *c* and 10.8*d* that show how the emerging structure develops in N steps terminating with the period-doubling-pitchfork

$$\frac{\partial f^{2^{N-1}}}{\partial x} = -1, \quad \text{(period-doubling)} \tag{10.2.56a}$$

$$\frac{\partial f^{2^N}}{\partial x} = 1, \quad \text{(pitchfork)} \tag{10.2.56b}$$

combination at 2^{N-1} stable-unstable fixed points marking the complete utilization of λ, with the slopes of f^{2^N} and $f^{2^{N-1}}$ simultaneously moving out of the stable unit interval in *opposite directions* into the unstable region $|x| > 1$, in the classic bidirectional competitive collaboration mode. In the absence of this typical double bound of the stable region for differential equations, the possible structures supported by these dynamical systems are comparatively simpler. Specifically it does not possess the hierarchal towered form that is the characteristic feature of two-component ill-posed maps such as the logistic where Eqs. (10.2.56a,b) actually determine the fixed-point x_* and the corresponding λ-value of the end of period 2^{N-1} and beginning of period 2^N. It is this distinction in the relationship between the stable and unstable points that is responsible for the difference between arbitrary complex systems and dissipative structures made below.

► $R_c \leq R \Leftrightarrow \lambda_1 < \lambda \leq 4$, panel (E). This R-ray symbolizing total chaos, is characterized as in the logistic case, by the complete lack of stabilizing effects, as the orbits generated by C_- and C_+ endlessly wander between them. Unlike the one-dimensional map, however, the three dimensional differential system does not display characteristic bifurcations beyond R_c, taking advantage instead of the added dimensional latitude in generating an entangled attractor with non-periodic orbits and sensitivity to initial conditions.

Although it is possible, as has been argued above, to establish an overall correspondence between the dynamics of discrete and continuous systems, a careful consideration reveals some notable fundamentally distinctive characteristics between the two that ultimately reflects on the higher number of space dimensions — (r, θ, φ) in the Lorenz case — available to the differential system[16]. This has the consequence that continuous time evolution governed by differential equations is reductionally well-defined and unique — unlike in the discrete case when ill-posedness and multifunctionality forms its defining character — with the system being severely restrained in its manifestation, not possessing a set of equivalent yet discernible possibilities to choose from. In fact, the dynamics of differential equations cannot generate attractors composed of isolated points like the Cantor set, and *it is our premise that the kitchen of Nature functions in an one-dimensional iterative analogue,*

[16] Thus, for example, as in Eq. (10.2.41a,b), the equivalent *Lorenz difference equation*

$$\begin{aligned} x_{n+1} &= x_n(1-\sigma) + \sigma y_n \\ y_{n+1} &= Rx_n - x_n z_n \\ z_{n+1} &= z_n(1-b) + x_n y_n \end{aligned}$$

would have a distinct and different dynamical evolution that is expected to have little bearing or similarity with the solution of its differential counterpart (10.2.49a–c).

not merely to take advantage of the multiplicities inherent therein, but more importantly to structure its dynamical evolution in a hierarchal canopy, so essential for the evolution of an interactive, non-trivial, complex system. The 3-dimensional serving table of physical space only provides a convenient and palatable presentation of nature's produce in its uni-dimensional kitchen. A closed system can gain overall order while increasing its entropy by some of the system's macroscopic degrees of freedom becoming more organized at the expense of microscopic disorder. In many cases of biological self-assembly, for instance metabolism, the increasing organization of large molecules is more than compensated by the increasing disorder of smaller molecules, especially water. At the level of whole organisms and longer time scales, though, biological systems are open systems feeding on the environment and dumping waste into it.

The special significance of one-dimensional dynamics relative to any other finds an appealing substantiation from the following interpretation of the Sharkovskii Theorem. Recall that the distinguished Sharkovskii ordering

$$3 \succ 5 \succ 7 \succ \cdots \succ 2\cdot 3 \succ 2\cdot 5 \succ 2\cdot 7 \succ \cdots \succ 2^n\cdot 3 \succ 2^n\cdot 5 \succ 2^n\cdot 7 \succ \cdots \succ 2^n \succ \cdots \succ 2^2 \succ 2 \succ 1$$

of positive integers implies the Sharkovskii Theorem which states that if $f:[a,b]\to\mathbb{R}$ is a continuous function having a n-periodic point, and if $n \succ m$, then f also has a m-periodic point: observe the significance of the upper and lower bounds of this ordering. Noting that the periodicity of an f-interaction between two spaces essentially denotes the number of independent degrees of freedom required to completely quantify the dynamics of f, it is inferred that while a fixed point of "dimension" 1 embodies the basic informations of all other periods, a period-3 embodies every other dimension within itself. Hence it can be concluded that dynamics on 1-dimension, by being maximally restrained compared to any other, allows for the greatest emergence of structures as mutifunctional graphical limits, while dimension 3 by being the least restrained is ideally suited for an outward well-defined, and aesthetically appealing, simultaneous expression of the multitude of eventualities that the graphical limits entail.

The convection rotating cells of the Lorenz system that appear spontaneously in the liquid layer when heated from outside is an example of Prigogine's *dissipative structure* [15]. At first when the temperature of the bottom plate T_h is equal to that of the top T_c, the liquid will be in equilibrium with its environment. Then as the temperature of the bottom is increased, the fluid resists the applied temperature gradient $\Delta T = (T_h - T_c) \backsim R$ by setting up a backward arrow of inter-molecular conductive dissipation, and the temperature increases linearly from top to bottom to establish thermal equilibrium in the fluid. If the temperature of the bottom is increased further, there will be a far from equilibrium temperature T_0 corresponding to R_0 of Fig. 10.6 at which the system becomes unstable, the incoherent molecular conduction yields place to coherent convection, and the cells appear increasing the rate of dissipation. The appearance of these ordered convective structures — a "striking example of emergent coherent organization in response to an external energy input" [28] — dissipates more energy than simple conduction, and convection becomes the dominant mode of heat transfer as R increases further. The microscopic random movement of conduction spontaneously becomes macroscopically ordered with a characteristic correlation length generated by convection. The rotation of the cells is stable and alternates between clockwise to counter-clockwise horizontally, and there is spontaneous symmetry breaking.

According to Schneider and Kay [28], the basic role of dissipative structures, like the Lorenz convection cells, is to act as *gradient dissipators* by "continually sucking orderliness from its environment" in hindering motion of the system away from equilibrium due to the increasing temperature gradients. The dissipative structures increase the rate of heat transfer in the fluid thereby utilizing this exergy in performing useful work in generating the structures. With increasing gradient, more work needs to be done to maintain the increased dissipation in the far-from-equilibrium state, more exergy must be destroyed in creating more entropy, the boundary layers become thinner, and the original vertically uniform temperature profile is restored in the bulk of the fluid. The structures developed in the Lorenz system thus organize the disorder of the backward convective cells by dissipation of an increasing amount of exergy in the activating, forward "sucking-orderliness" direction of heating.

Thermodynamics of Bidirectionality: Optimized Adaptation in Engine-Pump Duality

They know enough who know how to learn.

Henry Adams

This subsection is an investigation into the relationship of our steady state $X_{\leftrightarrow}$ to the entropy principle of non-equilibrium thermodynamics. In recent papers Dewar [9] establishes the Maximum Entropy Principle for stationary states of open, non-equilibrium systems by maximizing the path information entropy $S = -\sum_{\Gamma} p_{\Gamma} \ln p_{\Gamma}$ with respect to p_{Γ} subject to the imposed constraints. In this non-equilibrium situation, the maximum entropy principle amounts to finding the most probable history realizable by the largest number of microscopic *paths* rather than microscopic *states* typical of Boltzmann-Gibbs equilibrium statistical mechanics. This approach to non-equilibrium MEP is supported by many investigations: the earth-atmosphere global fluid system, for example, is believed to operate such that it generates maximum potential energy and the steady state of convective fluid systems, like that of the Lorenz model, have been suggested to represent a state of maximum convective heat transfer, [23].

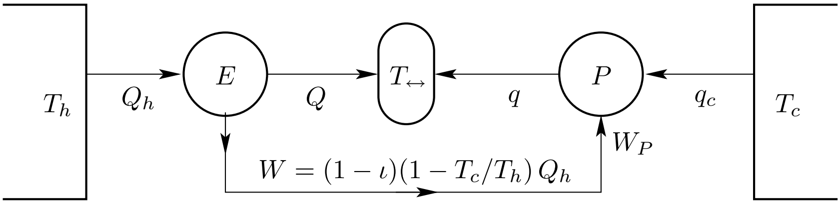

Fig. 10.7. Reduction of the dynamics of opposites of Fig. 10.5*b* to an equivalent engine-pump thermodynamic system. The fraction $W = (1-\iota)W_{\rm C}$ of the available maximum reversible work $W_{\rm C} = \eta_{\rm C}\, Q_h := (1 - T_c/T_h)\, Q_h$ of a reversible engine operating between $[T_c, T_h]$ is internally utilized to self-generate a heat pump P to inhibit, by gradient dissipation, the entropy that would otherwise be produced in the system. This permits decoupling natural irreversibility to a reversible engine-pump dual that uses the fraction ι of the available exergy in running the pump. The coefficient of performance $q/W = q/(q - q_c) = T_{\leftrightarrow}/(T_{\leftrightarrow} - T_c)$ of P establishes the reverse arrow of $q := q_c + W$. The two parameters $T_{\leftrightarrow}$ and ι are obtained as described in the text.

An effective reduction of the inverse-direct model of Fig. 10.5*b* as a coupled thermodynamic engine-pump system is illustrated in Fig. 10.7 in which heat transfer between temperatures $T_h > T_c$ is reduced to a engine E-pump P combination operating respectively between temperatures $T < T_h$ and $T_c < T$. We assume that a complex adaptive system is distinguished by the full utilization of the fraction $W := (1-\iota)W_{\rm C} = (1-\iota)\eta_{\rm C} Q_h = (1-\iota)(1 - T_c/T_h)Q_h$ of the work output of an imaginary reversible engine running between temperatures T_h and T_c, to generate a pump P working in competitive collaboration with a reversible engine E, where the irreversibility index

$$\iota \overset{\text{def}}{=} \frac{W_{\rm C} - W}{W_{\rm C}} \in [0, 1] \tag{10.2.57}$$

accounts for that part $\iota W_{\rm C}$ of available energy (exergy) that cannot be gainfully utilized but must be degraded in increasing the entropy of the universe. The self-induced pump effectively decreases the temperature gradient $T_h - T_c$ operating the engine to a value $T_h - T$, $T_c \le T < T_h$, thereby inducing a degree of dynamic stability to the system.[17] With $q = q_c + (1-\iota)W_{\rm C} = q_c + W_P$, the coefficient of performance $\zeta_P = q/W_P = T/(T - T_c)$ of P yields

$$q = (1-\iota)Q_h \left(\frac{T}{T_h}\right)\left(\frac{T_h - T_c}{T - T_c}\right).$$

Let the irreversibility ι be computed on the basis of dynamic equilibrium[18]

[17] More generally, W is to be understood to be indicative of the exergy of Eq. (10.2.2).

[18] Note that this is $W_E = W_P = \iota W_{\rm C}$.

$$Q_h\left(\frac{T_h-T}{T_h}\right):=W_E(T)=W_P(T):=q\left(\frac{T-T_c}{T}\right)$$

of the engine-pump system; hence

$$\begin{aligned}\iota &= \frac{T-T_c}{T_h-T_c} \\ &= \frac{(T_h-T_c)-(T_h-T)}{T_h-T_c}\end{aligned} \tag{10.2.58}$$

where $T_h - T_c$ represents the original reversible work that is split up into the non-entropic $T_h - T$ shaft output internally utilized to generate the pump P, and a $T - T_c$ manifestation of entropic work by P with the equilibrium temperature T defining this recursive dynamics. The irreversibility ι can be taken to have been adapted by the engine-pump system such that the induced instability due to P balances the imposed stabilizing effort of E to the best possible advantage of the system and its surroundings. This the system does by adapting itself to a state that optimizes competitive collaboration for the greatest efficiency consistent with this competitiveness. This distinguishing feature of the non-equilibrium situation with corresponding equilibrium case lies in the mobility of the defining temperature T: for the introverted self-adaptive systems, the dynamics organizes to the prevailing situation by best adjusting itself *internally* for maximum possible global advantage.

