Nonlinear Analysis: Real World Applications ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

# Is nature quantum non-local, complex holistic, or what?: II — Applications

A. Sengupta

*Department of Mechanical Engineering, Indian Institute of Technology Kanpur, Kanpur-208016, India*

ARTICLE INFO

*Article history:*
Received 27 August 2008
Accepted 18 September 2008

*Keywords:*
ChaNoXity
Entanglement
Quantum measurment
Logistic map
Rigged Hilbert space
Black hole

ABSTRACT

In Part I of this paper [A. Sengupta, Is nature quantum non-local, complex holistic, or what? I – Theory & analysis. Nonlinear Anal.: RWA (2009) (in press)] to be referenced "I-", we examined the linear–nonlinear divide of the natural world in an attempt to seek a rationale for the question "Is NATURE INTERACTIVELY NONLINEAR AND HOLISTIC, OR IS IT ADDITIVELY LINEAR AND REDUCTIONIST?": Is Nature governed by entanglements of linear superposition or does it represent the nonlinear holism of emergence, self-organization, and complexity? This second part carries the debate forward to propose that Quantum Mechanics is an effective linear representation of a fully chaotic, maximally illposed, multifunctional negworld that obviously is not just a mirror image of the functional real world we inhabit: in fact we argue that nonlinear complex holism represents a stronger form of entanglement than linear quantum non-locality. The bi-directionality of a self-organized, emergent, engine-pump system is analyzed with reference to the role of gravity as the compressive agent responsible for generation and maintenance of structures and life in Nature; we also explore the applicability of chanoxity to the metaphorical resolution of some of the long-standing paradoxes and puzzles in quantum measurement and non-locality, in Prigoginian intrinsic irreversibility, and in some core issues in cosmology and gravitational black holes.

Holism is to be seen as complementing mainstream reductionism – linear science has after all stood the test of the last 400 years as quantum mechanics is acknowledgedly one of the most successful yet possibly one of the most mysterious of scientific theories: the success lies in its capacity to classify and predict the physical world, the mystery in what this physical world must be like to behave quantum mechanically – providing a unified picture of the dialectics of the evolutionary dynamics of Nature.



## 1. Introduction

This concluding exposition of the two-part thesis initiated in Sengupta [2] completes its assertions by providing instances and arguments of the applicability of chanoxity to the metaphorical resolution of some of the long-standing paradoxes and puzzles in quantum measurement and non-locality, in the Prigoginian concept of intrinsic irreversibility, in the possible role of gravity in sustenance of the delicate balance of homeostatic dialectics in structures and life occurring in Nature, and in some core issues arising in cosmology and gravitational black holes. The central proposition of the thesis is the self-inducement of a bi-directional, moderating inhibition of the destructive eventuality of the Second Law of Thermodynamics in the spirit of a "go-slow" served by Nature itself. These emergent, self-organizing effects are purely nonlinear: in fact so severely nonlinear as to induce chaos and complexity in the evolving structure of the systems. This brings into focus the

*E-mail address:* osegu@iitk.ac.in.

1468-1218/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.nonrwa.2008.09.002

distinguishing role of nonlinearity in the evolving dialectics of Nature: Linear systems cannot be chaotic, hence complex, and therefore holistic.

The science of holism that considers "systems and processes that interact with themselves and produce themselves from themselves", it has been argued in [2], calls for radically different methods and philosophy from those of the mainstream reductionist Newtonian paradigm. Part I – Theory & Analysis enunciates the rules and methodologies of this new paradigm of a "Participatory Universe": holism attends to the entire field assigning causality to it, making relatively little use of categories and formal logic and relying more on "dialectic" reasoning, while analyticism pays attention primarily to the object and the categories to which it belongs using rules including formal logic, to understand its behavior. Holistic thought has been defined [3] as "involving an orientation to the context or field as a whole, including attention to relationships between a focal object and the field and a preference for predicting events on the basis of such relationships. Holistic approaches rely on experience-based knowledge rather than on abstract logic and are dialectical, meaning that there is emphasis on change, a recognition of contradictions and of the need for multiple perspectives and a search for the middle way between opposing propositions". Analytic thought, on the other hand, involve "detachment of the object from its context, a tendency to focus on attributes of the object to assign it to categories, and a preference for using rules about the categories to explain and predict the object's behavior. Inferences rest in part on the practice of de-contextualizing structure from content, the use of formal logic, and avoidance of contradictions".

In the traditional exophysical point of view, a complete description of the world is possible without interacting with it; endophysics differs in maintaining that the view from within a system generally differs from that obtained by an external observer. The possibly "heavy" and largely "unfamiliar" mathematical theory of chanoxity involving multifunctions, convergence in exclusion topology, jumps, discontinuities, multiplicities and ill-posedness, and inverse and direct limits as developed in I – with its attendant philosophical interpretations – lead us to believe that the bi-directional Hegelian *thesis-antithesis-synthesis* dialectic evolution of opposites in the emergence and chaotic self-organization of chanoxity might very well be a general principle that nature obeys in all its manifestations.

The second half of this series concludes this dispensation in the hope that the dielectics of opposites essentially expressed in the engine-pump duality of chanoxity, will induce serious introspection of the reconciliatory unification of the good in both synthetic holism and logical analyticity as the apparent dialectics of Nature.

## 2. Bi-directional chanoxity and · · ·

• *Every "it" — every particle, every force, even the spacetime continuum itself — derives its function, its meaning, its very existence entirely — even if in some contexts indirectly — from the apparatus-elicited answers to yes-or-no questions, binary choices, bits. "It from bit" symbolizes the idea that the physical world has at bottom — at a very deep bottom, in most instances — an immaterial source and explanation; that which we call reality arises in the last analysis from the posing of yes-no questions and the registering of equipment-evoked responses; in short, that all things physical are information-theoretic in origin and this is a participatory universe.* WHEELER [4]

### 2.1. · · · quantum non-locality, complex holism and the participatory universe

• *An experiment is an active intervention into the course of Nature. We set up this or that experiment to see how Nature reacts. If from such a description we can further distill a model of a free-standing "reality" independent of our interventions then so much the better. Classical physics is the ultimate example of such a model. However, there is no logical necessity for a realistic worldview to always be obtainable. If the world is such that we can never identify a reality independent of our experimental activity, then we must be prepared for that, too.* FUCHS AND PERES [5]

• *It has been suggested that quantum phenomena exhibit a characteristic holism or nonseparability, and that this distinguishes quantum from classical physics. The puzzling statistics that arise from measurements on entangled quantum systems demonstrate, or are explicable in terms of, holism or nonseparability rather than any problematic action at a distance.* STANFORD ENCYCLOPEDIA OF PHILOSOPHY

Possibly the most primitive of the tensions between classical and quantum perspectives of Nature is the conflict between the Copenhagen interpretation pioneered by the Bohr and Heisenberg that regards quantum mechanics to be intrinsically about awareness, observations, and measurements emanating from the unitary evolution of Schrodinger equation with precious little to offer on what it really is *ontologically*, or what in fact it seeks to describe,[1] and the deterministic world-views as advanced for example by Bohm [7] and 'tHooft [1]. According to this doctrine, the wavefunction is simply an auxiliary mathematical tool devoid of any physical significance, whose only import lies in its ability to generate probabilities: it compactly represents our knowledge of the preparation and subsequent evolution of a physical system. Niels Bohr persisted that only experimental results lie in the purview of physical theories; any ontological questions are unscientific and must be

[1] "It seems clear that quantum mechanics is fundamentally about atoms and electrons, quarks and strings, not those particular macroscopic regularities associated with what we call *measurements* of the properties of these things. But if these entities are not to be somehow identified with the wave function itself then where are they to be found in the quantum description?" [6].

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

eschewed. The $|\Psi\rangle$ function has only a "symbolic" significance in associating expectation values with dynamic variables and does not represent anything real; according to Bohr, who lays defining significance to it, the imaginary component in the state variable does not allow it to pictorially represent the real world although he held atoms to physically exist in the sense that they are "neither heuristic nor logical constructions". The quantum "object" defined by the state function $|\Psi\rangle$ is distinct from the classical "measuring device" and the association of these complementary classical concepts to the measurement process depend on the specific experimental "context" of the phenomena. These mutually exclusive experiments on the complete setup of the quantum and classical components provide an exclusively exhaustive objective knowledge of the system in terms of a complete set of orthogonal projectors $\{P_i\}$. The dynamics of the Schrodinger equation describes how the knowledge of the system changes as a function of time.

Our objective in the present section is to interpret this global view of the quantum-classical coexistence in the light of our negworld-world paradigm. For the sake of completeness, we provide below a quick overview of those basic features of quantum mechanics relevant in the present context.

The interpretative understanding of quantum measurements is based on the so-called *projection (collapse) postulate* arising from the linearity of superpositions that the mathematics of quantum mechanics must respect. Accordingly, on measurement of properties represented by operators $\mathscr{S}$ and $\mathscr{E}$ on systems $S$ and $E$, the state $|\Psi\rangle_{SE} \in \mathscr{H}_S \otimes \mathscr{H}_E$ of the 2-particle system in $2^2$-dimensional tensor product of the two 2-dimensional Hilbert spaces $\mathscr{H}_S$ and $\mathscr{H}_E$ with $|\phi\rangle \in \mathscr{H}_S$, $|\psi\rangle \in \mathscr{H}_E$ satisfying

> Linearity: $\alpha\,|\phi\rangle \otimes |\psi\rangle = \alpha\,(|\phi\rangle \otimes |\psi\rangle)$, Distributivity: $|\phi\rangle \otimes (|\psi_1\rangle + |\psi_2\rangle) = |\phi\rangle \otimes |\psi_1\rangle + |\phi\rangle \otimes |\psi_2\rangle$, Commutativity: $|\phi\rangle \otimes |\psi\rangle = |\psi\rangle \otimes |\phi\rangle$, Adjointness: $(|\phi\rangle \otimes |\psi\rangle)^{\mathrm{H}} = \langle\phi| \otimes \langle\psi|$, Inner Product: $(\langle\phi_1| \otimes \langle\psi_1|)\,(|\phi_2\rangle \otimes |\psi_2\rangle) = \langle\phi_1 \mid \phi_2\rangle\,\langle\psi_1 \mid \psi_2\rangle$, Outer Product: $(|\phi_1\rangle \otimes |\psi_1\rangle)\,(\langle\phi_2| \otimes \langle\psi_2|) = |\phi_1\rangle\langle\phi_2| \otimes |\psi_1\rangle\langle\psi_2|$. Note how linearity constrains the parts to interact only among themselves in the two product variants.

must "collapse" to an eigenvector $|s_i\rangle \otimes |e_i\rangle$ of the observable $\mathscr{S} \otimes \mathscr{E}$ with eigenvalue $\sigma_i\varepsilon_i$; thus

$$|\Psi\rangle_{SE} \mapsto \frac{P_i\,|\Psi\rangle_{SE}}{\sqrt{{}_{SE}\,\langle\Psi\,|P_i|\,\Psi\rangle_{SE}}}$$

represents the new "state of knowledge" acquired by the measurement. The difference from classicality is a consequence of the superposition principle that allows the quantum system to be in any of the $2^N$ basic states simultaneously leading to the non-classical manifestations of interference, non-locality and entanglement, a quantum mechanical phenomenon in which the quantum states of two or more objects have to be described with reference to each other, even though the individual objects may be spatially separated.