Define the equilibrium steady-state representing $X_{\leftrightarrow}$ of optimized E-P adaptability between E and P be given in terms of the adaptability function

$$\alpha_P(T_P):=\eta_E\zeta_P=\left(\frac{T_h-T_P}{T_P-T_c}\right)\left(\frac{T_P}{T_h}\right)$$

that represents an effective adaptive efficiency of the engine-pump system to the environment (T_c, T_h). Hence

$$T_P=\frac{1}{2}\left[(1-\alpha_P)T_h+\sqrt{(1-\alpha_P)^2T_h^2+4\alpha_PT_hT_c}\right]. \tag{10.2.59a}$$

Alternatively if the system induces P to act as a refrigerator rather than a pump then the defining equations, with $\zeta_R := q_c/(q-q_c) = T_c/(T_R-T_c)$, become

$$q_c=(1-\iota)Q_h\left(\frac{T_c}{T_h}\right)\left(\frac{T_h-T_c}{T_R-T_c}\right),$$

with the adaptability criterion

$$\alpha_R(T_R):=\eta_E\zeta_R=\left(\frac{T_h-T_R}{T_R-T_c}\right)\left(\frac{T_c}{T_h}\right)$$

leading to

$$T_R = \frac{(1+\alpha_R)T_h T_c}{T_c + \alpha_R T_h}. \quad (10.2.59b)$$

For the reversible $(\iota = 0) \Rightarrow (T = T_c)$, $\alpha \to \infty$ case, with no entropy production and no generation of P, the resulting inverse-system operates uni-directionally as an ordering agent, while in the absence of E at $(\iota = 1) \Rightarrow (T = T_h)$, $\alpha = 0$, the self-generation of P cannot, infact, occur. An intermediate, non-zero, finite value of α is what the self-emergent system seeks for its optimization that we take to be the maximum at $\alpha := 1 - T_c/T_h$. Hence

$$\alpha = \eta_{\mathrm{C}} \Longrightarrow \begin{cases} T_P = \dfrac{1}{2}\left[T_c + \sqrt{T_c^2 + 4T_c(T_h - T_c)}\right] \\ T_R = \dfrac{(2T_h - T_c)T_c}{T_h}, \end{cases} \quad (10.2.60)$$

leads to $\iota_R = T_c/T_h := 1 - \alpha$ for the E-R system. The original temperature gradient $T_h - T_c$ is shared by the $E{-}P$ system in the true spirit of synthetic cohabitation of opposites in the proportion $E : T_h - T$, $P : T - T_c$ thereby optimizing its adaptability to the environment.

	E-P	E-R		E-P	E-R
T	426.5860	412.5000	q_c	19.7791	28.1250
ι	0.7033	0.6250	S_1	0.09375	0.09375
W_{C}	28.1250	28.1250	$S_{\leftrightarrow}$	0.06593	0.05859
W_E	8.3459	10.5469	η	0.1113	0.1406
Q_c	66.6541	64.4531	$\eta_{\leftrightarrow}$	0.1113	0.1406
Q	55.6541	64.4531	ζ	8.9865	6.1111
q	28.1250	38.6719	$\zeta_{\leftrightarrow}$	3.3699	2.6667

Table 10.5. Comparison of engine-pump and engine-refrigerator bi-directionality. The equations used for E-R are (with corresponding ones for E-P): $\alpha = 0.375$, $W_{\mathrm{C}} = [1 - (T_c/T_h)]Q_h$, $W_E = [1 - (T/T_h)]Q_h$, $Q_c = Q_h - (1-\iota)W_{\mathrm{C}}$, $Q = Q_h - W_E$, $q = (1-\iota)(T/(T-T_c))W_{\mathrm{C}}$, $q_c = (1-\iota)(T_c/(T-T_c))W_{\mathrm{C}}$, $S_1 = W_{\mathrm{C}}/T_c$, $S_{\leftrightarrow} = \iota W_C/T_c = (Q_h/T_h)[(T/T_c) - 1]$, $\eta = (Q_h - Q_c)/Q_h$, $\eta_{\leftrightarrow} = (T_h - T)/T_h$, $\zeta = Q_c/(Q_h - Q_c)$, $\zeta_{\leftrightarrow} = q_c/(q_h - q_c) = T_c/(T_h - T_c)$. The role of the pump as a "gradient dissipator" is to decrease the irreversibility (and chanoxity) index from the metallic conduction value of 1 to $(T - T_c)/(T_h - T_c)$.

As an example, in the conduction of heat along a bar from $T_h = 480°$K to $T_c = 300°$K for $Q_c = Q_h - W(=0) = 75\,$kJ-min^{-1} involving an entropy increase of $S_{\iota=1} = -75/480 + 75/300 = 0.09375\,\text{kJ-(min-K)}^{-1}$. If the bar is replaced by a reversible $\iota = 0$ engine between the same temperatures, then $W_C = 28.125\,\text{kJ-min}^{-1}$, $Q_c = Q_h - W = (W_C) = 46.875\,\text{kJ-min}^{-1}$, and the entropy change of $S_{\iota=0} = -75/480 + 46.875/300 = 0$ precludes any emergence in this reversible case. If, however, bi-directionality of $X_{\leftrightarrow}$ is to be established by an induced pump or refrigerator then the results, summarized in Table 10.5, shows that the actual entropy increases are 70% of the unmoderated value S_1 with an increase of the shaft work to $(1-\iota)W_C$ from 0.

This self-generation of bi-directional stability is to be compared and contrasted with the entropy generation when a hot body is brought in thermal contact with a cold body: As in the bi-directional case, the entropy increase $m_1c_1\ln(T/T_h) + m_2c_2\ln(T/T_c)$ of the universe is maximum at $T = T_h$ and minimum for $T = T_c$. Unlike in self-organizing complexes however, the equilibrium system has a well-defined temperature $T = (m_1c_1T_h + m_2c_2T_c)/(m_1c_1 + m_2c_2)$ that is not amenable to adjustment by the system for its best possible advantage, with the resultant *negative* entropy $m_1c_1\ln(T_c/T_h)$ implying that order must be imported from outside if such a condition is to be physically realizable. Thus for $m_1/m_2 = 30\,\text{kg}/150\,\text{kg}$, $c_1/c_2 = 0.5\,\text{kJ/kg-}^\circ\text{K}/2.5\,\text{kJ/kg-}^\circ\text{K}$, and $T_h/T_c = 480^\circ\text{K}/300^\circ\text{K}$, whereas the equilibrium temperatute of $T = 306.92^\circ\text{K}$ generates $1.8477\,\text{kJ/K}$ of entropy, for a self-organizing system reversibility would impose $T = 305.472^\circ\text{K}$ as the solution of $0 = m_1c_1\ln(T/T_h) + m_2c_2\ln(T/T_c)$, import $7.05\,\text{kJ/K}$ of order from the enlarged environment at $T = T_c$, and export $176.25\,\text{kJ/K}$ of disorder when $T = T_h$.[19]

[19] In a revealing analysis of *What is Life?* [29], the theoretical biologist Robert Rosen contends [25] that it is precisely the duality between "how a given material system changes its own behaviour in response to a force, and how that same system can generate forces that change the behaviour of other systems" that Schrodinger was addressing in the context of Mendelian genes and molecules and "the mode of forcing of phenotypes (the actual physical properties of a molecule) by genotypes (the genetic profile of the molecule)". While the phenotype and genotype are related, they are not necessarily identical with the environment playing an important role in shaping the actual phenotype that results, Rosen proceeds to argue that "We cannot hope for identical relations between inertial and gravitational aspects of a system, such as are found in the very special realms of particle mechanics. Yet, in a sense, this is precisely what Schroedinger essay is about. Delbruck was seeking to literally reify a *forcing* (the Mandelian gene), *something 'gravitational'*, by clothing it in *something with 'inertia'*, by realizing it as a molecule. Schrodinger, on the other hand understood that this was not nearly enough, that we must be able to go the other way and determine the forcings manifested by something characterized inertially: just as we realize a *force* by a *thing*, we must also, perhaps more importantly, be able to realize a *thing* by a *force* (emphasis added). It was in this later connection that Schrodinger put forward the 'principle of order from order' and the 'feeding of negative entropy'. It was here that he was looking for the new physics".

In the Lorenz system, the potential energy of the top-heavy liquid created by the imposed temperature gradient $\Delta T = T_h - T_c$, taking T_c to be fixed, leads to conversion of the input heat energy to mechanical work of convective viscous mixing that acts as a gradient dissipator. Taking $Q_h = 1$, W_{r} corresponds to R and $\iota = (R - R_g)/R$ to that fraction of R that is not utilized in gravitational gradient dissipation through convection. In an arbitrary non-equilibrium steady state, the temperature induced upward potential energy production must be balanced by the dissipations which includes an atmospheric loss component also. In general for the non-equilibrium steady state $X_{\leftrightarrow}$, the increase in internal stability due to viscous dissipation leads to a backward-forward synthesis, when the direct arrow of entropy increasing emergence is moderated by the inverse arrow of order and self-organization. This is when all irreversible motivations guiding the system must cease, and the dead state of a "local non-equilibrium maximum entropy" — of magnitude less than that of the completely irreversible "global" equilibrium conductive state — consistent with the applied constraint of viscous damping, is reached. Refer Fig. 10.5*b*.

The earth-atmosphere system offers another striking example of this non-equilibrium local principle, in which the earth is considered as a two-region body of the hot equator at T_h and the cold poles at T_c, with radiative heat input at the equator and thermal dissipation at the poles. A portion of the corresponding W_{r} is utilized in establishing the P induces pole $\leftrightarrow$ equator atmospheric circulation resulting in internal stabilization, structuring, and inhibitory gradient dissipation. The radiative polar heat loss constitutes the entropy increasing direct arrow that is moderated by the that makes this planet habitable.

As a final illustration, mention can be made to the interesting example of frost heaving [22] as a unique model of a "reverse Lorenz system" where the temperature gradient is *along* the direction of gravity. A regular Lorenz under such conditions would be maximally irreversible, as an effective conductive entity, without any internal generation of P-stabilization. In frost heaving, however, ice and supercooled water are partitioned by a microporous material permeable to the water, the pressure of the ice on the top of the membrane being larger than that exerted by the water below: thus the temperature and pressure of the water below are less than that of the ice above. If the water is sufficiently supercooled however, it flows up against gravity due to P, into the ice layer, freezes and in the process heaves the ice column up.

Thus according to Rosen, Schroedinger supreme contribution in posing his now famous question elevated the object of his inquiry from a passive adjective to an active noun by suggesting the necessity of a "new physics" for investigating how in open, non-equilibrium systems, every forward-indirect arrow of phenotype inertia engine E is necessarily coupled to a backward-direct impulse from some genotype gravity pump P. For Schrodinger while a Mandelian gene was surely a molecule, it was more important to investigate when the molecule becomes a gene.

The non-equilibrium steady-state $X_{\leftrightarrow}$, Equation (10.2.48), is therefore a local maximum-entropy state that the dynamics of the non-linear system seeks as its most gainful eventuality, given the constraint of conflicting and contradictory demands of the universe it inhabits, with the constraints effectively lowering the entropic sum $S = -\sum_j p_j \ln p_j$. Accordingly while the entropy of a partition of unconstrained elementary events in the rolling of a fair die with $\{p_j\}_{j=1}^{6} = 1/6$ is $\ln 6 = 1.7918$, the entropy of a constrained partition satisfying $p_1 + p_3 + p_5 = 0.6$ and $p_2 + p_4 + p_6 = 0.4$ in the appearance of odd and even faces is $0.6 \ln(0.2) + 0.4 \ln(0.1333) = 1.7716$. The applied constraints therefore reduce the number of faces of the die to an unconstrained effective value of $\exp(1.7716) = 5.88$, thereby reducing the disorder of the system, which can be interpreted as a corresponding lowering of the temperature gradient ΔT of the irreversible $\iota = 1$ instance of $W = 0$. In the examples above the respective constraints are the convection rolls, atmospheric convection currents, and anti-gravity frost heaving. Without this component of the energy input, emergent internal structuring in natural systems would be absent. It may therefore be inferred that the two-component decomposition (10.2.1) of entropy corresponds to the break-up we propose here.

10.2.3 An Index of Nonlinearity

At the moment there is no formalization of complexity that enables it to overcome its current rather confused state and to achieve the objective of first becoming a method and then a bonafide scientific theory. The complexity approach that has recently appeared in modern scientific circles is generally still limited to an empirical phase in which the concepts are not abundantly clear and the methods and techniques are noticeable lacking. This can lead to the abuse of the term "complexity" which is sometimes used in various contexts, in senses that are very different from one another, to describe situations in which the system does not even display complex characteristics.

Formalizing complexity would enable a set of empirical observations, which is what complexity is now, to be transformed into a real hypothetical-deductive theory or into an empirical science. Therefore, at least for the moment, there is no unified theory of complexity able to express the structures and the processes that are common to the different phenomena that can be grouped under the general heading of complexity. There are several evident shortcomings in modern mathematics which make the application of a complexity theory of little effect. Basically this can be put down to the fact that mathematics is generally linear.

We are now faced with the following problem. We are not able to describe chaotic phenomenology or even that type of organized chaos that is complexity by means of adequate general laws; consequently we are not able to formulate effective long-term predictions on the evolution of complex *systems. The mathematics that is available to us does not enable us to do this in an adequate manner, as the techniques of such mathematics were essentially developed to describe linear phenomena in which there are no mechanisms that unevenly amplify any initial uncertainty or perturbation.*

Bertuglia and Vaio [3]

With ininality in the cartesian space $C \times C$ serving as the engine for the increase of evolutionary entropic disorder, we now examine how a specifically nonlinear index can be ascribed to chaos, nonlinearity and complexity to serve as the benchmark for chanoxity. For this, we first recall two non-calculus formulations of entropy that measure the complexity of dynamics of evolution of a map f.

Let $\mathcal{A} = \{A_i\}_{i=1}^{I}$ be a disjoint partition of non-empty subsets of a set X; thus $\bigcup_{i=1}^{I} A_i = X$. The entropy

$$S(\mathcal{A}) = -\sum_{i=1}^{I} \mu(A_i)\ln(\mu(A_i)), \qquad \sum_{i=1}^{I} \mu(A_i) = 1 \tag{10.2.61}$$

of the partition $\mathcal{A}$, where $\mu(A_i)$ is some normalized invariant measure of the elements of the partition, quantifies the uncertainty of the outcome of an experiment on the occurrence of any element A_i of the partition $\mathcal{A}$. A refinement $\mathcal{B} = \{B_j\}_{j=1}^{J \geq I}$ of the partition $\mathcal{A}$ is another partition such that every B_j is a subset of some $A_i \in \mathcal{A}$, and the largest common refinement

$$\mathcal{A} \bullet \mathcal{B} = \{C \colon C = A_i \bigcap B_j \text{ for some } A_i \in \mathcal{A}, \text{ and } B_j \in \mathcal{B}\}$$

of $\mathcal{A}$ and $\mathcal{B}$ is the partition whose elements are intersections of those of $\mathcal{A}$ and $\mathcal{B}$. The entropy of $\mathcal{A} \bullet \mathcal{B}$ is given by

$$\begin{aligned} S(\mathcal{A} \bullet \mathcal{B}) &= S(\mathcal{A}) + S(\mathcal{B} \mid \mathcal{A}) \\ &= S(\mathcal{B}) + S(\mathcal{A} \mid \mathcal{B}), \end{aligned} \tag{10.2.62}$$

where the weighted average

$$S(\mathcal{B} \mid \mathcal{A}) = \sum_{i=1}^{I} P(A_i)\, S(\mathcal{B} \mid A_i) \tag{10.2.63a}$$

of the conditional entropy

$$S(\mathcal{B} \mid A_i) = -\sum_{j=1}^{J} P(B_j \mid A_i)\ \ln(P(B_j \mid A_i)) \tag{10.2.63b}$$

of $\mathcal{B}$ given $A_i \in \mathcal{A}$, is a measure of the uncertainty of $\mathcal{B}$ if at each trial it is known which among the events A_i has occurred, and

$$P(B_j \mid A_i) = \frac{P(B_j \cap A_i)}{P(A_i)} \tag{10.2.63c}$$

yields the probability measure $P(B_j \cap A_i)$ from the conditional probability $P(B_j \mid A_i)$ of B_j given A_i, with $P(A)$ the probability measure of event A.

The entropy (10.2.61) of the refinement $\mathcal{A}^n$, rather than (10.2.62), that has been used by Kolmogorov in the form

$$h_{\mathrm{KS}}(f;\mu) = \sup_{\mathcal{A}_0}\left(\lim_{n\to\infty}\frac{1}{n}\,S(\mathcal{A}^n)\right) \tag{10.2.64}$$

to represent the complexity of the map as measuring the time rate of creation of information with evolution, yields $\ln 2$ for the tent transformation. Another measure — the topological entropy $h_{\mathrm{T}}(f) := \sup_{\mathcal{A}_0}\lim_{n\to\infty}(\ln N_n(\mathcal{A}_0)/n)$ with $N_n(\mathcal{A}_0)$ the number of divisions of the partition $\mathcal{A}^n$ derived from $\mathcal{A}_0$, that reduces to

$$h_{\mathrm{T}}(f) = \lim_{n\to\infty}\frac{1}{n}\ln \mathcal{I}(f^n) \tag{10.2.65}$$

in terms of the number of injective branches $\mathcal{I}(f^n)$ of f^n for partitions generated by piecewise monotone functions — also yields $\ln 2$ for the entropy of the tent map. For the logistic map,

$$\mathcal{I}(f^n) = \mathcal{I}(f^{n-1}) + \left\langle\{x\colon x = f^{-(n-1)}(0.5)\}\right\rangle \tag{10.2.66}$$

is the number of injective branches arising from the solutions of

$$\begin{aligned} 0 = \frac{df^n(x)}{dx} &= \frac{df(f^{n-1})}{df^{n-1}}\,\frac{df^{n-1}(x)}{dx} \\ &= \frac{df(f^{n-1})}{df^{n-1}}\,\frac{df(f^{n-2})}{df^{n-2}}\cdots\frac{df(f)}{df}\,\frac{df(x)}{dx} \end{aligned}$$

that yields

$$x = f^{-}(\cdots(f^{-}(f^{-}(0.5)))\cdots)$$

where $\langle\{\cdots\}\rangle$ is the cardinality of set $\{\cdots\}$. Note that in the context of the topological entropy, $\mathcal{I}(f)$ is only a tool for generating a partition on $\mathcal{D}(f)$ by the iterates of f.

Example 10.1. (1) In a fair-die experiment, if $\mathcal{A} = \{\text{even, odd}\}$ and the refinement $\mathcal{B} = \{j\}_{j=1}^{6}$ is the set of the six faces of the die, then for $i = 1, 2$

$$P(B_j \mid A_i) = \begin{cases} \dfrac{1}{3}, & j \in A_i \\ 0, & j \notin A_i, \end{cases}$$

and $S(\mathcal{B} \mid A_1) = \ln 3 = S(\mathcal{B} \mid A_2)$ by (10.2.63b). Hence the conditional entropy of $\mathcal{B}$ given $\mathcal{A}$, using $P(A_1) = 0.5 = P(A_2)$ and Eq. (10.2.63a), is $S(\mathcal{B} \mid \mathcal{A}) = \ln 3$. Hence

$$\begin{aligned} S(\mathcal{A} \bullet \mathcal{B}) &= S(\mathcal{A}) + S(\mathcal{B} \mid \mathcal{A}) \\ &= \ln 6. \end{aligned}$$

If we have access only to partition $\mathcal{B}$ and not to $\mathcal{A}$, then $S(\mathcal{B}) = \ln 6$ is the amount of information gained about the partition $\mathcal{B}$ when we are told which face showed up in a rolling of the die; if on the other hand the only partition available is $\mathcal{A}$, then $S(\mathcal{A}) = \ln 2$ measures the information gained about $\mathcal{A}$ on the knowledge of the appearance of an even or odd face.

(2) The dynamical evolution of Fig. 10.3 provides an example of conditional probability and conditional entropy. Here the refinements of basic partition $\mathcal{A}_0 = \{\text{matter, negmatter}\} = \{A_{01}, A_{00}\}$ generated by the inverses of the tent map, are denoted as $\mathcal{A}_n = \{t^{-n}(A_{0i})\}_{0,1}$ for $n = 1, 2, \cdots$ to yield the largest common refinements

$$\mathcal{A}^n = \mathcal{A}_0 \bullet \mathcal{A}_1 \bullet \mathcal{A}_2 \bullet \cdots \bullet \mathcal{A}_n, \qquad n \in \mathbb{N}, \tag{10.2.67}$$

where the refinements are denoted as indicated in the figure, and $\mathcal{A}^n = \mathcal{A}_n$. Taking the measure of the elements of a partition to be its euclidean length, gives

$$P(A_{nj} \mid A_{0i}) = \begin{cases} \dfrac{1}{2^{n-1}}, & j \in A_{0i} \\ 0, & j \notin A_{0i}, \end{cases}$$

$S(\mathcal{A}_n \mid A_{0i}) = (n-1)\ln 2$, $i = 0, 1$, (Equation 10.2.63b), $S(\mathcal{A}_n \mid \mathcal{A}_0) = (n-1)\ln 2$, and finally $S(\mathcal{A}_n \bullet \mathcal{A}_0) = n \ln 2$. In case the initial partition $\mathcal{A}_0$ is taken to be the whole of $\mathcal{D}(t)$, then (10.2.61) gives directly $S(\mathcal{A}_n) = n \ln 2$.

(3) Logistic map $f_\lambda(x) = \lambda x(1-x)$, [21]. For $0 \le \lambda < 3$, Fig. 10.8a, the dynamics can be subdivided into two broad categories. In the first, for $0 \le \lambda \le 2$, $\mathcal{I}(f_\lambda^n) = 2$ gives $h_{\mathrm{T}}(f_\lambda) = 0$. This is illustrated in Fig. 10.8a (i), (ii), and (iii) which show how the number of subsets generated on X by the increasing iterates of the map tend from 2 to 1 in the first case and to the set $\{\{0\}, (0,1), \{1\}\}$ for the other two. The figure demonstrates that while in (a) the dynamics eventually collapses and dies out, the other two cases are equally uneventful in the sense that the converged multifunctional limits — of $(0, [0, 1/2]) \cup ((0,1), 1/2) \cup (1, [0, 1/2])\}$ in figure (iii), for example — are as much passive and has no real "life"; this is quantified by the constancy of the lap number and the corresponding topological entropy $h_{\mathrm{T}}(f) = 0$. Although the partition induced on $X = [0,1]$ by the evolving map in (iv) is refined with time, the *stability of the fixed point* $x_* = 0.6656$ prevents the dynamics from acquiring any meaningful evolutionary significance with its multifunctional graphical limit being of the same type as in (ii) and (iii): as will be evident in what follows, *instability of fixed points is essential for the evolution of meaningful complexity.* $\lambda^{(0)} = 2$ of (iii) — obtained by solving the equation $f_\lambda(0.5) = 0.5$ — is special because its super-stable fixed point $x = 0.5$ is the only point in $\mathcal{D}(f)$ at which f is injective and therefore well-posed by this criterion.

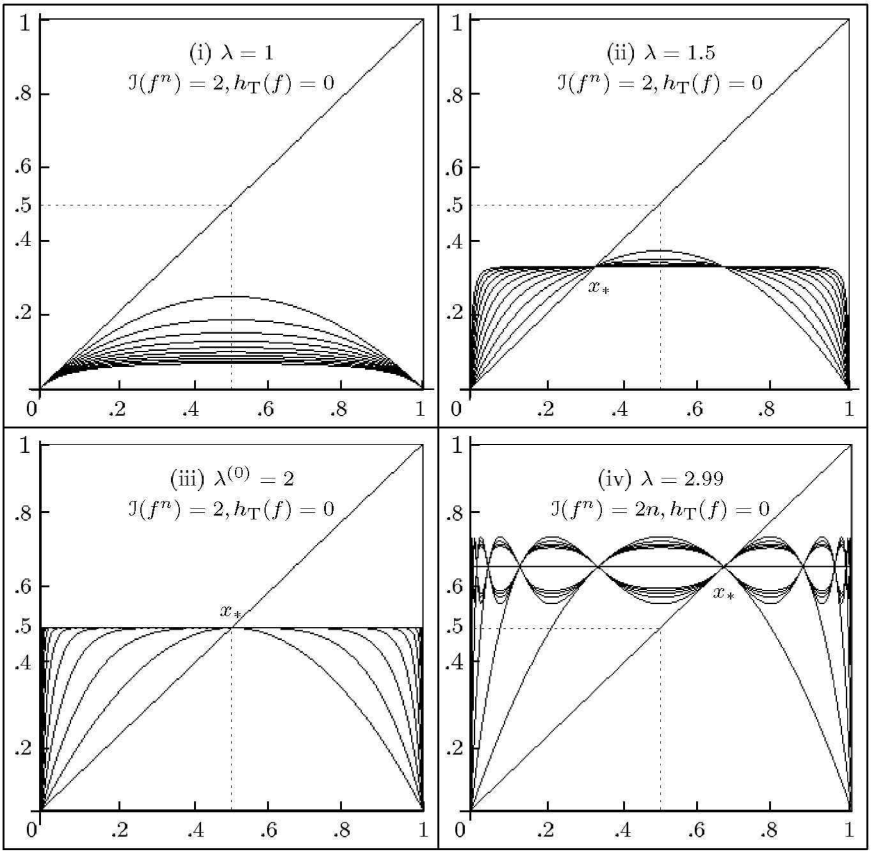

Fig. 10.8*a*. Non-life dynamics of the first 10 iterates of the logistic map $f_\lambda = \lambda x(1-x)$ generated by its only stable fixed point $x_* = (\lambda - 1)/\lambda$.