**Definition 1** (*Entanglement*). Any

$$|\Psi\rangle_{SE} = \sum_{i,j} \alpha_{ij}\,|\phi_i\rangle \otimes \left|\psi_j\right\rangle$$

in $\mathscr{H}_S \otimes \mathscr{H}_E$ that cannot be factored as a tensor product of vectors of its parts $\{|\phi_i\rangle\} \in \mathscr{H}_S$ and $\left\{\left|\psi_j\right\rangle\right\} \in \mathscr{H}_E$, that is $|\Psi\rangle_{SE} \neq |\phi\rangle \otimes |\psi\rangle$, is said to be **entangled** *(non-local);* $|\Psi\rangle_{SE}$ is **unentangled** *(separable)* if it is factorisable into its components.

**Example 1.** Choose orthonormal bases $\{|\uparrow\rangle_S\,, |\downarrow\rangle_S\}$ and $\{|\uparrow\rangle_E\,, |\downarrow\rangle_E\}$ in $\mathscr{H}_S$ and $\mathscr{H}_E$ so that $\mathscr{H}_{SE}$ is spanned by the vectors $|\uparrow\rangle_S\,|\uparrow\rangle_E, |\uparrow\rangle_S\,|\downarrow\rangle_E, |\downarrow\rangle_S\,|\uparrow\rangle_E$, and $|\downarrow\rangle_S\,|\downarrow\rangle_E$ in $\mathscr{H}_{SE}$. Then for the qubits[2] $|\Phi\rangle_S = \sigma_1\,|\uparrow\rangle_S + \sigma_2\,|\downarrow\rangle_S$ and $|\Upsilon\rangle_E = \varepsilon_1\,|\uparrow\rangle_E + \varepsilon_2\,|\downarrow\rangle_E$, $|\Psi\rangle_{SE} = \sigma_1\varepsilon_1\,|\uparrow\uparrow\rangle + \sigma_1\varepsilon_2\,|\uparrow\downarrow\rangle + \sigma_2\varepsilon_1\,|\downarrow\uparrow\rangle + \sigma_2\varepsilon_2\,|\downarrow\downarrow\rangle = |\Phi\rangle_S\,|\Upsilon\rangle_E$ is a ***separable*** state, an example of ***nonseparable entangled*** state being $|\Psi\rangle_{SE} = \alpha_1\,|\uparrow\uparrow\rangle + \alpha_2\,|\downarrow\downarrow\rangle \neq |\Phi\rangle_S\,|\Upsilon\rangle_E$. A 2-qubit state is separable iff $ad = bc$ for $|\Psi\rangle_{SE} = a\,|\uparrow\uparrow\rangle + b\,|\uparrow\downarrow\rangle + c\,|\downarrow\uparrow\rangle + d\,|\downarrow\downarrow\rangle$. An entangled state does not define vectors in the individual factor spaces $\mathscr{H}_S$ and $\mathscr{H}_E$ unless the state is actually unentangled.

Thus for physically separated $S$ and $E$, a measurement outcome of $|\uparrow\rangle$ on $S$ implies that any subsequent measurement on $E$ in the same basis will always yield $|\uparrow\rangle$. If $|\downarrow\rangle$ occurs in $S$, then $E$ will be guaranteed to return $|\downarrow\rangle$; hence system $|E\rangle$ has been altered by local random operations on $|S\rangle$. This non-local puzzle of entangled quantum states — the orthodox Copenhagen doctrine maintains that neither of the particles possess any definite position or momentum before they are measured — is resolved by bestowing quantum mechanics with non-local properties determined by Bell's inequality.

In the linear setting of quantum mechanics, multipartite systems modeled in $2^N$-dimensional tensor products $\mathscr{H}_1 \otimes \cdots \otimes \mathscr{H}_N$ of 2-dimensional spin states, correspond to the $2^N$ "dimensional space" of unstable fixed points of chanoxity, see [8], page 336. This formal equivalence illustrated in Fig. 1 while clearly demonstrating how holism emerges in $2^N$-cycle complex systems, also focuses on the significant differences between complex holism and quantum non-locality that can eventually be traced to the constraints of linear superposition and the consequent reductionism. The converged holistic

2 Unlike a *bit* which must be either of the two possible values "on" ($|\uparrow\rangle$) or "off" ($|\downarrow\rangle$), the *qubit* can be either $|\uparrow\rangle$, or $|\downarrow\rangle$, or a superposition $\alpha\,|\uparrow\rangle + \beta\,|\downarrow\rangle\big(\text{with } \alpha^2 + \beta^2 = 1\big)$ of both.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

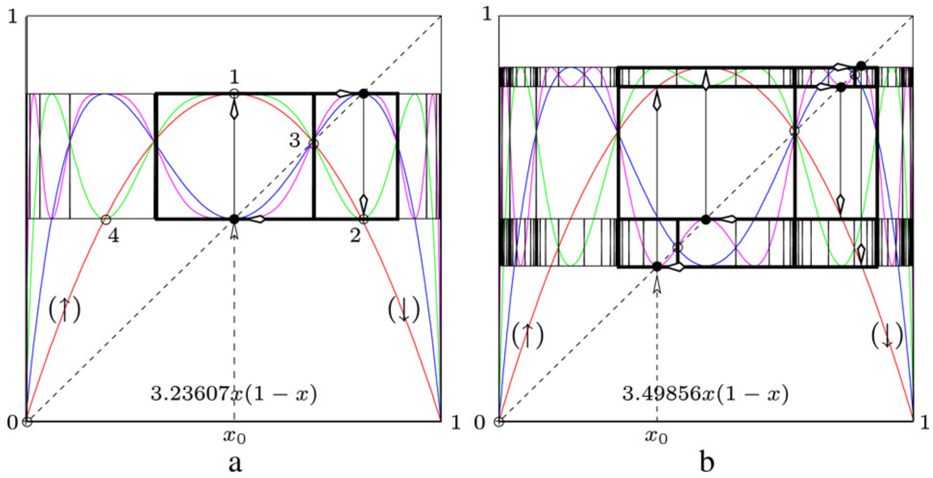


**Fig. 1.** "Entanglement" and "non-locality" in discrete dynamical systems. Panels (a) and (b) demonstrate the increasing complexity of evolution with increasing λ: the significant point is that the stable dynamics generated by the respective emergent $2^N$-periodic cycle display "entanglement" of the ↑ and ↓ components that spreads out in space and time as the system self-organizes itself to the graphically converged multifunctional limiting sets indicated by the heavy lines. Thus the parts surrender their individuality to the holism induced by the periodic cycles. The model 1 → 2, 2 → 1, 4 → 1, 3 → 3 shows that "if information is allowed to dissipate we have to treat the equivalence classes of states as the basis of a quantum Hilbert space" [1], see Section 4.

behavior of complex "entanglement" reflects the fact that the subsystems have combined nonlinearly to form an emergent, self-organized system – denoted by the heavy $2^1$ and $2^2$ cycles in Fig. 1(a) and (b) – that cannot be decoupled without destroying the structure itself; contrast with the quantum concept of Definition 1 and the notion of partial tracing for obtaining properties of individual components from the whole in Eqs. (4a) and (4b). Periodic cycles are the "eigenfunctions" of the *generalized nonlinear eigenvalue equation* $f^{(p)}(x) = x$ with "eigenvalue" $p$; unlike the linear case, however, these composite cycles are not linearly superposed but appear as emergent, self-organized, holistic entities. *In this sense complex holism represents a stronger form of "entanglement" than Bell's nonlocality*: linear systems cannot be chaotic, hence complex, and therefore holistic. While quantum non-locality is a paradoxical manifestation of linear tensor products, complex holism is a natural expression of the nonlinearity of emergence and self-organization. Nature uses chaos as an intermediate step in the attainment of states that would otherwise be inaccessible to it. Well-posedness of a system is an extremely inefficient way of expressing a multitude of possibilities as this requires a different input for every possible output; nature chooses to express its myriad manifestations through the multifunctional route. The countably many outputs arising from the non-injectivity of $f$ for a given input is interpreted to define complexity because in a nonlinear system each of these possibilities constitute a experimental result in itself that may not be combined in any definite predetermined manner. This is in sharp contrast to linear systems where a linear combination, governed by the initial conditions, always generate a unique end result. This multiplicity of possibilities that have no predetermined combinatorial property is the basis of the diversity of Nature.

Bipartite states $|\Psi\rangle_{SE}$, entangled or not, are relevant only when maximal knowledge of the preparation and evolution of the system is available and intrinsic uncertainty from basis superposition is the only source of incomplete knowledge. Very often though, a more fundamental source of uncertainty is represented by a *density matrix* on the Hilbert space: if one has a large number of copies of the same system, then the state of this ensemble is described by a density matrix, an idempotent, Hermitian projection $P$ on some ray of $\mathcal{H}$. If $\{p_m, P_m\}$ is a family of projections projecting onto rays of $\mathcal{H}$, then

$$\rho \triangleq \sum_{m=1}^{M} p_m P_m$$

is a density operator provided $p_m \geq 0$ and $\sum p_m = 1$; a density that is not itself a projector, $\rho^2 \neq \rho$, can be expressed in an infinite number of ways as a weighted sum of projections onto rays. An operator

$$\rho = \sum_{m=1}^{M} p_m \, |\Psi_m\rangle\langle\Psi_m| \tag{1}$$

associated with an ensemble $\{p_m, |\Psi_m\rangle\}$ projecting onto $|\Psi_m\rangle$ with probability $p_m$ is a density operator iff, it is positive of unit trace.[3] When $M = 1$, $\rho = |\Psi\rangle\langle\Psi| = \rho^2$ projecting $\mathcal{H}$ onto the one-dimensional span of $|\Psi\rangle$ is a **pure state** if $\mathrm{Tr}(\rho^2) = 1$; for $M > 1$ the *convex combination of pure states* is a density matrix of **mixed state** on $H_S \otimes H_E$ with $\mathrm{Tr}(\rho^2) < 1$. The $p$ and $\Psi$ can be conveniently chosen to be the eigenvalues and eigenfunctions of $\rho$.