For $3 \le \lambda \le 4$, $h_T(f_\lambda) = 0$ whenever $\mathfrak{I}(f_\lambda) \le 2n$ which occurs, from Fig. 10.8*b*, for $\lambda \le \lambda^{(1)} = 1 + \sqrt{5} = 3.23607$; here $\lambda^{(m)}$ is the λ value at which a super-stable 2^m-cycle appears. The super-stable λ for which $x = 0.5$ is fixed for f^n, $n = 2^m$, $m = 0, 1, 2, \cdots$ leads to a simplification of the dynamics of the map, possessing as they do, the property of the stable horizontal parts of the graphically converged multifunction being actually tangential to all the turning points of every iterate of f. The immediate consequence of this is that for a given $3 < \lambda < \lambda_* = 3.5699456$, the dynamics of f attains a state of basic evolutionary stability after only the first $\{2^m\}_{m=0,1,\cdots}$ time steps in the sense that no new spatial structures *emerge* after this period, any further temporal evolution being fully utilized in spatially *self-organizing* this basic structure throughout the system by the generation of equivalence classes of the initial 2^m time steps. As seen in Fig. 10.8*b*, the unstable fixed point x_* is directly linked to its stable partners of f^2 that report back to x_*. Compared

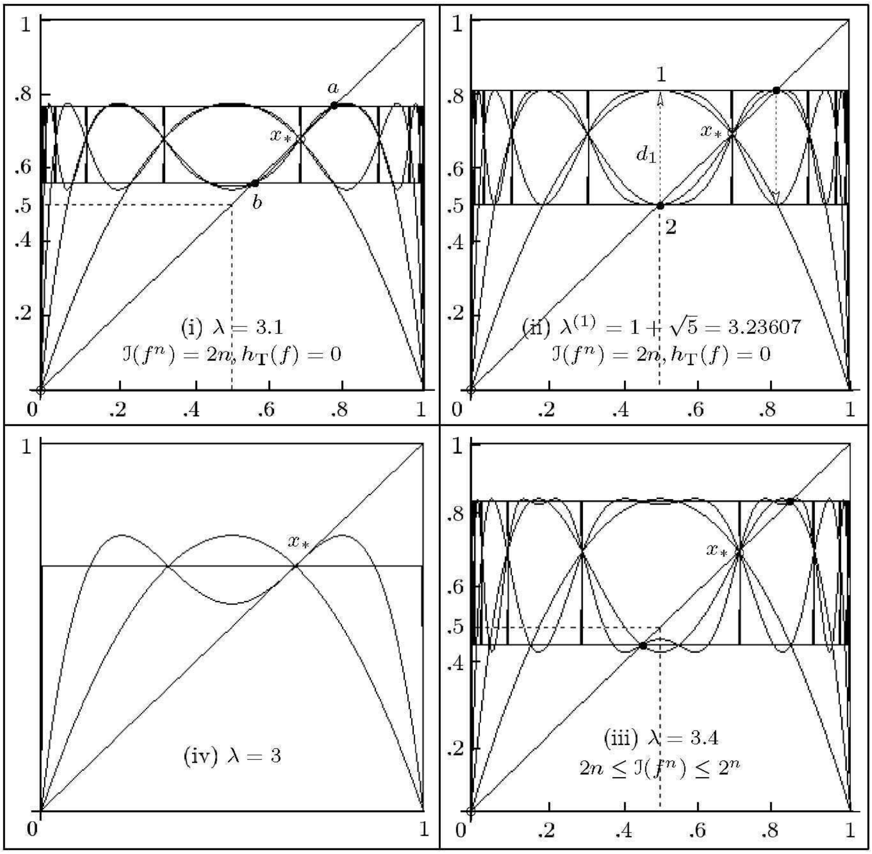

Fig. 10.8*b*. Dynamics of stable 2-cycle of the logistic map, where each panel displays the first four iterates superposed on the graphically converged multifunction represented by iterates 1001 and 1002. Panel (iv) in this and the following two figures, illustrates Eqs. (10.2.56a, b) in the birth of new period doubling cycles. The $d_i := f_{\lambda_i}^{2^{i-1}}(0.5) - 0.5$ in these figures define the universal Feigenbaum constant $-\alpha := \lim_{i\to\infty} d_i/d_{i+1} = 2.502907\cdots$, while the super-cyclic parameters $(\lambda_i)_i$ generate the second constant $\delta := \lim_{i\to\infty}(\lambda_i - \lambda_{i-1})/(\lambda_{i+1} - \lambda_i) = 4.669201\cdots$ of period doubling.

to (i) however, where the relative simplicity of the instability of x_* allows its stable partners to behave monotonically as in Fig. 10.8a (ii), the instability of 10.8b (iii) is strong enough to induce the oscillatory mode of convergence of 10.8a (iv). Case (ii) of the super-stable cycle for $\lambda^{(1)} = 1 + \sqrt{5}$ — obtained by solving the equation $f_\lambda^2(0.5) = 0.5$ — reflecting well-posedness of f at $x = 0.5$ represents, as in Fig 10.8a (iii), a mean of the relative simplicity of (i) and the complex instability of (iii) that grows with increasing λ.

When $\lambda > \lambda^{(1)}$ as in Figs. 10.8b (iii) and 10.8c, the number of injective branches lie in the range $2n \leq \mathcal{I}(f_\lambda^n) \leq 2^n$ and the difficulty in actually obtaining these numbers for large values of n is apparent from Eq. (10.2.66). The unstable basic fixed point x_* in Fig. 10.8c is now linked to its *un*stable partners denoted by open circles arising from f^2, who report back to the overall controller x_* the information they receive from their respective stable subcommittees. Compared to the 2-cycle of Fig. 10.8b, the instability of principal x_* is now serious enough to require sharing of the responsibility by two other instability governed partners who are further constrained to delegate authority to the subcommittees mentioned above. Case (ii) of the super-stable cycle for $\lambda^{(2)} = 3.49856$ is obtained by solving $f_\lambda^4(0.5) = 0.5$ denotes as before the mean of the relative simplicity of (i) and the large instability of (iii). For $\lambda = 4$, however $\mathcal{I}(f_4^n) = 2^n$ and the topological entropy reduces to the simple $h(f_4) = \ln 2$; $h_{\mathrm{T}}(f) > 0$ is sufficient condition for f_λ to be chaotic. The tent map behaves similarly and has an identical topological entropy, see Fig. 10.10a.

The difficulty in evaluating $\mathcal{I}(f^n)$ for large values of n and the open question of the utility of the number of injective branches of a map in actually measuring the complex dynamics of nonlinear evolution, suggests the significance of the role of *evolution of the graphs of the iterates of* f_λ in defining the dynamics of natural processes. It is also implied that the dynamics can be simulated through *the partitions induced on* $\mathcal{D}(f)$ *by the evolving map* as described by graphical convergence of the functions in accordance with our philosophy that the dynamics on C derives from the evolution of f in C^2 as observed in $\mathcal{D}(f)$. The following subsection carries out this line of reasoning, to define a new index of chaos, nonlinearity and complexity, that is of *chanoxity.*

ChaNoXity

The really interesting comparison (of Windows) *is with Linux, a product of comparable complexity developed by an independent, dispersed community of programmers who communicate mainly over the internet. How can they outperform a stupendously rich company that can afford to employ very smart people and give them all the resources they need? Here is a possible answer: Complexity.*

Microsoft's problem with Windows may be an indicator that operating systems are getting beyond the capacity of any single organization to handle them. Therein may lie the real significance of Open Source. Open Source is not a software or a unique group of hackers. It is a way of building complex things. Microsoft's struggles with Vista suggests it may be the only way to do operating systems in future.

John Naughton, Guardian Newspapers Limited, May 2006.

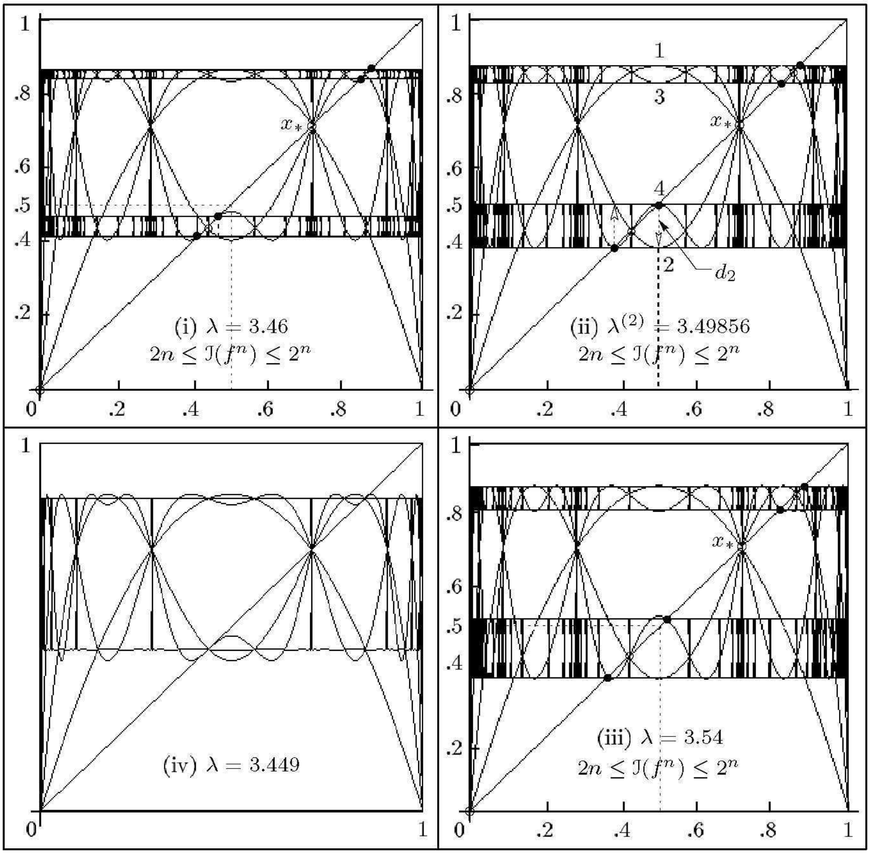

Fig. 10.8*c*. Dynamics of stable 4-cycle of the logistic map, where each panel displays the first four iterates superposed on the graphically "converged" multifunction represented by iterates 1001-1004.

The magnified view of the *stable* 8-cycle, Fig. 10.8*d*, graphically illustrates evolutionary dynamics of the logistic interaction. The 2^3 *unstable* fixed points marked by open circles interact among themselves as indicated in the figure to generate the stable periodic cycle, providing thereby a vivid illustration of competitive collaboration between matter-negmatter effects. The increasing iterations of irreversible urge toward bijective simplicity of ininality constitutes the activating backward-direct direction of increasing entropic disorder that is effectively balanced by restraining forward-inverse exergy destruction of expansion, increasing order, and self-organization that eventually leads to the stable periodic orbit. The activating effect of the direct limit appears in the figure as the negative slope associated with each unstable fixed points except the first at $x = 0$ which must now be paired with its equivalent image at $x = 1$. Display (iii) of the partially superimposed limit graphs 1001-1008 on the first 8 iterates — that remain invariant with further temporal evolution — illustrate that while nothing new emerges after this initial period, *further increasing temporal evolution propagates the associated changes throughout the system as self-generated equivalence classes guiding the system to a state of local* (that is spatial, for the given λ) *periodic stasis.* As compared to Fig. 10.3 for the tent interaction, this manifestation of coeffects in the logistic for $\lambda < \lambda_* = 3.5699456$ has a feature that deserves special mention: while in the former the negative branch belongs to distinct fixed points of equivalence classes, in the later matter-negmatter competitive-collaboration is associated with each of the 2^N generating branches possessing bi-directional characteristics with the activating effect of negmatter actually initiating the generation of the equivalence class. In the observable physical world of $\mathcal{D}(f)$, this has the interesting consequence that whereas the tent interaction generates matter-negmatter intermingling of disjoint components to produce the homogenization of Fig. 10.3, for the logistic interaction the resulting behaviour is a consequence of a deeper interplay of the opposing forces leading to a higher level of complexity than can be achieved by the tent interaction.

This distinction reflects in the interaction pair $(f, \mathfrak{f})$ that can be represented as

$$x \longmapsto 2x \longmapsto \begin{cases} 2x, & \text{if } 0 \leq x < 0.5 \\ 2(1-x), & \text{if } 0.5 \leq x \leq 1 \end{cases}, \qquad x \longmapsto 2x \longmapsto 4x(1-x), \tag{10.2.68}$$

which leads — despite that "researchers from many disciplines now grapple with the term *complexity*, yet their views are often restricted to their own specialties, their focus non-unifying; few can agree on either a qualitative or quantitative use of the term" [6] — to the

Definition 10.2 (Complex System, Complexity). *The couple* $((X, \mathcal{U}), f)$ *of a compound topological space* $(X, \mathcal{U})$ *and an interaction* f *on it is a* complex system $\mathcal{C}$ *if* (see Fig. 10.9 and Eq. (10.2.72))

(CS1) *The* algebraic structure *of* $\mathcal{D}(f)$ *is defined by a* finite *family* $\{A_j\}_{j=0}^{n}$, $n = 1, 2, \cdots, N$, *of progressively refined hierarchal partitions of non-empty subsets induced by the iterates of* f, *with increasing evolution building on this foundation the overall configuration of the system.*

This family interacts with each other through

(CS2) *The* topology of $(\mathcal{D}, \mathcal{U})$ *such that the subbasis of* $\mathcal{U}$ *at any level of refinement is the union of the open sets of its immediate coarser partition and that generated by the partition under consideration where all open sets are saturated sets of equivalence classes generated by the evolving iterates of the interaction.*

The complexity of a system is a measure of the interaction between the different levels of partitions that are generated on $\mathcal{D}(f)$ under the induced topology on X. Thus as a result of the constraint imposed by (CS1), under

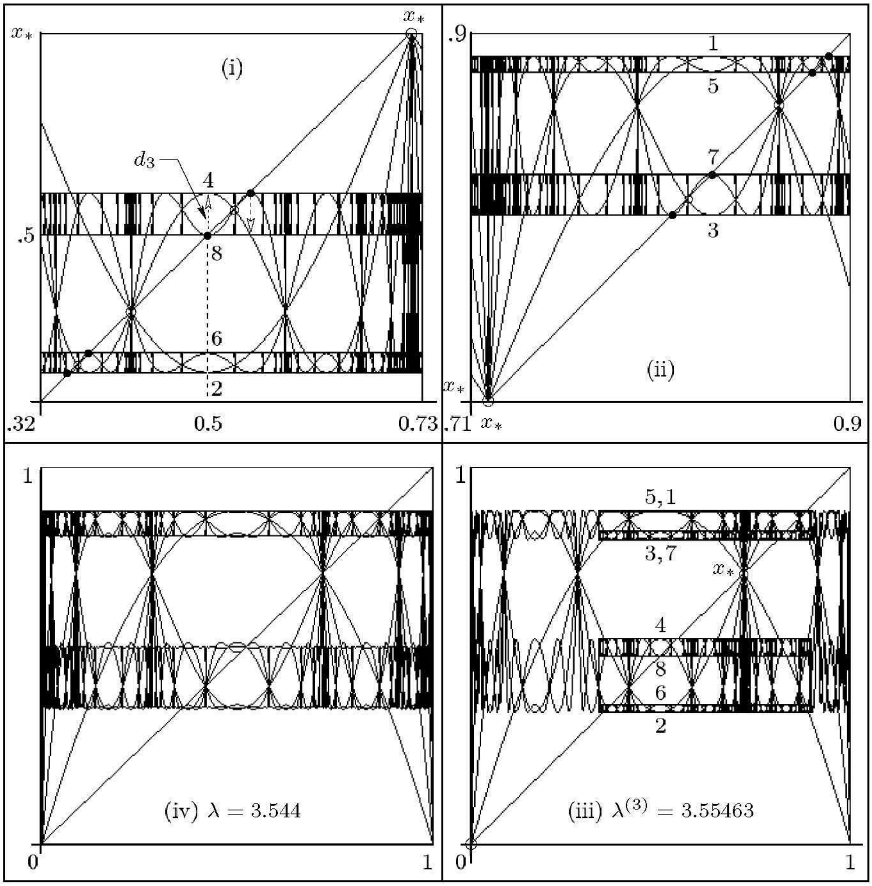

Fig. 10.8*d*. Magnified view of the *stable* 8-cycle graphically illustrates how the evolutionary dynamics of the logistic interaction, under the synthetic influence of its stable-unstable components, spontaneously produces for any given resource $3 \leq \lambda < \lambda_*$, a set of 2^n uniquely stable configurations between which it periodically oscillates. Thus in this case the "unpredictability" of nonlinear interactions manifests as a "surprise" in the autonomous generation of a set of well-defined stable states, which as we shall see defines the "complexity" of the system.

the logistic interaction complex structures can emerge only for $3 \leq \lambda < \lambda_*$ which in the case of the stable 2-cycle of Fig. 10.8*b*(ii), reduces to just the first 2 time steps that is subsequently propagated throughout the system by the increasing ill-posedness, thereby establishing the global structure as seen in Fig. 10.9. With increasing λ the complexity of the dynamics increases as revealed in the succeeding plots of 4- and 8-cycles: compared to the single refinement for the 2-cycle, there are respectively 2 and 3 stages of refinements in the 4- and 8-cycles and in general there will be N refining partitions of

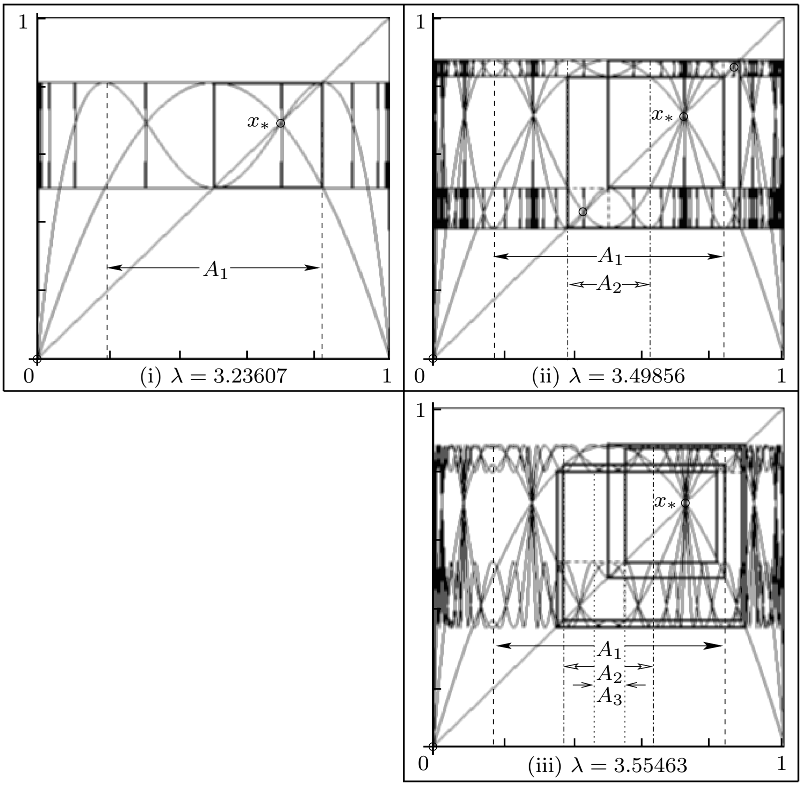

Fig. 10.9. The role of unstable fixed points in generating the partitions $\{A_j\}_{j=0}^{n}$, $n = 1, 2, \cdots, N$ required in the definition of complexity, where the $\{A_j\}$ are appropriately defined as the inverse images of $f_{i,j} := \left|f^i(0.5) - f^j(0.5)\right|$, refer Eq. (10.2.72), and $A_0 = \mathcal{D}(f)$. The open circles in (i) and (ii) represent the unstable fixed points that have been omitted from (iii) for the sake of clarity. The converged multifunctional graphical limits are also shown for the 2- and 4-cycles.

$\mathcal{D}(f)$ for the 2^N-stable cycle. The equilibrated $X_{\leftrightarrow}$, by Fig. 10.5b and the subsequent discussion, corresponds to the $\{\mathcal{D}, f(\mathcal{D}), \cdots, f^{2^N}(\mathcal{D})\}$ on $\mathcal{D}(f)$. Below $\lambda = 3$, absence of instabilities allows no emergence of new features, while above $\lambda = \lambda_*$ the absence of stabilizing effects prevent self-organization from moderating the dynamics of the system.The *motivating* saturated open sets of X on $\mathcal{D}(f)$ and $\mathcal{R}(f)$ are the projections of the boxes of the converged multi-limits in Figs. 10.8b, c, d onto the x- and $y-$axes, with their boundary being represented by the members of the equivalence class $[x_*]$ of the unstable fixed point x_*.

Complexity therefore, represents a state of dynamical balance between a catabolic emergent, destabilizing, backward, bottom-up pump direction, opposed by an anabolic top-down, stabilizing forward engine arrow of self-organization. This may be represented, with reference to Fig. 10.5a, b, as

$$\begin{gathered}\left.\begin{array}{r}\underline{\text{FORWARD-INVERSE ARROW}}\\ \text{Synthesis of } E\text{-expansion,}\\ \text{order, entropy decreasing top-}\\ \text{down } \textit{self-organization } C_{\leftarrow}\end{array}\right\} \oplus \left\{\begin{array}{l}\underline{\text{BACKWARD-DIRECT ARROW}}\\ \text{Analysis of } P\text{-contraction,}\\ \text{disorder, entropy increasing}\\ \text{bottom-up } \textit{emergence } {}_{\rightarrow}C\end{array}\right. \\ \Longleftrightarrow \\ \text{Synthetic cohabitation of opposites } \mathcal{C} = C_{\leftrightarrow},\end{gathered} \tag{10.2.69}$$

with $\oplus$ denoting a non-reductionist sum of the components of a top-down engine and a bottom-up pump as elaborated in Sec. 10.2.2. A complex system behaves in an organized collective manner with properties that cannot be identified with any of the individual parts but arise from the entire structure acting as a whole: these systems cannot be dismantled into their components without destroying itself. Analytic methods cannot simplify them because such techniques do not account for characteristics that belong to no single component but relate to the parts taken together with all their interactions. This analytic base must be integrated into a synthetic whole with new perspectives that the properties of the individual parts fail to add up to. A complex system is therefore a

► dynamical, C-interactive, interdependent, hierarchal homeostasy of *P-emergent, disordering instability* competitively collaborating with adaptive *E-self-organized, ordering stability* generating thereby a non-reductionist structure that is more than the sum of its constituent parts.

Emergence implies instability inspired (and therefore "destructive", anti-stabilizing) generation of overall characteristics that do not reduce to a linear composition of the interacting parts: complexity is a result of the "failure of the Newtonian paradigm to be a general schema through which to understand the world", [3], and in fact "if there were only Newton's laws, there could never have been any motion in the earth" [22].[20] As noted earlier, complexity can be distinguished into two subclasses depending on which of the two limits of Eq. (10.2.69) serve as *activating* and which *restraining* and our classification of "life" will be based on this distinction.

A complexity supporting interaction will be distinguished as C-interaction. Examples of C- and non-C-interactions that will be particularly illuminating

[20] Darwinian theory of *natural selection* is different from complexity generated emergence and self-organization. Selection represents a competition between different systems for the limited resources at their disposal: it signifies an *externally* directed selection between competing states of equilibria that serves to maximize the "fitness" of the system with respect to its environment. Complexity, on the other hand, typifies an *internally* generated process of "continuous tension between competition and cooperation".

in our work are respectively the λ-logistic map and its “bifurcated” ($\lambda/2$)-tent counterpart

$$\lambda x(1-x) \longmapsto \begin{cases} \frac{\lambda}{2}x, & 0 \le x \le 0.5 \\ \frac{\lambda}{2}(1-x), & 0.5 \le x \le 1. \end{cases} \tag{10.2.70}$$

It will be convenient to denote a complex system $\mathcal{C}$ simply as $(A, \mathfrak{B})$, with the interaction understood from the context. The distinguishing point of difference between the dissipative structures $\mathcal{D}$ of multi-dimensional differential system and evolutionary complex dynamics of a C-interaction is that the former need not possess any of the hierarchal configuration of the later. This tiered structure of a complex system is an immediate consequence of the partitioning refinements imposed by the interaction on the dynamics of the system with emergence and self-organization being the natural outcome when these refinements, working independently within the global framework of the interaction, are assembled together in a unifying whole. Hence it is possible to make the distinction

- a *dissipative structure* $\mathcal{D}$ is a special system of spatially multidimensional, non-tiered, forward-backward synthesis of opposites that attains dynamic equilibrium largely through self-organization without significant instability inspired emergence

from a general complex system.

A Measure of ChaNoXity

The above considerations allow us to define, with reference to Fig. 10.5*b*, the *chanoxity index* of the interaction to be the *constant* $0 \le \chi \le 1$ that satisfies

$$f(x) = x^{1-\chi}, \qquad x \in \mathcal{D}(f). \tag{10.2.71a}$$

Thus if $\langle f(x) \rangle$ and $\langle x \rangle$ are measures that permit (10.2.71*a*), then in

$$\chi = 1 - \frac{\ln \langle f(x) \rangle}{\ln \langle x \rangle} \tag{10.2.71b}$$

we take

(a) $\langle x \rangle$ to be the number of *basic unstable fixed points* of f responsible for *emergence*. Thus for $1 < \lambda \le 3$ there is no *basic* unstable fixed point at $x = 0$, followed by the familiar sequence of $\langle x \rangle = 2^N$ points until at $\lambda = \lambda_*$ it is infinite.