[3] The *outer product* $|w\rangle\langle v|$ defines a linear operator $V \to W$ as $(|w\rangle\langle v|)\,|x\rangle = \langle v \mid x\rangle\,|w\rangle$.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

When two quantum systems interact they are entangled (correlated), destroying the coherence of superposition of the individual states of the subsystems. In a composite system, it is impossible to ascribe definite states to any of the components: typically such subsystems are represented by mixed density operators representing superposition of the individual states. Mixed states represents incomplete knowledge in the preparation of $|\Psi\rangle$; alternatively mixed states arise for an ensemble $\{p_m, |\Psi_m\rangle\}$. To investigate density matrices we need the

**Theorem 1** (*Schmidt Decomposition of Composite States*). *Let $\mathcal{H}_S$ and $\mathcal{H}_E$ be Hilbert spaces of dimensions n and m respectively. For any state $|\Psi\rangle_{SE}$ of a composite system $\mathcal{H}_S \otimes \mathcal{H}_E$, entangled or not, there exist orthonormal bases $\{|\phi_i\rangle\}_{i=1}^r \subset \mathcal{H}_S$, $\{|\psi_i\rangle\}_{i=1}^r \subset \mathcal{H}_E$ and scalars $r \le \min\{m, n\}$, $\{\lambda_i\}_{i=1}^r \ge 0$, $\sum_{i=1}^r \lambda_i = 1$, such that*

$$|\Psi\rangle_{SE} = \sum_{i=1}^{r} \sqrt{\lambda_i}\, |\phi_i\rangle\, |\psi_i\rangle\, . \tag{2}$$

*The number r in the* **Schmidt decomposition** (2) *is the* **Schmidt rank**, *and $\sqrt{\lambda_i}$ are the* **Schmidt coefficients** *of $|\Psi\rangle_{SE}$.*

Technically, the $m \times n$ matrix $\Gamma = (\gamma_{kl}) : \mathcal{H}_S \to \mathcal{H}_E$ "diagonalized" as $\Gamma = U \Sigma W^{\mathrm{H}}$ ($^{\mathrm{H}}$ is the Hermitian conjugate) by the unitary matrices $U_{m\times m}$, $W_{n\times n}$ with orthonormal columns of eigenvectors of $\Gamma\Gamma^{\mathrm{H}}$ and $\Gamma^{\mathrm{H}}\Gamma$, and $\Sigma_{m\times n}$ a rectangular diagonal matrix of $r \le \min\{m, n\}$ "singular values" of the positive square roots of their common non-zero eigenvalues, it can be shown that [9]

- The last $n - r$ columns of $W$ generate the null-space of $\Gamma$.
- The first $r$ columns of $W$ generate the row space of $\Gamma$.
- The last $m - r$ columns of $U$ generate the left null-space of $\Gamma$.
- The first $r$ columns of $U$ generate the column space of $\Gamma$.

Thus a quantum state in its Schmidt representation represents a joint state in the enlarged $mn$-dimensional space $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E$ of effective dimension $r \le \min\{m, n\}$ with the matrix $\Gamma$ reducing $\mathcal{H}_S$ and $\mathcal{H}_E$ both to the same dimension $r$; the free dimensions $n - r$ and $m - r$ imply irreversible non-invertibility leading to ill-posedness and multiplicities; recall a linear algebraic system of $m$ equations in $n$ unknowns of rank $r$.

For the composite system with a pure density matrix

$$\rho_{SE} \triangleq |\Psi\rangle_{SE}\, \langle\Psi| = \sum_{i,j=1}^{r} \sqrt{\lambda_i\lambda_j}\, |\phi_i\rangle\, |\psi_i\rangle\, \langle\phi_j|\, \langle\psi_j| \tag{3}$$

corresponding to (2), the reduced operator for subsystem $S$ is[4]

$$\begin{aligned}\rho_S &\triangleq \mathrm{Tr}_E\left(|\Psi\rangle_{SE}\, \langle\Psi|\right)\\ &= \sum_{i,j=1}^{r} \sqrt{\lambda_i\lambda_j}\, |\phi_i\rangle\, \langle\phi_j|\, \mathrm{Tr}_E\left(|\psi_i\rangle\, \langle\psi_j|\right)\\ &= \sum_{i,j=1}^{r} \sqrt{\lambda_i\lambda_j}\, |\phi_i\rangle\, \langle\phi_j|\, \langle\psi_i \mid \psi_j\rangle\, .\end{aligned}$$

Hence the diagonal mixtures in the Schmidt basis

$$\rho_S = \sum_{i=1}^{r} \lambda_i\, |\phi_i\rangle\, \langle\phi_i| \tag{4a}$$

$$\rho_E = \sum_{i=1}^{r} \lambda_i\, |\psi_i\rangle\, \langle\psi_i|\, , \tag{4b}$$

for $\lambda_i > 0$, $\sum_{i=1}^r \lambda_i = 1$ are expressed in terms of the identical non-zero diagonal elements $\{\lambda_i\}_{i=1}^r$ of $\Gamma\Gamma^{\mathrm{H}}$ (or $\Gamma^{\mathrm{H}}\Gamma$); hence any of the non-pure states $\rho_S$ or $\rho_E$ can be "purified" as (3) in interaction with the other. In fact, $|\Psi\rangle_{SE}$ is entangled (separable) iff, $r > 1$ ($r = 1$), iff, any of its reduced states $\rho_S$ and $\rho_E$ (and hence both) are mixed (pure) states.

The entanglement of a pure density state can be parametrized by its *entropy of entanglement* defined as the von Neumann entropy

$$\begin{aligned}E &\triangleq S(\rho_S) = -\mathrm{Tr}\left(\rho_S \log \rho_S\right)\\ &= -\mathrm{Tr}\left(\rho_E \log \rho_E\right) = S(\rho_E)\\ &= -\sum_{i=1}^{r} \lambda_i \log \lambda_i\end{aligned} \tag{5}$$

[4] $\mathrm{Tr}(|\psi\rangle\langle\psi|) = \langle\psi \mid \psi\rangle$, $\mathrm{Tr}(A|\psi\rangle\langle\psi|) = \langle\psi \mid A \mid \psi\rangle$.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

of either of its reduced densities $\rho_S$ or $\rho_E$ and quantified by the number of e-bits that can be produced ("distilled") per copy of $\rho_{SE}$ from an asymptotically large ensemble. For a product state with $r = 1$, there is no uncertainty about its preparation, $\lambda = 1$, and the entropy is 0. The maximum entanglement $E(|\Psi\rangle_{SE}) = \log N$ appears for the Schmidt e-bit

$$|\Psi\rangle_{SE} = N^{-1/2} \sum_{i=1}^{N} |\phi_i\rangle\, |\psi_i\rangle \tag{6}$$

randomly prepared in any of the $r = N$ equally likely possibilities. In addition to $r$, mixed states unlike the pure, also depend on the additional parameter $M > 1$ that effectively introduces irreversibility in the quantum system by regarding entanglement to be of two reciprocal varieties: The *entanglement of formation* $E_F(\rho_{SE})$ denotes the minimum number of Schmidt bits (6) required to generate one nonmaximally entangled $\rho_{SE}$ for infinitely many copies, and the *entanglement of distillation* $E_D(\rho_{SE})$ corresponds to the maximum number of e-bits that can be distilled from an asymptotically large ensemble of $\rho_{SE}$. If $\rho_{SE}$ is pure, the measures coincide, $E_D = E = E_F$; for mixed states $E_D(\rho_{SE}) \le E_F(\rho_{SE})$.[5]

Using these bounds, a *thermodynamic analogy of entanglement* has been proposed [11] through the identifications

$E_F \leftrightarrow$ Internal Energy $U$

$E_D \leftrightarrow$ Free Energy $A$

$E_F - E_D \leftrightarrow TS$

for a formal equivalence with Eqs. (I-15a, b). This allows a generalization of Eq. (I-12a) to define an irreversibility of entanglement

$$\iota_{\mathrm{E}} = \frac{1}{E_F}\,(E_F - E_D) \tag{7}$$

and to treat it in the engine-pump perspective with $E_F := W_{\mathrm{rev}}$ and $E_D := W$ corresponding to the engine and the pump respectively: the twirling operation producing $2 \times 2$ Werner states [11] is an instance of increasing irreversibility through a decrease in $E_D$. As Horodecki argue [11], for the reversible $\iota_{\mathrm{E}} = 0$ corresponding to $E_D = E_F$ of a dispersionless, noiseless channel, it is sufficient to invest only $E_F$ ebits to generate a state of this entanglement, whereas if the channel is noisy with $\iota_{\mathrm{E}} > 0$, "the number of ebits must be larger: some amount of the sent entanglement will be spread over the system and environment". Thus the useful entanglement $E_D$ is less than the number of exchanged qubits $E_F$ and the lost entanglement is unavailable in performing useful work and must change to "uncontrolled" entropic form. However, this linear non-local resource cannot match the holistic outreach of nonlinear complexity: the indicator of entanglement $r$ at most as large as the smaller of the dimensions of the entangling Hilbert spaces compares with the exponentially increasing $2^N$ degrees of freedom available to the emergent holistic patterns. With non-locality it is possible to predict with certainty the outcome of a separated distant partner by observations made on the other; it cannot self-generate patterns or structures, it cannot teleport.

The reduced density matrix plays a key role in *decoherence* [12], the mechanism by which open quantum systems interact with their environment leading to spontaneous suppression of interference and hence appearance of classicality. In this approach, utilized principally to explain the *measurement problem* involving transition from the quantum world of superpositions to the definiteness of the classical objectivity, ("why does the world appear classical to us, in spite of its supposed underlying quantum nature, which would in principle, allow for arbitrary superpositions?", [12] partial tracing over the environment of the total density operator (here "environment" includes the measuring apparatus) produces an "environment selected" basis in which the reduced density is diagonal. This irreversible decay of the off-diagonal terms is the basis of decoherence that effectively bypasses "collapse" of the state on measurement to one of its eigenstates. This "derivation of the classical world from quantum-mechanical principles", restrained by the simplicity of linear superpositions, is to be compared with nonlinearly-induced emergence of complex patterns and structures through the multifunctional graphical convergence route. Multiplicities inherent in this mode, liberated from the strictures of linear superposition and reductionism, allow interpretation of objectivity and definiteness as in classical probabilistic systems through a judicious application of the Axiom of Choice. Hence

**Proposition 1.** *To define a choice function*[6] *is to conduct an experiment.*

Moreover, unlike in the quantum-classical transition, complex evolving systems are in a state of homeostasy with the environment with "measurement" providing a record of such interaction; probing holistic systems for its parts and

[5] "The answer of Quantum Field Theory (QFT) to Einstein–Podolsky–Rosen is very simple: the particles one is dealing with here would possess their own element of reality only if quantum mechanics were a fundamental theory. In QFT the quantum particles have no independent physical reality, nor a well-defined physical location; they just correspond to the peculiarity of the localized quantum measurement that can only reveal the presence of the quantum field in a quantized way. Thus in QFT the elements of reality only belong to the quantum field which is neither localized nor separable" [10].

[6] **Definition 2** (*Choice Function*). Let $X$ be a set. A choice function for $X$ is a many-to-one functional relation $c : \mathcal{P}(X) - \emptyset \to X$ such that $\forall A \in \mathcal{P}(X) - \emptyset$, the image $c(A) \in A$.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

components is expected to lead to paradoxes and contradictions.[7] A complex system represents a state of dynamic stasis between the opposites of bottom-up pump induced synthesis of concentration, order and emergence, and top-down engine dominated analysis of dispersion, disorder, and self-organization. Since these are self-organizing, emergent systems, the demonic pump actually succeeds in deceiving the Second Law by promoting order through entropy reduction and gradient dissipation that in the ultimate analysis amounts to a trade-off between the different topologies induced by these opposites.