(b) for $f(x)$ the estimate

$$\langle f(x) \rangle = 2f_1 + \sum_{j=1}^{N} \sum_{i=1}^{2^{j-1}} f_{i,i+2^{j-1}}, \qquad N = 1, 2, \cdots, \tag{10.2.72}$$

λ	N	$\langle f(0.5)\rangle$	χ_N	λ	N	$\langle f(0.5)\rangle$	χ_N
(1, 3]	–	1.000000	0.000000	3.5699442	9	3.047727	0.821363
3.2360680	1	1.927051	0.053605	3.5699454	10	3.053571	0.838950
3.4985617	2	2.404128	0.367243	3.5699456	11	3.056931	0.853447
3.5546439	3	2.680955	0.525751	3.5699457	12	3.058842	0.865585
3.5666676	4	2.842128	0.623257	3.5699457	13	3.059855	0.875887
3.5692435	5	2.935294	0.689299	3.5699457	14	3.060524	0.884730
3.5697953	6	2.988959	0.736726	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$?
3.5699135	7	3.019815	0.772220	λ_*	∞	3.??????	1.000000
3.5699388	8	3.037543	0.799637				

Table 10.6*a*. In the passage to full chaoticity, the system becomes increasingly complex and nonlinear (remember: chaos is maximal nonlinearity) such that at the critical value $\lambda = \lambda_* = 3.5699456$, the system is fully chaotic and complex with $\chi = 1$. For $1 < \lambda \leq 3$ with no generated instability of which $\lambda = 2$ is representative, $\chi = 1 - \ln(1/2 + 1/2)/0 = 0$. The expression for $\langle f(x)\rangle$ reduces to $2f_1 + f_{12}$, $2f_1 + f_{12} + (f_{13} + f_{24})$, $2f_1 + f_{12} + (f_{13} + f_{24}) + (f_{15} + f_{26} + f_{37} + f_{48})$ for $N = 1, 2, 3$ respectively.

with $f_i = f^i(0.5)$ and $f_{i,j} = |f^i(0.5) - f^j(0.5)|$, to get the measure of chanoxity as

$$\chi_N = 1 - \frac{1}{N \ln 2} \ln \left[2f_1 + \sum_{j=1}^{N} \sum_{i=1}^{2^{j-1}} f_{i,i+2^{j-1}} \right], \tag{10.2.73}$$

that we call the *dimensional chanoxity of* f_λ[21]; notice how Eq. (10.2.72) effectively divides the range of f into partitions that progressively refine with increasing N. In the calculations reported here, λ is taken to correspond to the respective superstable periodic cycle, where we note from Figs. 10.8*b*, *c* and *d*, that the corresponding super-stable dynamics faithfully reproduces the features of emergence during the first N iterates, followed by self-organization of the emerging structure for all times larger than N.

The numerical results of Table 10.6*a* suggest that

$$\lim_{N\to\infty} \chi_N = 1$$

[21] Recall that the fractal dimension of an object is formally defined very similarly:

$$D = \frac{\ln(\#\text{ self-similar pieces into which the object can be decomposed})}{\ln(\text{magnification factor that restores each piece to the original})}.$$

at the critical $\lambda = \lambda_* = 3.5699456$. Since $\chi = 0$ gives the simplest linear relation for f, a value of $\chi = 1$ indicates the largest non-linearly emergent complexity so that the logistic interaction is maximally complex at the transition to the fully chaotic region. It is only in this region $3 \leq \lambda < \lambda_*$ of resources that a global synthesis of stability inspired self-organization and instability driven emergence lead to the appearance of a complex structure.

λ		N						
		12	14	16	18	20	$\to$	∞
3.5700	$\langle f(0.5)\rangle$	5.057857	10.69732	31.38651	119.4162	468.8398		
	χ_N	0.805123	0.755773	0.689245	0.616675	0.556352	$\stackrel{?}{\to}$	0.0000
3.6000	$\langle f(0.5)\rangle$	275.7782	1125.908	4480.310	17996.46	72205.91		
	χ_N	0.324386	0.275938	0.241914	0.214699	0.193009	$\stackrel{?}{\to}$	0.0000
3.7000	$\langle f(0.5)\rangle$	885.4386	3683.121	14863.74	59511.41	236942.7		
	χ_N	0.184146	0.153806	0.133781	0.118840	0.107291	$\stackrel{?}{\to}$	0.0000
3.8000	$\langle f(0.5)\rangle$	1167.886	4597.633	18266.08	73197.48	293016.6		
	χ_N	0.150860	0.130952	0.115195	0.102249	0.091969	$\stackrel{?}{\to}$	0.0000
3.9000	$\langle f(0.5)\rangle$	1381.043	5595.363	22404.29	89472.39	358001.9		
	χ_N	0.130705	0.110713	0.096782	0.086158	0.077520	$\stackrel{?}{\to}$	0.0000
3.9999	$\langle f(0.5)\rangle$	1691.944	6625.197	26525.88	106254.9	424020.1		
	χ_N	0.106294	0.093304	0.081555	0.072379	0.065311	$\stackrel{?}{\to}$	0.0000
4.0000	$\langle f(0.5)\rangle$	14.00000	16.00000	18.00000	20.00000	22.00000	$\to$	$N+2$
	χ_N	0.682720	0.714286	0.739380	0.759893	0.777028	$\to$	1

Table 10.6*b*. Illustrates how the fully chaotic region of $\lambda_* < \lambda < 4$ is effectively "linear" with no self-organization, and only emergence. The jump discontinuity in χ at λ_* reflects a qualitative change in the dynamics, with the energy input for $\lambda \leq \lambda_*$ being fully utilized in the generation of complex internal structures of the system of emerging patterns and no self-organization .

What happens for $\lambda > \lambda_*$ in the fully chaotic region where emergence persists for all times $N \to \infty$ with no self-organization, is shown in Table 10.6*b* which indicates that on crossing the chaotic edge, the system abruptly transforms to a state of *effective linear simplicity* that can be interpreted to result from the drive toward ininality and effective bijectivity on saturated sets and on the component image space of f. This jump discontinuity in χ demarcates order from chaos, linearity from (extreme) nonlinearity, and simplicity from complexity. This non-organizing region $\lambda > \lambda_*$ of deceptive simplicity characterized by dissipation and irreversible "frictional losses", is to be compared with the nonlinearly complex domain $3 \leq \lambda < \lambda_*$ where irreversibility generates self-organizing useful changes in the internal structure of the system in order to attain the levels of complexity needed in the evolution. While the state of eventual evolutionary homeostasy appears only in $3 \leq \lambda < \lambda_*$, the relative linear simplicity of $\lambda > \lambda_*$ arising from the dissipative losses characteristic of this region conceals the resulting self-organizing thrust of the higher periodic windows of this region, with the smallest period 3 appearing at $\lambda = 1 + \sqrt{8} = 3.828427$. By the Sarkovskii ordering of natural numbers, there is embedded in this fully chaotic region a backward arrow that induces a chaotic tunnelling to lower periodic stability eventually terminating with the period doubling sequence in $3 \leq \lambda < \lambda_*$. This decrease in λ in the face of the prevalent increasing disorder in the over-heated scorching $\lambda > \lambda_*$ region reflecting the negmatter effect of "letting off steam", is schematically indicated in Fig. 10.10*a* and is expressible as

$$x \longrightarrow f_\lambda(x) \left\{ \begin{array}{r} \text{self-organizing complex system} \\ 3 \leq \lambda < \lambda_*,\ 0 < \chi \leq 1, \\ \\ \xrightarrow{\text{ininality}} \lambda_* \leq \lambda \leq 4,\ \chi = 0, \\ \text{chaotic complex system} \end{array} \right\} \begin{array}{r} \overset{\text{regulating}}{\longleftarrow} \\ \text{Sarkovskii} \uparrow \\ \underset{\text{effects}}{\longrightarrow} \end{array} \qquad (10.2.74)$$

Under normal circumstances dynamical equilibrium is attained, as argued above, within the temporal, iterational, self-organizing component of the loop above. If, however, the system is spatially driven by an increasing λ into the fully chaotic region, the global negworld effects of its periodic stable windows acts as a deterrent and, prompted by the Sarkovskii ordering induces the system back to its self-organizing region of equilibration. This condition of dynamical homeostasy is thus marked by a balance of *both the spatial and temporal effects,* with each interacting synergetically with the other to generate an optimum dynamical state of stability, with Figs. 10.8*b, c*, *d* clearly illustrating how new, distinguished and non-trivial features of the evolutionary dynamics occur only at the 2^N unstable fixed points of f_λ, leading to emerging patterns that characterize the net resources λ available to the interaction.

Panels (i), (ii), and (iii) of Fig. 10.10*a* magnifies these features of the defining fixed points and their classes for $3 \leq \lambda < \lambda_*$ that generates the stable-unstable signature in the graphically convergent limit of $t \to \infty$, essentially reflecting the synthetic cohabitation of the matter-negmatter components associated with these points. This in turn introduces a sense of symmetry with respect to the input-output axes of the interaction that, as shown in panel (iii), is broken when $\lambda > \lambda_*$ with the boundary at the critical $\lambda = \lambda_*$ signaling this physical disruption with a discontinuity in the value of the chanoxity index χ. Fig. 10.10*b* which summarizes these observations, identifying the self-organizing emergent region $3 \leq \lambda < \lambda_*$ as the "life" supporting complex domain of the logistic interaction f_λ. Below $\lambda = 3$, the resources of f_λ are insufficient in generating complexity, while above $\lambda = \lambda_*$ too much "heat" is

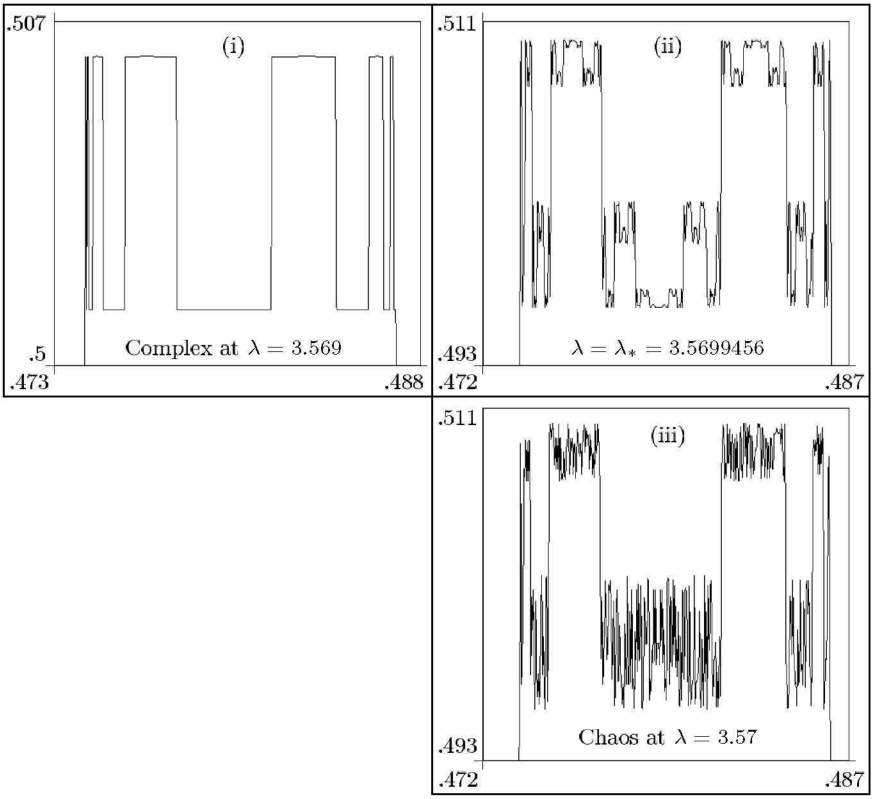

Fig. 10.10*a*. In contrast with the relatively tame (i) and (ii), panel (iii) illustrates the property of fully chaotic maximal ill-posedness and instability.

produced for support of constructive competition between the opposing directions, with the drive toward uniformity of ininality effectively nullifying the reverse competition. χ is in fact the irreversibility index ι in the complexity range $3 \leq \lambda < \lambda_*$. Both these parameters lie in the identical unit interval [0,1], with absence of disorder-inducing P at $(\iota = 0)(T = T_c)$ corresponding to the order-freezing $\lambda = 3$ and absence of order-generating E at $(\iota = 1)(T = T_h)$ consistent with the disorder-disintegrating $\lambda = \lambda_*$. The later case is effectively indistinguishable from the former because when the engine is not present no pump can be generated that shows up as an identical $\chi = 0$ for $\lambda > \lambda_*$. Significantly, however, while the former represents stability with reference to $\mathcal{D}(f)$ the later is stability with respect to $\mathcal{R}(f)$, and in the absence of an engine direction at $\iota = 1$ with increasing irreversibility and chanoxity, control effectively passes from the forward stabilizing direction to the backward destabilizing sense, thereby bringing the complementary neg-world effects into greater prominence through the appearance of singularities with respect to $\mathcal{D}(f)$. Finally, Fig. 10.10c which is a plot of the individual increasing and de-

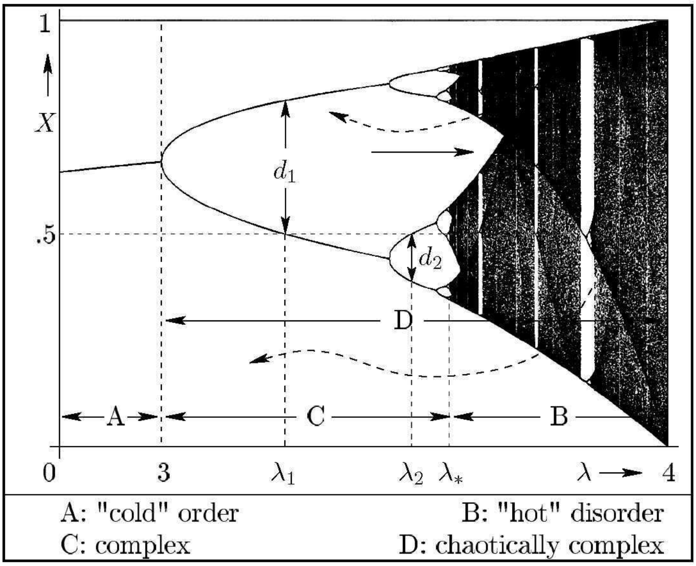

Fig. 10.10*b*. The dynamics of panels (i)-(iii) generates this division of the available resource into order, $0 \leq \lambda < 3$; chaos, $\lambda_* \leq \lambda$; and complex, $3 \leq \lambda < \lambda_*$. This complex region C is distinguished as a synthetic cohabitation of the stable-unstable opposites of A and B. The feedback of the chaos and the order regions constitutes the required synthesis to the higher level of complexity.

creasing parts of the logistic map confirms the observation that independent reductionist evolution of the component parts of a system cannot generate chaos or complexity. This figure, illustrating the unique role of non-injective ill-posedness in defining chaos, complexity and "life", clearly shows how the individual parts acting on their own in the reductionist framework and not in competitive collaboration, leads to an entirely different simple, non-complex, dynamics.

The figures of the dynamics in regions $\lambda < 3$ and $\lambda_* < \lambda$ of *actual* and *deceptive* simplicity can be interpreted in terms of symmetry arguments as follows [3]. In the former stable case of symmetry in the position of the individual parts of the system, the larger the group of transformations with respect to which the system is invariant the smaller is the size of the part that can be used to reconstruct the whole, and *symmetry is due to stability in the positions.* By comparison, the unstable chaotic region displays *statistical symmetry in the sense of equal probability of each component part that, without any fixed position, finds itself anywhere in the whole,* and symmetry is in the spatial or spatio-temporal averages.

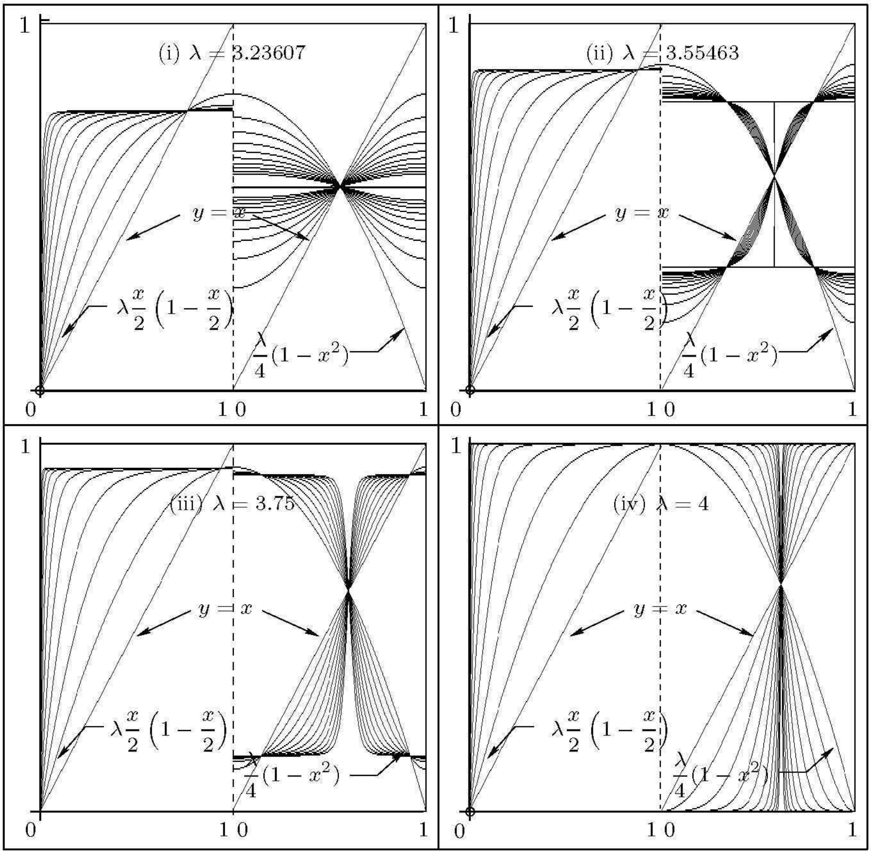

Fig. 10.10*c*. Reductionism cannot generate chaos or complexity or "life". This figure clearly illustrates the unique role of non-injective ill-posedness in defining chaos, complexity and "life", how the individual parts acting independently on their own in the reductionist framework not in competitive collaboration, leads to an entirely different simple, non-complex, dynamics.

10.3 What Is Life?

This 1944 question of Erwin Schroedinger [29, "one of the great science classics of the twentieth century"] credited with "inspiring a generation of physicists and biologists to seek the fundamental character of living systems" [13], suggests that "the essential thing in metabolism is that the organism succeeds in freeing itself from all the entropy it cannot help producing while alive", thereby maintaining order by consuming the available free energy in generating high entropy waste. In biology, "life" might mean the ongoing process of which living things are a part, or the period between birth and death of an organism, or the state of something that has been born and is yet to die. Living organisms require both energy and matter to continue living, are composed of at least one cell, are homeostatic, and evolve; life organizes matter into increasingly complex forms in apparent violation of the tenet of the second law that forbids order in favour of discord, instability and lawlessness.

Among the various characterizations of life that can be found in the literature, the following are particularly noteworthy.

- *Everything that is going on in Nature (implies) an increase of entropy of the part of the world where it is going on. A living organism continually increases its entropy and thus tends to approach the dangerous state of maximum entropy, which is death. It can only keep aloof from it, i.e. stay alive, by continually drawing from its environment "negative entropy". The essential thing in metabolism is that the organism succeeds in freeing itself from all the entropy it cannot help producing while alive by attracting, as it were, a stream of negative entropy upon itself (in order) to compensate the entropy increase it produces by living. It thus maintains itself stationery at a fairly high level of orderliness (= fairly low level of entropy) (by) continually sucking orderliness from its environment. In the case of higher animals we know the kind of orderliness they feed upon: the extremely well-ordered state of matter in more or less complicated organic compounds, which serve them as foodstuff. After utilizing it they return it in a very much degraded form — not entirely degraded, however, for plants can still make use of it. These, of course, have their most powerful supply of negative entropy in the sunlight.* Schroedinger [29].
- *Life is a far-from-equilibrium dissipative structure that maintains its local level of self-organization at the cost of increasing the entropy of the larger global system in which the structure is imbedded.* Schneider and Kay [27].
- *A living individual is defined within the cybernetic paradigm as a system of inferior negative feedbacks subordinated to (being at the service of) a superior positive feedback.* Korzeniewski [16].
- *Living things are systems that tend to respond to changes in their environment, and inside themselves, in such a way as to promote their own continuation; this may be interpreted to mean that a living system continuously computes the solution to the problem of its own continued existence through a process of internal adjustments to external causation.* Morales [19].

The message of bidirectional homeostasy implicit in the above passages forms the basis of Cinquin and Demongeot's *Positive and Negative Feedback: Striking a Balance Between Necessary Antagonists* [7] in a wide class of biological systems that possess multiple steady states. To deal with such classes of nonequilibrium systems, Schneider and Kay's [27] reformulation of Kestin's Unified Principle of Thermodynamics [14] implies that thermodynamic gradients drive self-organization, and chemical gradients lead to autocatalytic self-organizing dissipative reactions with positive feedback, with the activity of the reaction augmenting itself in self-reinforcing reactions, stimulating the global activity of the whole. Seen in this perspective, "life is a balance between the imperatives of survival and energy degradation" identifiable respectively with the backward and forward directions of Eq. (10.2.69). In the present context, it is more convenient and informative to view these arrows not by affine translation as was done in Sec. 10.2.2, but by considering the two worlds in their own reference frames with their forward arrows opposing each other and establishing a one-to-one correspondence between them; the activating and its regulating spaces are then equivalent[22]. This equivalence of the forward with its corresponding backward will serve to differentiate "life" from the normal complex system as suggested below.

All multicellular organisms are descendants of one original cell, the fertilized egg (or zygote) with the potential to form an entire organism through a process of bifurcation called mitosis. The function of mitosis is to first destabilize the zygote by constructing an exact copy of each chromosome and then to distribute, through division of the original (mother) cell, an identical set of chromosomes to each of the two progeny (daughter) cells. The two opposites involved in this process are the male — modeled by the increasing positive slope half of the logistic map — sperm cell (represented by the fixed point $x_{\mathrm{M}} = 0$) and the female — modeled by the decreasing, negative slope of the map — egg (represented by the fixed point $x_{\mathrm{F}} = (\lambda - 1)/\lambda$). The first cell division of the fertilized egg for $\lambda = 3$, initiates a chain of some 50 bifurcations to generate the approximately 10^{14} cells in an adult human, with each division occurring at equal intervals of approximately twenty hours. All of the approximately 200 distinct types of cells are derived from the single fertilized egg x_{F} through a process known as differentiation and specialization by which an unspecialized cell specializes into one of the many cooperating types, such as the heart, liver and muscle, each with its own individually distinctive role collaborating with the others to make up the whole living system. During this intricately regulated stage of self-organization, certain genes are turned on, or become activated, while other genes are switched off, or deactivated, so that a differentiated cell develops specific characteristics and performs specific functions. Differentiation involves changes in numerous aspects of cell physiology: size, shape, polarity, metabolic activity, responsiveness to signals, and gene expression profiles can all change during differentiation. Compare this with the emerging patterns of partitioning induced by the logistic map for number of iterates $\leq N$ in the 2^N stable cycle that resulted in the definition (10.2.73) of the chanoxity index in Sec. 10.2.3, followed by the self-organizing iterates for times larger than N. This sequence of destabilizing-stabilizing cell divisions

[22] Thus in $\mathbb{R}$, $\mid a \mid = \mid -a \mid$ defines an equivalence, and if $a < b$ then $-b < -a$ when viewed from $\mathbb{R}_+$, but $\mathfrak{a} < \mathfrak{b}$ in the context of $\mathbb{R}_-$. The basic fact used here is that two sets are "of the same size", or *equipotent*, iff there is a one-to-one correspondence between them.

represent emerging self-organization in the bidirectional synthetical organization (10.2.69) of a complex system: *through cell cooperation, the organism becomes more than merely the sum of its component parts.*

Abnormal growth of cells leading to cancer occur because of malfunctioning of the mechanism that controls cell growth and differentiation, and the level of cellular differentiation is sometimes used as a measure of cancer progression. A cell is constantly faced with problems of proliferation, differentiation, and death. The bidirectional control mechanism responsible for this decision is a stasis between cell regeneration and growth on the one hand and restraining inhibition on the other. Mutations are considered to be the driving force of evolution, where less favorable mutations are removed by natural selection, while more favorable ones tend to accumulate. Under healthy and normal conditions, cells grow and divide to form new cells only when the body needs them. When cells grow old and die, new cells take their place. Mutations can sometimes disrupt this orderly process, however. New cells form when the body does not need them, and old cells do not die when they should. Each mutation alters the behavior of the cell somewhat. This cancerous bifurcation, which is ultimately a disease of genes, is represented by the chaotic region $\lambda \geq \lambda_*$ where no stabilizing effects exist. Typically, a series of several mutations is required before a cell becomes a cancer cell, the process involving both oncogenes that promote cancer when "switched on" by a mutation, and tumor suppressor genes that prevent cancer unless "switched off" by a mutation.

Life is a specialized complex system of homeostasis between these opposites, distinguishing itself by being "alive" in its response to an ensemble of stratified hierarchal units exchanging information among themselves so as to maintain its entropy lower than the maximal possible for times larger than the "natural" time for decay of the information-bearing substrates. Like normal complex systems, living matter respond to changes in their environment to promote their own continued existence by resisting "the gradients responsible for the nonequilibrium condition". A little reflection however suggests that unlike normal complex systems, the activating direction in living systems corresponds not to the forward-inverse arrow of the physical world but to the backward-direct component with its increase of entropic disorder generating collaborative support from the restraining self-organizing effect of the forward component in an equilibrium of opposites. Thus it is the receptor "yin" egg x_{F} that defines the activating direction of evolution in collaboration with the donor "yang" sperm x_{M}, quite unlike the dynamics of the Lorenz equation, for example, that is determined by the activating temperature gradient acting along the forward arrow of the physical world.