The maximally ill-posed multifunctional limit of the time-irreversible logistic difference equation can be correlated with evolutions of the time reversible Schrodinger equation: nonlocality and entanglement correspond to the graphically converged $2^N$ periodic cycles; qubits and "contextual objectivity" of quantum states endowed with objective reality in the context of observed classical macro experiments to non-injectivity of the positive-negative slope curve, compare Ref. [13] with the interpretation of the neg-world in inducing competitively collaborating homeostasy between the two worlds as outlined above. While quantum non-locality is a natural consequence of quantum entanglement that endows multi-partite systems with definite properties at the expense of the individual constituents thereby rendering it impossible to reconstruct the state of a composite from knowledge of its parts, for chanoxity the effective power law [8]

$$f(x) = x^{1-\chi}$$
$$\therefore\ \chi = 1 - \frac{\ln\langle f(x)\rangle}{\ln\langle x\rangle}, \quad 0 \le \chi \le 1, \tag{8}$$

with

$$\langle x\rangle \triangleq 2^N \xrightarrow{\lambda=\lambda_*} \infty \tag{9a}$$

defining the degeneracy of microstates representing the basic unstable fixed points corresponding to the $N+1$ macrostates $\{f^i\}_{i=0}^N$ that contribute a net feedback[8]

$$\langle f(x)\rangle \triangleq 2f_1 + \sum_{j=1}^{N}\sum_{i=1}^{2^{j-1}} f_{i,i+2^{j-1}}, \quad N = 1, 2, \ldots \tag{9b}$$

to emergence and self-organization, bestows the complex system with the necessary composite holistic character. Hence

$$\chi_N = 1 - \frac{1}{N\ln 2}\ln\left[2f_1 + \sum_{j=1}^{N}\sum_{i=1}^{2^{j-1}} f_{i,i+2^{j-1}}\right] \tag{9c}$$

is the measure of chanoxity, where $f_i = f^i(0.5)$ and $f_{i,j} = |f^i(0.5) - f^j(0.5)|$. Most significantly, as our calculations in [8] demonstrate, the dynamics of the logistic map undergoes a dramatic discontinuous transition from the monotonically increasing $0 \le \chi < 1$ in $3 \le \lambda < \lambda_*$ of region (I) to a disjoint different world at $\chi = 0$ in the fully chaotic sub-range $\lambda_* \le \lambda < 4$ of (II), Table I-2. This reduces the chaotic world to one of effective linear simplicity and suggests integration of quantum mechanics with chanoxity by identifying $\langle x\rangle$ of Eq. (9a) with the dimension of the resulting Hilbert space leading to the[9]

**Conjecture 1.** *Quantum Mechanics is an effective linear representation of the fully chaotic, maximally illposed* $\mathrm{Multi}_{\|}$ *boundary* $\lambda_* \le \lambda < 4$ *that manifests itself only through a bi-directional, contextually objective, inducement of* $W$ *in adapting to the Second Law of Thermodynamics. The opposites of the (pump) preparation of the state and the subsequent (engine) measurement collaborate to define the contextual reality of the present.*

Together with the observations following Definition 1, it is possible to speculate that the quantum mechanical eigenvalues and eigenfunctions correspond to the emergent periodic cycles suitably modified for linearity and superpositions. We can now read Conjecture 1 with Proposition 1 for the

7 "This system-environment dualism is generally associated with quantum entanglement which always describes a correlation between the parts of the universe. As long as the universe is not resolved into individual subsystems there is no measurement problem. Only when we decompose the total Hilbert-state space of the universe into a (tensor) product of two spaces and want to ascribe an individual state to one of the two systems (say, the apparatus), does the measurement problem arise" [12].

**Question:** Can partial tracing over the remaining 200 odd organs of the adult human body define the current state of any of its constituents?

8 Thus for $N = 3$, $\langle f(x)\rangle = (2f_1 + f_{12}) + (f_{13} + f_{24}) + (f_{15} + f_{26} + f_{37} + f_{48})$ gives the number of degenerate microsates as 2, 4, 8 for $n = 1, 2, 3$ with the groupings shown.

9 "The notions of localization and separation that realism demands of any physical theory and that are so patently violated by both quantum mechanics and Nature, imply that in any realistic physical theory of the quanta their clearcut objective definition must be structurally and logically impossible. This happens in QFT where localization and separation are (approximate) physical properties of the measuring apparatus, and are in no way intimately connected with the reality of the field; the space coordinates $(x_1, \ldots, x_N)$ can in no way be attributed to the $N$ indistinguishable quantum particles but are in a one-to-one correspondence with the positions in space where a possible observer cares to put his measuring device. Whether the system is observed or not, the wave-function as well as physical reality does not care, it continues to develop in time according to the Schrodinger equation until it reaches the ground state, the state of minimum energy of the stationary Schrodinger equation" [10].

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

**Corollary 1.** *Quantum mechanical "collapse" of the wave function is a linear objectification of the measurement choice function.*

**Example 2** (*Bohmian Mechanics*). While the objective of this paper has been to make a rational case for bi-directional direct and inverse limits in the framework of Chaos-Nonlinearity-compleXity as the operational mechanism of Nature with a focus on the limitations imposed by linearity, the present example taken from Durr, Goldstein and Zanghi [14] explicitly demonstrates the limitations of the orthodox quantum dynamics of Schrodinger *differential* equation as compared to Bohmian interpretation. In this theory the state of evolution is given not simply by the wave function $\Psi$ but by $(Q, \Psi)$, where $Q$ the actual position of the particle.

The equations of Bohmian mechanics for a single particle of mass $m$ in one dimension, are

$$\mathrm{i}\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right)\Psi(q, t) \tag{10a}$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\hbar}{m}\,\mathrm{Im}\left(\frac{1}{\Psi}\frac{\partial\Psi(Q, t)}{\partial Q}\right) \tag{10b}$$

where $Q$, the actual position of the particle determined by the *guidance* equation (10b), is to be distinguished from the generic configuration variable $x$. Durr, Goldstein and Zanghi consider the double humped potential $V(x) = \lambda(q^4 - q^2)$ for $\lambda$ so very large that the wells are effectively separated by a high barrier between them. Then the two lowest eigenfunctions $\Psi_0, \Psi_1$ are symmetric and antisymmetric for eigenenergies $E_0$ and $E_1$, and the combinations

$$\Psi_{\mathrm{L}} = \frac{1}{\sqrt{2}}(\Psi_0 - \Psi_1), \qquad \Psi_{\mathrm{R}} = \frac{1}{\sqrt{2}}(\Psi_0 + \Psi_1)$$

are supported on the left and right respectively. Schrodinger evolution leads to

$$\Psi_{\mathrm{R}} \to \Psi_t = \frac{(\Psi_0 + \mathrm{e}^{-\mathrm{i}\omega t}\Psi_1)}{\sqrt{2}}$$

with a time period $T = h/(E_1 - E_0)$; let $\tau = T/4$. A measurement of $\sigma := \mathrm{sgn}(x) = |\Psi_{\mathrm{R}}\rangle\langle\Psi_{\mathrm{R}}| - |\Psi_{\mathrm{L}}\rangle\langle\Psi_{\mathrm{L}}|$ probing the state at time $\tau$ generates $\sigma = +1$ with 50% probability of projecting onto R and $\sigma = -1$ with 50% probability for $L$. On repeated iteration, the outcome at any time $n\tau$ is uniformly distributed between the two halves, independent on the step at which it was conducted.

With the additional guidance equation (10b) in Bohmian evolution, the situation is quite different, however. In this case dividing $L$ and $R$ each into two parts, measurement of $\sigma$ at time $\tau$ depends on the particular half $Q$ is initially located, and [14] demonstrates that the time evolution of (10a) and (10b) can be fully represented by symbolic dynamics of the shift map $x \mapsto 2q \pmod 1$, and hence by the iteration of one-dimensional maps. This has the significant implication that while unpredictability is characteristic of quantum phenomena, randomness associated with measurement and collapse in Bohmian mechanics derives entirely from the uncertainty of initial condition, there being in the quantum orthodoxy of unitary, reversible and regular evolution of Schrodinger equation "absolutely nothing from which this randomness can be said to derive". Bohm's theory is patently non-local and violates Bell's inequality.

Bohmian Mechanics admits an intuitive interpretation of the measurement paradox in terms of the guiding Eq. (10b). The wave function $\Psi$ of the universe, as a solution of the Schrodinger equation, does not interact with the guiding trajectory equation: $\Psi$ is independent of the actual configuration $Q$ of the particles, "it is what it is". In the situation under consideration of a system in interaction with its environment, let $q = (s, e)$ be the generic variables and $Q = (S, E)$ the actual configuration of the particle. Then the "conditional wave function" $\psi(s) := \Psi(s, E) \in \mathcal{H}_S \otimes \mathcal{H}_E$ [14] of the system satisfying $\psi_t(s) := \Psi_t(s, E_t)$ with $E$ obtained from Eq. (10b) need not be wave-like corresponding to any Schrodinger equation. However, the "effective wave function" defined by the superposition (10b)

$$\Psi_t(s, E_t) = \sum_i a_i\psi_i(s)\Phi_i(E_t), \tag{11}$$

with the supports of $\{\Phi_i\}$ disjoint and the guiding equation choosing some $E_0$ in the support of a $\Phi_0$ at initial time, quickly decoheres to

$$\psi_t(s) \to a_k\psi_k(s)\Phi_k(E_t),$$

thereby resolving the "measurement problem" through the intermediary of evolution of the actual configuration $E_t$ of the environment. The system and environment are therefore intimately correlated in the Bohmian universe with the effective system state

$$\psi(s) = \sum_i a_i\psi_i(s) \mapsto \psi_k(s)$$

being actually determined, unlike $\Psi$, by its participating environment in accordance to $\Psi \to E \to \psi$ of Eqs. (10b) and (11) respectively: the guiding equation effectively defining the choice function of the universe.