In the present context, let us identify the backward-direct, catabolic, yin component $\mathfrak{M}$ of life $\mathfrak{L} := \{B, \mathfrak{M}\}$ as its *mind* collaborating competitively with the forward-inverse, anabolic, yang *body* B, and define

Definition 10.3 (Life). *Life is a special complex system of activating mind and restraining body.*

In this terminology, a non-life complex system (respectively, a dissipative structure) is a hierarchal (respectively, non-hierarchal) compound system with activating body and restraining mind. To identify these directions, the following illustrative examples should be helpful.

Example 10.2. (a) In the Lorenz model the forward-inverse arrow in the direction of the positive z-axis is, according to Fig. 10.5b, the activating direction of increasing order and self-organization. The opposing gravitational direction, by setting up the convection cells that reduces the temperature gradient by increasing the disorder of the cold liquid, marks the direction of entropy increase. Since the forward-inverse body direction is the activating direction, the Lorenz system denotes a non-life complex system. Apart from these organizing rolls representing "the system's response to move it away from equilibrium", availability of the angular variables prevents the Lorenz system from generating any additional emerging structures in the body of the fluid.

The familiar prototypical example of uni-directional entropy increase required by equilibrium Second law of Thermodynamics of the gravity dominated egg crashing off the table never to reassemble again is explained, in terms of Fig. 10.5b, as an "infinitely hot reservoir" dictating terms leading to eventual "heat death": unlike in the Lorenz case, the gravitational effect is not moderated here for example by the floor rising up to meet the level of the table, with the degree of disorder of the crashed egg depending on the height of the table.

(b) For the logistic map in the complexity region of λ, the activating backward-direct arrow $\{\mathfrak{D}, \{\mathfrak{D}, f(\mathfrak{D})\}, \{\mathfrak{D}, f(\mathfrak{D}), f^2(\mathfrak{D})\} \cdots\}$ is of increasing iterations, disorder, and entropy, while the restraining, expanding direction of self organization corresponds to decreasing non-injectivity of the increasing inverse iterates. Because the activating direction is that of the mind, the logistic dynamics is life-like.

The dominance of the physical realization M of $\mathfrak{M}$ as the *brain* in determining the dynamics of $\mathfrak{L}$ is reflected by the significance of sleep in all living matter. While there is much debate and little understanding of the evolutionary origins and purposes of sleep, there appears, nevertheless, to be a consensus that one of the major functions of sleep is consolidation and optimization of memories. However, this does not explain why sleep appears to be so essential or why mental functions are so grossly impaired by sleep deprivation. One idea is that sleep is an anabolic state marked by physiological processes of growth and rejuvenation of the organism's immune and nervous systems. Studies suggest sleep restores neurons and increases production of brain proteins and certain hormones. In this view, the state of wakefulness is a temporary hyperactive catabolic state during which the organism acquires nourishment and procreates: "sleep is the essential state of life itself". Anything that an organism does while awake is superfluous to the understanding of life's metabolic processes, of the balancing states of sleep and wakefulness. In support of this idea, one can argue that adequate rest and a properly functioning immune system are closely related, and that sleep deprivation compromises the immune system by altering the blood levels of the immune cells, resulting in a greater than normal chance of infections. However, this view is not without its critics who point out that the human body appears perfectly able to rejuvenate itself while awake and that the changes in physiology and the immune system during sleep appear to be minor. Nevertheless the fact that the brain seems to be equally — and at times more — active during sleep than when it is awake, suggests that the sleeping phase is not just designed for relaxation and rest. Experiments of prolonged sleep deprivation in rats led to their unregulated body temperature and subsequent death, is believed to be due to a lack of REM sleep of the dreaming phase. Although it is not clear to what extent these results generalize to humans, it is universally recognized that sleep deprivation has serious and diverse biological consequences, not excluding death. In the context of our two-component activating-regulating formulation of homeostasy and evolution, it is speculated that sleep, particularly its dreaming REM period, constitutes a change of guard that hands over charge of $\mathfrak{L}$ to its catabolic $\mathfrak{G}$ component from the anabolic M that rules the wakeful period. It is to be realized that all living matter are constantly in touch with their past through the mind; thus anything non-trivial that we successfully perform now depends on our ability to relate the present to the past involving that subject. In fact an index of the quality life depends on its ability to map the past onto the present and project it to the future, and the fact that a living body is born, grow and flourish without perishing (which an uni-directional second law would have), thanks to anabolic synthesis due to its immune system, is a living testimony to the bi-directionality of the direct-inverse arrow manifesting within the framework of the backward-forward completeness of the living world.

10.4 Conclusions: The Mechanics of Thermodynamics

In this paper we have presented a new approach to the nonlinear dynamics of evolutionary processes based on the mathematical framework and structure of multifunctional graphical convergence introduced in [30]. The basic point we make here is that the *macroscopic* dynamics of evolutionary systems is in general governed by strongly nonlinear, non-differential laws rather than by the Newtonian Hamilton's linear differential equations of motion

$$\frac{d\mathbf{x}_i}{dt} = \frac{\partial H(\mathbf{x})}{\partial \mathbf{p}_i}, \quad \frac{d\mathbf{p}_i}{dt} = -\frac{\partial H(\mathbf{x})}{\partial \mathbf{x}_i}, \qquad -\infty < t < \infty \tag{10.4.1}$$

of an N particle isolated (classical) system in its phase space of microstates $\mathbf{x}(t) = (\mathbf{x}_i(t), \mathbf{p}_i(t))_{i=1}^N$. As is well known, Hamiltonian dynamics leads directly to the microscopic-macroscopic paradoxes of Loschmidt's time-reversal invariance of Eq. (10.4.1), according to which all forward processes of mechanical system evolving according to this law must necessarily allow a time-reversal that would require, for example, that the Boltzmann H-function decreases with time just as it increases, and Zarmelo's Poincare recurrence paradox which postulates that almost all initial states of isolated bounded mechanical system must recur in future, as closely as desired. One approach — [10], [24] — to the resolution of these paradoxes require

(1) A "fantastically enlarged " phase space volume as the causative entropy increasing drive. Thus, for example, a gas in one half of a box equilibrates on removal of the partition to reach a state in which the phase space volume is almost as large as the total phase space available to the system under the imposed constraints, when the number of particles in the two halves becomes essentially the same. In this situation, *for a dilute gas* of N particles in a container of volume V under weak two-body repulsive forces satisfying the linearity condition $V/N \gg b^3$ with b the range of the force, Boltzmann identifies the thermodynamic Clausius entropy with $S_B = k \ln |\Gamma(M)|$, where $\Gamma(M)$ is the region in $6N$-dimensional Lioville phase space of the microstates belonging to the equilibrium macrostate M in question; the second law of thermodynamics then simply implies that an observed macrostate is the most probable in the sense that it is realizable in more ways than any other state. When the system is not in equilibrium, however, the phase space arguments imply that the relative volume of the set of microstates corresponding to a given macrostate for which evolution leads to a macroscopic decrease in the Boltzmann entropy *typically* goes exponentially to zero as the number of atoms in the system increases. Hence for a macroscopic system "the fraction of microstates for which the evolution leads to macrostates with larger Boltzmann entropy is so close to one that such behaviour is exactly what should be seen to always happen", [18]. A more recent interpretation[9] is to consider not the number of microstates of a macrostate M, but the most probable macroscopic history as that which can be realized by the greatest number of microscopic paths compatible with the imposed constraints. Paths, rather than states, are more significant in non-equilibrium systems because of the non-zero macroscopic fluxes whose statistical description requires consideration of the temporal causative microscopic behaviour.

(2) The statistical techniques implicit in the foregoing interpretation of macroscopic irreversibility in the context of microscopic reversibility of Newtonian mechanics rely fundamentally on the conservation of Lioville measures of sets in phase space under evolution. This means that if a state $M(t)$ evolves as $M(t_1) \stackrel{t_1<t_2}{\longrightarrow} M(t_2)$ such that the evolved phase space $\Gamma_{t_2}(M(t_1))$ of $M(t_1)$ is necessarily contained in $\Gamma(M(t_2))$ by the arguments in (1), then the preservation of measures requires that $\Gamma_{t_2}(M(t_1)) \subseteq \Gamma(M(t_2))$ by the law of increasing

S_B. Conversely, even as $M(t_2) \overset{t_1<t_2}{\longrightarrow} M(t_1)$ is not prohibited by the microscopic laws of motion, the exact identification of the subset $\Gamma_{t_2}(M(t_1)) \subseteq \Gamma(M(t_2))$ cannot be ensured *a priori* to enable the system to eventually end up in $\Gamma(M(t_1))$; although the macroscopic reverse process is permissible, it is improbable enough never to have actually occurred. Identifying the macrostate of a system with our image $f(x)$ of a microstate x in "phase space" $\mathcal{D}(f)$ that generates the equivalence class $[x]$ of microstates, invariance of phase space volume can be interpreted to be a direct consequence of the *linearity assumption of the Boltzmann interaction for dilute gases* that is also inherent in his *stosszahlansatz* assumption of molecular chaos which neglects all correlations between the particles.

(3) Various other arguments like cosmological big bang and the relevance of initial conditions preferring the forward arrow to the reverse are invoked to argue a justification for macroscopic irreversibility, that in the ultimate analysis is a "consequence of the great disparity between microscopic and macroscopic scales, together with the fact (or very reasonable assumption) that what we observe in nature is typical behaviour, corresponding to typical initial conditions", [10].

In comparison the multifunctional graphical convergence techniques, founded on difference rather than differential equations, adapted here avoids much of the paradoxical problems of calculus-based Hamiltonian mechanics, and suggests an alternate specifically nonlinear dynamical framework for the dissipative dynamical evolution of Nature supporting self-organization, adaption, and emergence in complex systems in a natural manner. The significant contribution of the difference equations is that evolution at any time depends explicitly on its immediate predecessor — and thereby on all its predecessors — leading to non-reductionism, self-emergence, and complexity.

To conclude, we recall the following passages from Jordan [11] as a graphic testimony to chanoxity:

Approximately one hundred participants met for three days at a conference entitled "Uncertainty and Surprise: Questions on Working with the Unexpected and unknowable". The diversity of the conference was vital (as) bringing together people with very different views strengthened the probability of extraordinary explanatory behaviour and the hope of producing entirely new structures, capabilities, and ideas. Out of our interconnections might emerge the kind of representation of the world that none of the participants, individually, possess or could possess. One purpose of the conference was to develop the capacity to respond to our changing science and to new ideas about the nature of the world as they relate to the unexpected and unknowable.

Participants recognized early on their difficulties in communicating with one another across the diversity of their backgrounds. One of the issues the group tried to resolve was differences in levels of understanding and experience related to the theme of uncertainty and surprise. The desire for a *common language was a reoccurring theme among conference participants as they tried to work out questions and ambiguities regarding even the fundamental themes of the conference, including the definitions of complexity, emergence, and uncertainty. Can we name or label what complexity is? Emergence was an idea that wove itself throughout much of the informal conversation, yet emergence as a term created confusion among the participants. There was acknowledgment of a need to state more clearly our assumptions with regard to fixed structure versus emergence. If you use "emergence" to mean in the complexity sense, it implies some sort of scale shift having to do with a fundamentally different structure of the organization of interactions, or a shift in the nature of the network, or of knowing, or awareness. Some conference participants cautioned the group not to equate emergence with miraculous magic.*

(It was) recognized that there are tendencies toward stability and tendencies toward variance. Our assumption about the value of stability may lead us to to our assumption of the value of permanence. There is evidence that the value of permanence may be a socially constructed Western trap that is not shared by Eastern philosophies. Complexity science leads us to understand that the degree of variability in the distribution of fluctuations in system dynamics is more important than any average quantity, which is counter to the traditional paradigms of medicine, management, and scientific research. We used to believe that equilibrium was the optimal for systems. Complexity science leads us to believe that stability is death and survivality is in variability. The tension between stability and variability is similar to the tension in the social sciences between exploitation and exploration. We often think of exploitation as a strategy for maintaining stability and exploration as a strategy for exploiting variability. We may need a balance between exploration and exploitation, stability and variability, convergence and divergence within a state.

An issue that resurfaced several times throughout the conference was the relationship between individual elements and collective elements. Traditionally Western thought has tended toward the individual over the collective; the opposite view is often taken by Eastern thought. It is not a question of either the individual or the collective, but the interaction of the two that is needed; $\cdots$ the individual and the group are the singular and plural of the same process. In order to honor the tension between the individual and the collective, a good model might be "If you win I win; if I lose, you lose". One participant felt that you can design an organization in such a way that people profited or lost together based upon how well they all did. One of our best levers for facing uncertainty and surprise might be to encourage quasi-autonomy (individuality) but at the same time willingness to cooperate across disciplines because this kind of collaboration gives us more capabilities and skills.

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