## 2.2. ⋯ rigged hilbert spaces, intrinsic irreversibility

In this paper we have adapted the "engine-pump" paradigm of [8] to argue for increasing organization and complexity of *open non-equilibrium systems* that clearly fail to abide by the prescription of the Second Law of Thermodynamics. The basic point is a resolution of the *time-symmetry, or reversibility, paradox*: although the fundamental *microscopic* laws of nature, as encapsulated by Newtonian mechanics, are time-reversible, an isolated *macroscopic* real system is anything but so. Time reversibility is to be understood in the context of evolution of a system of $N$ particles in its $6N$ dimensional phase space $\Omega = \mathbb{R}^{6N}$ of positions and momenta represented by the "curve" $x(t) = f_t(x)$ that associates with each initial point $x(0) \in \Omega$ the solution of the differential equations of motion at time $t$. Time reversibility then implies that [15]

$$f_t i f_t(x) = \mathrm{i}(x), \quad \mathrm{i}^2 = \mathbf{1}$$

where the velocity-involution $i$ reverses the velocity of a particle. Time symmetry can therefore be interpreted to involve: (i) forward evolution of the particle from $x(0)$ to $x(t)$ as induced by $f$ in the inclusion topology of $x(t)$, (ii) the artifact of neg-spaces allows $i$-reversing this forward thermodynamic arrow in establishing the reverse $f$-direction with respect to the exclusion topology of $x(0)$, and (iii) allowing the reversed gravitational arrow to completely retrace the forward path of the particle back to its initial point.[10] The natural intrinsic irreversibility of the evolution of the iterates of one-dimensional unimodal maps is a consequence of the function-multifunction asymmetry of the forward-backward iterates of the map with the equivalence class of inverse images corresponding to the macrostate of their image. This asymmetry, of course, lies at the heart of the probabilistic multifunctional (in the sense of the Axiom of Choice) extension of the functional material world we inhabit. The coarser topology[11] of the later actuality, as compared with the finer topology of the former, leads to the intrinsic irreversibility of nonlinear evolution. The objective dynamical reality of Prigoginian "intrinsic" irreversibility is to be contrasted with the subjective Boltzmannian view of "extrinsic" irreversibility that seeks to interpret irreversibility as the direct consequence of explicit introduction of ignorance and approximations by the observer into the laws of nature resulting from the interaction of the system with its environment. The Prigogine school holds irreversibility to be intrinsic to the dynamics of the microsystem rather than induced by macroscopic influences of external reservoirs or measuring devices. Considering the universe as a large quantum system, the dynamics are governed by quantum mechanical laws of Hermitian operators in Hilbert spaces $\mathcal{H}$ that are not equipped to deal with physical processes such as resonance scattering decay resulting from *complex eigenvalues*.[12]

The rigged Hilbert space or Gelfand triplet $(\Phi, \mathcal{H}, \Phi^{\times})$ [17], with $\mathcal{H}$ infinite dimensional and separable, is a specific construct designed to link distribution (test function) with square-integrable functions fully endorsing situations beyond the restrictive confines of Hilbert spaces that do not support either eigenvectors corresponding to the continuous spectra of self-adjoint operators or plane waves and Dirac delta functions. This is generically portrayed in Fig. I-2b; here $X$ corresponds to $\mathcal{H}$, $\{X_{-k}\}$, $\{X^k\}$ represent nested lattices of subspaces $\mathcal{H}_k$ and the associations

$$\Phi \leftrightarrow X_{\leftarrow}$$
$$\Phi^{\times} \leftrightarrow {}_{\rightarrow}X$$

formally identify the inverse and direct limit spaces with $\Phi$ and $\Phi^{\times}$ respectively. Hence $(\Phi, \mathcal{I})$ is the projective limit of the sequence $\{(\mathcal{H}^k, \mathcal{T}^k)\}$ in which the induced norm-topologies get coarser with increasing $k$ and is characterized as a test-function space of well-behaved, continuous, $k$-times differentiable functions having a bounded domain or decaying exponentially beyond some finite range, and $(\Phi^{\times}, \mathcal{F})$ the topological dual of $(\Phi, \mathcal{I})$ is the inductive limit of $\{(\mathcal{H}_{-k}, \mathcal{T}_{-k})\}$ with topologies getting finer with increasing $k$ constitute the set of distributions (of continuous linear functionals) on $(\Phi, \mathcal{I})$. Because the elements of $\Phi$ are so "well-behaved" its dual $\Phi^{\times}$ consists of elements that need not at all be so, some being singular or improper multifunctions like the Dirac $\delta$. The natural supremum topology $\mathcal{I} = \sup(\pi^{k-}(\mathcal{T}^k))$ generated by the subbasis $\bigcup_{i\in\mathbb{D}} \pi^{k-}(\mathcal{T}^k) := \{\pi^{k-}(U^k) : U^k \in \mathcal{T}^k, \iota \in \mathbb{D}\}$ on $\Phi$ is the coarsest *initial topology* that the makes induced inclusions $\pi^k : \Phi \to X^k$ continuous, while the infimum topology $\mathcal{F} = \inf(\pi_{-k}(\mathcal{T}_{-k})) := \bigcap_{i\in\mathbb{D}} \mathcal{T}_{-i}$ on the disjoint union $\Phi^{\times}$ of $\{\mathcal{H}_{-k}\}$ is the finest *final topology* for which the induced inclusions $\eta_{-k} : \mathcal{H}_{-k} \to \Phi^{\times}$ are continuous, [18].

Intrinsic irreversibility corresponds to the expansive arrow of "cold" inverse limits of the natural reverse direction of decreasing subsets along the *thermodynamic forward* direction of $\mathbb{D}$ required in the topological theory of convergence, see Fig. I-2b. The bra-ket formalism of interaction between $\Phi^{\times}$ and $\Phi$ generating the phenomenon of actual physical observations corresponds to our engine-pump modeled state of dynamical homeostatic equilibrium of nonequilibrium systems. In essence, this Bohm–Prigogine [19,20], "time-asymmetric quantum mechanics school" establishes the intrinsic arrow of time, and thereby intrinsic irreversibility, from the Hardy classes of functions

$$\mathcal{H}_{\pm} \triangleq \{f_{\pm} : \mathbb{R} \to \mathbb{C} \mid f_{\pm}(x) = \lim_{\epsilon\to 0} f(x \pm \mathrm{i}\epsilon), \text{a.e. } \epsilon > 0\}$$

[10] In non-technical terms, therefore, an irreversible process is one that cannot reverse itself without a consequent change in the surroundings: the probability of an irreversible process *spontaneously* reversing itself without outside interference is zero.

[11] If $\mathcal{S}$ and $\mathcal{T}$ are two topologies on a set $X$ such that $\mathcal{S} \subseteq \mathcal{T}$, that is every set open in $\mathcal{S}$ is also open in $\mathcal{T}$, then the topology $\mathcal{S}$ is said to be *coarser* (*weaker* or *smaller*) than $\mathcal{T}$ and $\mathcal{T}$ *finer* (*stronger* or *larger*) than $\mathcal{S}$. Hence, $i : (X, \mathcal{T}/\mathcal{S}) \to (X, \mathcal{S}/\mathcal{T})$ is continuous/open $\Leftrightarrow \mathcal{S} \subseteq \mathcal{T}$ for the inclusion map $i$.

[12] See Bishop [16], especially Section 5 titled "Matter Meets Mind" for an almost identical interpretation as presented here and in Section 3 "What is Life?" of [8].

representing boundary values of functions $f(x \pm i\epsilon)$, $\epsilon > 0$ in the upper and lower half planes, analytic in their respective domains, constituting a direct sum decomposition of $\mathscr{H}$. Taking recourse to an externally imposed "time ordering rule", Antoniou and Prigogine [20] require terms corresponding to *excitation* processes to be oriented towards the past for advanced propagators and terms corresponding to *de-excitations* to be future oriented through retarded propagators as general boundary conditions necessary to uniquely solve the dynamical system. The preparation/excitation, gravitational backward-direct pump then decays as $e^{(\Gamma/2)t}$ "in the past" $-\infty < t \le 0$ and is described by the rigged Hilbert space $\Phi_- \subset \mathscr{H} \subset \Phi_-^\times$, while $\Phi_+ \subset \mathscr{H} \subset \Phi_+^\times$ models the registration/de-excitation, thermodynamic forward-inverse engine decomposing like $e^{-(\Gamma/2)t}$ "in the future" $0 \le t < \infty$,[13] with $\Phi_\pm$ and $\Phi_\pm^\times$ corresponding to the spaces $\mathscr{S} \cap \mathscr{H}_\pm|_{\mathbb{R}_+}$ and $(\mathscr{S} \cap \mathscr{H}_\pm|_{\mathbb{R}_+})^\times$ for $\mathscr{S}$ the space of smooth complex valued Schwartz functions of the real variable $x$ that tend to 0 faster than any power of $x$. While the states $\phi_\pm \in \Phi_\pm$ correspond to smooth wave functions in the Schrodinger representation, $\Phi_\pm^\times$ represent generalized functionals on $\Phi_\pm$ that can contain, beside singular Dirac kets and eigenkets of real eigenvalues, also eigenvectors with complex eigenvalues as in Eq. (13) below. Thus

$$\left\langle e^{iH_+ t}\phi_+ \mid G_- \right\rangle = \left\langle \phi_+ \mid e^{-iH_+^\times t} G_- \right\rangle, \quad \forall \phi_+ \in \Phi_+, G_- \in \Phi_+^\times, t \ge 0 \tag{12a}$$

$$\left\langle e^{iH_- t}\phi_- \mid G_+ \right\rangle = \left\langle \phi_- \mid e^{-iH_-^\times t} G_+ \right\rangle, \quad \forall \phi_- \in \Phi_-, G_+ \in \Phi_-^\times, t \le 0, \tag{12b}$$

where $H_\pm$ are restrictions of the Hamiltonian $H$ to the subspaces $\Phi_\pm$, introduces the positive and negative semigroups without identity that manifest in the forward-backward directions in the evolution of time. With

$$H_\pm^\times G_\mp = \left( E_R \mp i\frac{\Gamma}{2} \right) G_\mp \tag{13}$$

the eigenvalue equations in $\Phi_\pm$, intrinsic irreversibility implies that

$$\lim_{t\to\pm\infty} \left\langle \phi_\pm \mid e^{-iH_\pm^\times t} G_\mp \right\rangle = \lim_{t\to\pm\infty} \langle \phi_\pm \mid G_\mp \rangle\, e^{-iE_R t} e^{\mp \frac{\Gamma}{2} t} = 0 \tag{14}$$

in their respective domains. Note that unlike for unitary operators with an identity evolving in the full range $-\infty < t < \infty$, the half-range semi-groups are defined only for positive and negative values, with their corresponding negative and positive extension being undefined. Thus as $t \to \pm\infty$, the factors $e^{\mp(\Gamma/2)t}$ arising from the complex part of the eigenvalues dampen the evolution with increasing $t$, approaching the limiting Second Law dead state of thermodynamic equilibrium on the real axis that demarcates the uni-directional semi-groups.

The foundations of our approach are based on the interesting duality principle of partially ordered sets: the inverse $\succeq$ of an order relation $\preceq$ on a partially ordered set $A$ is also an order relation on $A$ such that the replacements "$\preceq$" $\leftrightarrow$ "$\succeq$", "maximal" $\leftrightarrow$ "minimal", "greatest" $\leftrightarrow$ "least", "upper bound" $\leftrightarrow$ "lower bound", "sup" $\leftrightarrow$ "inf" results in the transformation of $A$ to its dual $A^\times$. The neg-world $\mathfrak{W}$ therefore can be viewed as this set-theoretic dual of the real world $W$ (with due consideration to the fact that it is not restricted to linearity), and *since the pairs initial-final, preimage-image, inverse-direct do indeed exist in the strict mathematical sense,* the pair $W - \mathfrak{W}$ can reasonably be taken to make sense too, with all the resultant consequences as set out here and in [8]. The pair $(\Phi, \Phi^\times)$ is "fully equipped" in the sense that unlike a Hilbert space, the rigged space is technically competent to represent important operators like unbounded operators corresponding to position and momentum which have no eigenvalues or eigenvectors in a separable Hilbert space. In the rigged space, these operators have complete set of generalized eigenfunctions according to the rule that if $A$ is a symmetric linear operator defined on the space $\Phi$ and admits a self-adjoint extension to $\mathscr{H}$, then $A$ possesses a complete set of generalized eigenfunctions in $\Phi^\times$. Hence $A$ can be extended by duality to $\Phi^\times$ according to $\langle A\varphi|\varphi^\times\rangle = \langle \varphi|A^\times\varphi^\times\rangle$, for all $\varphi \in \Phi$ and $\varphi^\times \in \Phi^\times$ and the extension $A^\times$ is continuous on $\Phi^\times$ in its operator topology.

These observations serve as a substantive corroboration of our contention that although the dual neg-world – like the Dirac delta multifunction – cannot be observed directly, it is only through its effect on this real world that Nature displays its unbounded beauty, diversity, and omnipresent ubiquity. Note that intrinsic irreversibility in either world is established by its corresponding own arrow of expansion; open systems attain homeostasy, however, as a result of dynamic equilibrium of these opposites. The distinguishing role of Hardy functions in deducing irreversibility has been analyzed in [21] who argue that "even if time reversal invariance of a theory is broken by means of semigroup laws, this does not affect the reversible character of the evolutions if they are still described by unitary operators", and interpret Eq. (14) to signify that $G_\pm$ decays in a "coarse-grained sense *from an observational point of view,* that is from the perspective given by the observable $|\phi_+\rangle\langle\phi_+|$ for any $\phi_+ \in \Phi_+$", with the coarse-grained expectation value of the observable $|\phi_+\rangle\langle\phi_+|$ in the generalized state $G_-$ actually decaying as $t \to \infty$. This tends to support our characterization of collaborative-competition between the of the world-negworld opposing duals in the outward manifestation of Nature.

[13] Compare with Example I-4 and Fig. I-6.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

## 3. Gravity and the logistic map

Gravity, the universal long-range attractive force, is the compressive pump that Nature uses to balance its own mandate of the Second Law of Thermodynamics. In this section we examine this unique tool in the light of dynamical homeostasy.

The equation of motion

$$y(x) = \tan\theta\, x - \frac{g}{2v^2\cos^2\theta}x^2, \quad 0 \le \theta \le \frac{\pi}{2} \tag{15}$$

of a projectile of velocity $\mathbf{v} = (v, \theta)$, with $x(t) = vt\cos\theta$, implies that

$$y(x) = 0 \Leftrightarrow x_0 = 0, \qquad x_1 = \frac{v^2}{g}\sin 2\theta.$$

Defining a non-dimensional distance $\widehat{x}(t) = x/x_1 \in [0, 1]$, Eq. (15) can be put in the logistic form

$$\widehat{y}(\widehat{x}) = \tan\theta\,\widehat{x} - \frac{gx_1}{2v^2\cos^2\theta}\widehat{x}^2; \tag{16a}$$

$$\therefore\ y(x) = \lambda x(1-x) \tag{16b}$$

in terms of the three adjustable parameters $(g, v, \theta) \mapsto (\gamma, w, \phi)$ that are determined under the mapping $\widehat{x} = 1$ of the range. Hence equality of the two coefficients in Eq. (16a) defines

$$\gamma = \frac{w^2}{x_1}\sin 2\phi \tag{17a}$$

an effective gravitational acceleration at ground level $y = 0$ in terms of the yet unknown parameter $x_1$, where[14]

$$\phi = \sin^{-1}\sqrt{\frac{\lambda\gamma}{2w^2}} \tag{17b}$$

for a given logistic parameter $\lambda$. The outcome of this parametrization is to determine an effective gravitational acceleration in terms of the projectile range $x_1$, demonstrating that Eq. (15) can be put in the standard form (16b) so that evolutionary dynamics of gravitational systems can be explored in terms of this iconical one-dimensional map.

Based on the equality of gravitational and inertial forces

$$F_{\text{grav}} = \frac{GMm_{\text{grav}}}{r^2}, \quad G = 6.6742\times 10^{-11}\ \text{m}^3/\text{kg s}^2$$
$$F_{\text{inertia}} = m_{\text{inertia}}g,$$

and of the masses $m_{\text{inertia}} = m_{\text{grav}}$ implying

$$g = \frac{GM}{r^2}, \tag{18a}$$

we now identify the critical gravitational acceleration $\gamma_S$ at $\lambda_S = 3.5699456$ marking the transition to fully chaotic region to be

$$\gamma_S \triangleq g_S = 1.0866\times 10^{13}\ \text{m s}^{-2} \tag{18b}$$

at the Schwarzschild radius[15] $r = r_S = 2GM/c^2 = 4135.7151$ m for a Chandrasekhar mass[16] $M = 1.4M_\odot$, $M_\odot = 1.9891\times 10^{30}$ kg being the solar mass. Hence

$$x_1 = 2r_S\sin 2\phi_S = 242.9733\ \text{m} \tag{19}$$

corresponding to a $\phi_S = 0.8417°$ follows from Eq. (17a).

[14] If $a_1x - a_2x^2 = \lambda\widehat{x}(1-\widehat{x})$, then $\lambda = a_1(x/\widehat{x}) = a_2(x/\widehat{x})^2$ gives $a_1/a_2 = x/\widehat{x} = x_1$; hence (17b) with $\lambda = x_1\tan\theta$.

[15] The *Schwarzschild radius*, obtained by equating the escape velocity $v_e = \sqrt{2GM/r}$ from a massive body $M$ of radius $r$ to the speed of light, is the characteristic radius for which if the mass were to be compressed within this value, it would continue to collapse into a gravitational singularity. An object smaller than its Schwarzschild radius is a *black hole*.

[16] The maximum nonrotating mass that can be supported against gravitational collapse by electron degeneracy pressure.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

**Remark 1.** As is well known the trajectory of a particle moving under the sole influence of the gravitational field of a spherical body is a conic with the center of the body forming the distant focus of an ellipse, and only a small part of the ellipse near its apogee constituting the trajectory of the particle. The ellipse, however, degenerates into a parabola as the focus moves out to infinity thereby reducing the central-force ellipse to a flat-earth parabolic approximation. In an analysis of how the parabolic approximation to an ellipse arises, Burko and Price [22] show that replacement of a spherical surface by a flat approximation holds only if the curvature of the trajectory at the apogee exceeds the curvature of the earth's surface. Defining two parameters $\varepsilon = h/R$ and $\alpha = v^2/gh$, with $R$ the radius of the earth, $h$ and $v$ the height above the earth's surface and the (horizontal) velocity at the top of the trajectory, [22] demonstrate that the condition $\varepsilon\alpha = v^2/gR \ll 1$ meets this criterion, with $\varepsilon \ll 1$ further reducing the resulting flat-earth *ellipse* to a flat-earth *parabola*. Our correspondence of the actual trajectory with the logistic profile for $x \in [0, 1]$, $\lambda \in [0, 4]$, is justified by the small range, small peak height property of the map that results in its increasing nonlinearity with increasing $\lambda$. We have therefore taken advantage of this characteristic of the logistic function to map the flat-earth ellipse to an equivalent flat-earth logistic parabola, thereby generating chaos and complexity in the reduction.

In the next section we investigate our identification of a black hole with the region $\lambda \geq \lambda_*$ of the logistic map, and consider its significance in relation to the intrinsic vacuum energy density $\rho$ of empty space and *positive cosmological constant* $\Lambda = (8\pi G/c^2)\,\rho$, representing a form of *gravitational repulsion* supporting expansion; this can in fact also be negative or zero allowing thereby the possibility of a dynamically static universe, with a negative constant and/or ordinary matter tending to decelerate it. Einstein's original theory of relativity admitted only expanding or contracting solutions; no "static" equilibrium universes were permissible. A positive cosmological constant for empty space implies *positive vacuum energy* and *negative pressure* of the collapsing (complimentary) environment leading to expansion of the system that opposes its own gravitation, thereby leading to the feasibility of a "static" universe, compare the arguments in Example I-4, however. In actuality, observations indicate that the universe is neither contracting as it would if it were completely matter-dominated, nor is it in the equilibrium state (that prompted introduction of the constant in the first place), but is in fact expanding. However, distance-redshift relations indicate that the expansion rate of the universe is *increasing,* a phenomenon that can be explained by assuming a very small positive cosmological constant of negative vacuum pressure equal to its energy density, $p \sim -\rho$.[17] Being proportional to its volume, the vacuum pressure $p$ actually decreases to supply the increased demand in energy on increasing volume work done by it, hence $p = -\rho c^2$ with $\rho$ the vacuum energy density of negative pressure. A positive vacuum energy of expansion caused by a collapsing gravitating environment induces negative pressure in the vacuum as it tries to pull back generating the anti-gravity effect in the complement. A major outstanding problem, nevertheless, is that most quantum field theories predict a huge cosmological constant from the energy of the quantum vacuum, of some $10^{120}$ orders of magnitude larger than the tiny positive observed value. This is the unresolved "conundrum" of the *cosmological constant problem*: there is no known acceptable way to adjust the cosmological constant to its infinitesimally vanishing observed value. The *dark vacuum energy* pervading all space with strong negative pressure – the exact nature of which is not understood – is believed not to interact with any of the fundamental forces except, of course, gravity. If the actual energy density of the universe is less than a certain critical value, the universe will continue to expand forever. If, however, the energy density is more than the critical, the universe will eventually stop expanding and begin to collapse under the influence of gravity. It is thought, however, that the actual energy density should be just about equal to the critical density resulting in a dynamically steady state of the universe, compare and contrast again with chanoxity of Example I-4, according to which the discrepancy in the magnitudes of the cosmological constant might have the simple resolution of a very small, near zero, irreversibility $\iota$.

### 3.1. Gravity cosmology and complexity

Let $(\mathcal{M}, g)$ be a Lorentzian manifold with $g$ a $(-, +, +, +)$ metric. A tangent vector $v$ classified according to

$$g(v, v) \lesseqqgtr 0, \begin{cases} \text{timelike} \\ \text{null} \\ \text{spacelike} \end{cases}$$

naturally induces forward and backward arrows on $\mathcal{M}$, intrinsically opposing each other, naturally induces thermodynamic-forward and gravitational-backward arrows on $\mathcal{M}$, intrinsically opposing each other. At each point in $\mathcal{M}$ therefore, timelike tangent vectors can be divided into the classes of "future directed" and "past directed" according to whether their first component is positive or negative, and for points $v, w \in \mathcal{M}$ the vector $v$ is said to*chronologically precede* $w$, $v \prec w$, if there exists a future directed timelike curve joining $v$ to $w$. The *chronological future*(forward arrow) and *chronological past* (backward arrow) of $v \in \mathcal{M}$

$$I^+(v) = \{w \in \mathcal{M} : v \prec w\}$$
$$I^-(v) = \{w \in \mathcal{M} : w \prec v\}$$

17 Positive-pushing/Negative-pulling pressure by environment ⇒ Gravitational attraction/Anti-gravity repulsion in system.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

are open sets containing $w$ such that $v$ chronologically precedes $w$, respectively chronologically succeeds $w$; clearly $v \in I^-(w)$ iff, $w \in I^+(v)$. The points in the interior of the future/past light cones $I^{\pm}(v)$ at $v$ are those that can be reached from $v$ by future/past directed timelike curves $\lambda(t)$ with $\lambda(0)/\lambda(1) = v$ and $\lambda(1)/\lambda(0) = w$, obviously $I^{\pm}(V) = \bigcup_{v \in V \subseteq W} I^{\pm}(v)$. These opposing arrows are conveniently studied in terms of new scaled coordinates $T$ and $R$ that satisfy

$$-\pi \le T \pm R \le \pi$$
$$0 \le R < \pi$$

in $ds^2 = -dT^2 + \left[dR^2 + \sin^2 R\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$.[18] Then the following definitions of infinities

- ***Future Infinity:*** *Timelike*, $i^+ : R = 0,\ T = \pi$; *Null*, $\mathscr{I}^+ : 0 < R < \pi,\ T = -R + \pi \in (0, \pi)$.
- ***Spatial Infinity:*** $i^0$: $R = \pi,\ T = 0$. is neither positive nor negative; it is directionless.
- ***Past Infinity:*** *Null*, $\mathscr{I}^- : 0 < R < \pi,\ T = R - \pi \in (-\pi, 0)$; *Timelike*, $i^- : R = 0,\ T = -\pi$.

formally establishes the forward and backward arrows in Minkowsky space-time with respect to the "origin" at $i^0 = (R, T) = (\pi, 0)$, refer Fig. 11.1 of Ref. [23].

The definition of a black hole [24]

$$\mathfrak{B} \triangleq \mathcal{M} - I^-(\mathscr{I}^+) \tag{20}$$

induced by an asymptotically flat spacetime manifold $\mathcal{M}$ as that subset "not contained in the chronological past $I^-$ of the future null infinity $\mathscr{I}^+$" — which demarcates a region of spacetime $\mathcal{M}$ that cannot link with the future through outgoing light signals, and defines a subset not in the chronological past of any point of the infinite future — decomposes for example, due to the opposing directions of the two halves, the one-dimensional timelike manifold $\mathbb{R}$ with $\mathscr{I}^{\pm} = \mathbb{R}_{\pm}$ into $\mathbb{R} = \mathbb{R}_+ \bigcup \{0\} \bigcup \mathbb{R}_-$; hence $I^-(\mathscr{I}^+) = \mathbb{R}_+$ and $\mathfrak{B} = \mathbb{R}_- \bigcup \{0\}$. A black hole, in this sense, is the neg-world $\mathfrak{W}$ that interacts with the real world complement $W$ through its real-world manifestation of "gravity". For $\lambda \ge 4$ the neg-world $\mathfrak{W}$ completely dominates its real-life partner $W$ with specific Cantor-like logistical evolution in the unit square $[0, 1]^2$. The gravitational black hole is the provider of emergent instability in the form of $P$-induced (potential) energy input to the system, spontaneously opposing the second-law dictated naturally expansive, self organizing, effect of the engine $E$ of $W$.[19] Hence the gravity-induced non-equilibrium steady state, with the blackhole acting as the generator of gravitational compressive emergence, inhibits the global expansive equilibrium of nature prescribed by the Second Law, see Fig. I-6. The corresponding real-world manifestation of entropy-reducing gravitational compression is then available to $W$ as the gradient dissipator $(T - T_c) < (T_h - T_c)$ inhibiting the maximum-entropic dictates of equilibrium thermodynamics, with the part $(T_h - T)$ being usefully utilized in the self-generation of the Maxwell-demon $P$.

This implies that any object inside the hole can only gravitate toward its center along the positive direction of time: *all future directed paths are in the direction of decreasing r and past-directions act along increasing r inside the black hole.* The event horizon of the Schwarzschild black hole spanning the quantum region $\lambda_* \le \lambda < 4$ is characterized by convergence of the period doubling sequence to a state of maximal ill-posedness as $\lambda \to \lambda_*$, and involves a piling-up of the iterates – as evident from the successive $\lambda_N$ values at periods $2^N$ – that manifests itself as a "hovering" of the object at the horizon, without "actually getting there". This also suggests a possible rationale for the hugely significant and uniquely uni-polar nature of gravity: *this neg-world component exists to provide the required dynamical stability in building the rich tapestry of Nature through competitive collaboration and complexity*. Being a constituent of the complimentary dual world, we of course cannot observe it directly (thereby explaining why a black hole is indeed so "black"), but only through its manifestation as the induced engine-pump system providing "completeness" to our existence.

As a definite illustration of this bi-directional expression of Nature, consider the following.

- **Big Bang**. The beginning of time $t = 0$, defined by the vacuum singularity $W \cup \mathfrak{W} = \emptyset$ generated by the functional simple $W := \lambda \in (2, 3)$ (region III) in partnership with the multifunctional simple $\mathfrak{W} := \lambda \in (0, 2]$, (region IV), Table I-2, at $\alpha = 1$ and $T_c = 0$, reduces (I-17) to $T_{\pm} = 0$ implying that the universe comprising the two opposites of $W$ and $\mathfrak{W}$ are identically at the same compressed null-state marked by △ in Fig. I-4. The resulting metastable state containing "everything" of the eventual concentration-dispersion pair encapsulated into the infinitely unstable "nothing" of $\emptyset$, that can only expand under the slightest fluctuation with the vacuum singularity degenerating spontaneously to a temperature singularity immediately on formation.
- **Black Hole and Steady State of the Universe.** As a result of this transition at $\alpha = 1$, $T_c = 0$, the far-from-equilibrium universe undergoes a dramatic and instantaneous transition to a state defined by $T_+ := T_c = 0$, $T_- = -\infty$ at the maximum possible reversible adaptability $\alpha = \infty$ generating an infinite temperature singularity $T_+ - T_-$. This

[18] Note the significant fact that both quantum mechanics and relativity treat time on a different footing with respect to space thereby implicitly accounting for the effect of the imaginary on real.

[19] Recall from Eqs. (1)–(11) that the inverse iterates of the logistic map constitute its thermodynamic forward direction of $E$-expansion and self-organization while the increasing iterates its gravitational backward direction of $P$-contraction and emergence.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

thermodynamically quasi-static, reversible, condition of $\iota = 0$ prohibits any measurable interaction between the real world $W$ and its black hole negative $\mathfrak{W}$. Inside the hole, there is a $W$-*increasing* temperature gradient $T_- - T_0$; hence a natural $W$-compressive ($\mathfrak{W}$-expansive) backward (forward) $P(\mathfrak{E})$-arrow $T_- \to T_0 := 0$ in $\mathfrak{W}$ that generates its complementary arrows in $W$, see Fig. I-6. This defines a corresponding temperature $T_h$ in accordance with Eqs. (I-20) and Fig. 1-4. In this sense *the black hole is a manifestation of maximal temperature discontinuity of the big-bang universal vacuum singularity*, that induces gravitational compression in $W$ that is responsible for its real-life dynamical productivity of a self-organized, emergent, state defined by $0 < \iota \leq 1$ and $0 < T_c \leq T \leq T_h$. Tension between the heating emergence of the Maxwell pump and the cooling self-organization of the engine is how nature adapts itself to the challenge of the Second Law. The engine-pump duality chooses an effective steady state $T$ of the evolving system with gravitational compressive heating playing an authoritative role in its foundation; see Example I-4.

Finally, we recall the loop quantum cosmology (LQC) approach [25,26] and compare with the picture presented above. Classical cosmology considers an arbitrary scale-factor $a(t)$ satisfying the Friedmann non-linear differential equation

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho(a)a^2(t), \tag{21a}$$

where $\rho(t)$ the total mass energy density (the cosmological constant $\Lambda$ and curvature $\kappa$ are both set equal to 0), with the size of the universe solely determined by the scale factor $a(t)$ describing expansion or contraction. A class of models assuming $\rho(t) \sim a(t)^{-1}$ for times $t \geq t_0$ leads to the solution

$$a(t) = \left(\sqrt{a_0} - \frac{1}{2\sqrt{3}}(t - t_0)\right)^2, \quad t \geq t_0 \tag{21b}$$

for a gravitationally collapsing cloud of matter-energy. For $a > a_0$ corresponding to $t > t_0$, the collapse of the spherical ball continues according to the laws of classical mechanics, with the density $\rho(t) \sim a(t)^{-1}$ approaching a singularity with diminishing $a$, and the mass, proportional to $a^2$, decreasing with the collapse due to loss to the exterior. As the collapse proceeds, $a(t) \to 0$ and $\rho(t) \to \infty$ leading to a singularity. Near the singularity, when densities are close to Planckian values, evolution breaks down due to the divergence of energy density with the right hand side of Eq. (21a) tending to 0.

This classical setting is modified by incorporating loop quantum effects to offset the two consequences of a singularity. Thus the main thrust of quantum cosmologies is to surmount the effects of a divergent density and to access "the other side" of the singularity. In terms of the Ashtekar variables, space becomes inherently discrete with quantum features becoming increasingly dominant as nonlinear curvature of space-time increases with the reduction in $a$, changing to the normal continuous description in the large eigenvalue limit. The significance of these variables is that the components $|p| = a^2$ playing the role of the scale factor $a(t)$ can, unlike $a$ itself, assume *both signs* introducing an orientation of space. Evidently what we see here is a disguised form of bi-directionality: whereas $a(t)$ can take only positive values, its loop-quantum incarnation is bi-directional.

In order to overcome the singularity problem at $a = a_0$ (that is at $t = t_0$) in the regime of strongly nonlinear curvatures, it is necessary to switch to a quantized Friedmann *difference equation* in terms of an internal discrete time $\mu$ which are eigenstates of $\mathbf{p}$, see [26]. This discrete quantization of the Friedmann equation is remarkably different from the normal Wheeler DeWitt quantization in the Planck region and leads directly to nonsingular evolution past the classical big-bang singularity. Whereas classical evolution in $p$ collapses at $p = 0$, the quantum difference equation uniquely evolves the wave function from positive to negative $\mu$: evolution thus does not terminate at the classical singularity but is extensible beyond it. Thus *decrease* of the volume eigenvalue parameter $V_\mu \propto |\mu|^{3/2}$ with decreasing negative $\mu$ changes to a monotonic increase for positive $\mu$ resulting in the initial gravitational collapse being halted at big-bang, and reversing thereafter to an expanding phase. As the sign of $\mu$ denotes the orientation of space through the classical singularity, it is suggested [26] that during its passage through the (two sides of the) singularity, the "universe turns its inside out" leading to a *repulsive* hard core that prevents gravitational collapse. In contrast to classical gravity which is always attractive with nothing to inhibit eventual collapse, loop quantum cosmology produces a repulsive core where attraction succumbs to repulsion thereby annulling singularities. In this scenario, the classical big-bang is replaced at Planckian scales by a "quantum bounce" [25] with the scale factor $a(t)$ starting on an expansive, rather than the usual classical contracting, phase.

The scenario presented above supports the bi-directional expansion compression thermodynamics of gravity dominated systems developed in this paper. Our approach, in addition, provides a possible self-contained rationale of the origin of this Second-Law defying behavior of natural systems in the dynamical framework of self-adapting complex systems, where complexity represents the state of dynamical balance between a emergent, destabilizing, bottom-up pump, opposed by an top-down, stabilizing engine of self-organization.

## 4. Discussions and conclusion

• *The lack of satisfying progress in the numberless attempts at the unification of physical theories, in spite of the fundamental changes produced by quantum mechanics in the interpretation of nature, seems to indicate that the exophysical approach has already reached its limits. If this is the case, it will be necessary that both the interaction of man with nature and the role of subjectivity find a proper place in science, to be framed in a perspective other than the exophysical attitude.* Buccheri [27].

For the benefit of the reader, we recount here the approach of Gerard 't Hooft who considers quantum mechanics and gravity from the point of dissipative dynamical systems. The principal feature of the 't Hooft analysis of deterministic systems [1] is the identification of equivalence classes of non-invertible, non-injective maps of the type we consider, to constitute a base of the quantum Hilbert spaces. As a specific example, he considers the equivalence classes $[1] = \{1\}$, $[2] = \{2, 4\}$, $[3] = \{3\}$ generated by the points 1, 2, 3, 4 of Fig. 1(a) under the mapping $1 \to 2 \to 1, 3 \to 3, 4 \to 1$ to define the 2-dimensional quantum space spanned by $\{([1] + [2])/\sqrt{2}, [3]\}$. The significance of this deterministic construction of quantum mechanics is information loss through dissipation: the information content in the primordial ontological states $\{1, 2, 3, 4\}$ is dissipated in the smaller number of equivalence classes: "equivalence classes tend to form discrete quantum sets only if one allows information to dissipate". The stable $2^N$ limit cycles of our complex states of the referenced figure represent the discrete, finite set of equivalence classes of quantum states possessing the same distant future in the 't Hooft universe. In this setting, with black holes representing massive sets of equivalence classes in extreme situation of information loss, the smaller number of classes compared to the primordial states is thought to explain how the entropy of a black hole is proportional only to its area rather than to the volume that depends on the number of ontological states.

The binary two-spin system, a perfect analogue of unimodal maps with rising and falling branches, makes the correspondences sketched above naturally intuitive. The main issue between them however revolves around their very basic premises; while one is based in the linearity of superpositions and tensor products to generate non-locality, the other is a strongly nonlinear discrete system of equivalence classes and ill-posedness, increasingly evolving into the future. This antithetical difference in the structure of the systems – of smoothness, continuity, reversibility and regularity on the one hand and singularity, jump, irreversibility and discontinuity on the other – warrants the search for a possibly more general foundation of the simple, especially in view of the current interest in the complexity paradigm of Nature. Abandonment of the severe restrictions imposed by linearity also helps toward a natural resolution of many of the long-standing paradoxes associated with it; thus, for example, as multiple degeneracy is a natural consequence of non-injective ill-posed induced equivalence classes, preparation-measurements now turn out to be paradox-free because experimental contextuality directly follows from the Axiom of Choice.

One of the simplest ways to model order is through symmetry: in symmetric patterns one part of the pattern is sufficient to reconstruct the whole. Complexity on the other hand is characterized by lack of symmetry or "symmetry breaking", by the fact that no part or character of the complex entity can provide sufficient information to actually or statistically predict the properties of the others parts; spontaneous symmetry breaking usually brings the system from the more probable higher entropy dispersed state into a less probable lower entropy organized form; "Nature is not symmetric: Nature abhors symmetry" [28]. Phase transitions often take place between phases with different symmetry, as for example the transition between a fluid and a crystalline solid, generally one phase in a phase transition is more symmetrical than the other. At low temperatures, the system tends to be confined to the low-energy states. At higher temperatures, thermal fluctuations allow the system to access states in a broader range of energy, and thus more of the symmetries. When symmetry is broken, one needs to introduce one or more extra variables as order parameters to describe the state of the system: an order parameter is a measure for the degree of order in a system with extremes 0 for total disorder, absolute symmetry, indistinguishability and maximum entropy and 1 for complete order, broken symmetry, and zero entropy.

Symmetry breaking phase transitions play an important role in cosmology. It has been speculated that, in the hot early universe the vacuum possessed high entropy with a large number of symmetries. As the universe expanded and cooled, the vacuum underwent a series of symmetry breaking transitions that generated the clusters of matter that Nature is composed of.

Patterns and structure exist in Nature because of lowering of symmetry — symmetry breaking creates form and substance. In the period doubling sequence, the current symmetrical stable form breaks to generate new types of symmetry (that appear as asymmetry of the former kind), and absence or the breaking of some form of symmetry is essential for the existence of something physical. The state of maximum entropy of indistinguishability, perfect and absolute symmetry corresponds to the sole effect of the "engine" that must be moderated by a self-generated "pump" breaking this symmetry to produce something tangibly real. This preliminary discussion suggests the possibility of a deeper study of a link between chanoxity, symmetry and phase transitions that we hope to undertake in the future: is complex holism a manifestation of a saturated two phase mixture of liquid-order and vapor-disorder with $\chi$ qualifying as a quality of the mixture?

If gravity is to play the distinguished role in the emergence of complex systems, study of the effects of prolonged weightlessness in space flights, for example, can lead to significant understanding of the issues involved. The human brain being one of the most profoundly evolved complex systems in nature composed of billions of cells with individual electrical activity and interconnected in highly intricate networks, sustained zero-gravity data tend to suggest that its functioning is fundamentally affected by weightlessness, with its capacity of response significantly declining under these conditions. While the basic cognitive processes tend to remain largely unimpaired during short-term space flights, substantial deviations in fine psychomotor processes may emerge under high load conditions involving deterioration of action related to the mind or will. Psychomotor activities are the physical gestures that result from mental processes and are a product of the psyche, with retardation manifesting as a slowing down of thought, coordination, speech, and impaired articulation, and a reduction of physical movements in a person; a state characterized by sluggishness, hesitancy and confusion in speech and intention. The view of life as a complex system of an entropy-decreasing mind-pump in cohabitation with an entropy-increasing body-engine, with the brain representing a physical manifestation of the mind, [8], acquires new significance on the basis of these investigations.

The immune system, seen as Nature's moderation to the inevitable second law degradation of the open, far from equilibrium life system, is also affected by weightlessness, astronauts becoming quite susceptible to illness in space. The human immune response lowers, and the quantity of infection-fighting cells decreases after prolonged exposure to microgravity, with this syndrome characterized by nausea, headache, lethargy and sweating. Even when astronauts undergo strenuous exercise routines daily to try and maintain bone and muscle mass during a long space mission, some still have to be carried on stretchers on return to earth. It is interesting to note the striking similarities between aging and the effects of space flight on a person's health. Like astronauts after a prolonged space mission, the elderly in a state of increased entropy, experience weakening muscles and bones, insomnia, difficulty in balancing and depressed immune response. The key significant difference between aging and micro-gravity is that the changes in astronauts are, for the most part, reversible, with the period of recovery proportional to the duration of the mission. This strongly suggests the role of gravity in the creation, maintenance, and regulation of the structure of life.

## Appendix. A one-dimensional representation of the henon map

The parametrized Henon map

$$H_a(x, y) = (f_a(x, y), g_b(x, y)) := (1 - ax^2 + y, bx), \quad b = 0.3 \tag{22}$$

leads to the two-dimensional iterative scheme

$$x_{n+1} = 1 - ax_n^2 + y_n$$
$$y_{n+1} = bx_n,$$

$n = 0, 1, 2, \ldots$ with fixed points $f(x, bx) = x$ given by

$$x_\pm = \frac{1}{2a}\left[b - 1 \pm \sqrt{(b-1)^2 + 4a}\right] \tag{23a}$$

$$y_\pm = bx_\pm \tag{23b}$$

which limit the admissible $a > -0.25(b-1)^2$ to values larger than $-0.1225$. This two dimensional map is known to undergo [29] a period doubling sequence to chaos at the critical value $a = 1.058049\cdots$; thus the first period doubling to a period-two orbit occurs when $a$ satisfies the slope condition

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} + 1 & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} + 1 \end{vmatrix} = 0 \tag{24a}$$

of $H_a$ at $a := 0.75(1-b)^2 = 0.3675$ for $x_+ = 0.5(1-b)/a = 0.952381$. Significantly, however, the integrated period doubling-pitchfork bifurcation condition

$$\begin{vmatrix} \dfrac{\partial f^2}{\partial x} - 1 & \dfrac{\partial f^2}{\partial y} \\ \dfrac{\partial g^2}{\partial x} & \dfrac{\partial g^2}{\partial y} - 1 \end{vmatrix} = 0 \tag{24b}$$

at the initiation of new periodic cycles at the fixed point $(x_+, y_+)$ is not satisfied, the value of determinant (24b) being 0.42. This apparently characteristic failure of multidimensional systems by which the new dimensions endow the system additional latitude in absorbing qualitative evolving properties that need no longer show up as emerging patterns in the structure of the system has been used in [8] to propose a one-dimensional *Kitchen of Nature* where all evolutionary processes are conceived to take place involving non-unique, multifunctional, structurally emergent features which are subsequently displayed in a more agreeable and pleasant final form in the rich tapestry of multidimensional Nature.

Here we propose a mechanism to reduce a multidimensional mapping to its basic one dimensional "kitchen" form using the tools of *lexicographic* or *dictionary ordering* [30] on the Cartesian product of ordered sets $(A_1, \leq_1) \times (A_2, \leq_2) \times \cdots \times (A_n, \leq_n)$ defined as

$$(a_{11}, a_{21}, \ldots, a_{n1}) <_D (a_{12}, a_{22}, \ldots, a_{n2}) \Leftrightarrow (\exists\, m > 0) : (\forall\, k < m)(a_{k1} = a_{k2}) \wedge (a_{m1} <_m a_{m2}) \tag{25}$$

the order $\leq_D$ on the Cartesian product follows the rule of arrangement of words in a dictionary. This order is utilized to generate a one-dimensional equivalent of the $n$-dimensional unit hypercube so that

$$(0.a_1a_2\ldots,\ 0.b_1b_2\ldots,\ldots,\ 0.n_1n_2\cdots) \in \mathbb{R}^n \mapsto 0.a_1b_1\cdots n_1a_2b_2\cdots n_2a_3b_3\cdots \in \mathbb{R} \tag{26}$$

is the *unique* one-dimensional representation in $[0, 1)$ of a point $(a, b, \ldots, n) \in [0, 1)^n$, where 1 is to be represented either as an infinite string of 9's in $0.99\cdots$ or as equivalent to 0. Clearly the one-dimensional equivalent respects the order of the original; thus

$$(a, b, c) <_D (x, y, z) \Leftrightarrow 0.a_1b_1c_1\, a_2b_2c_2\, a_3b_3c_3 < 0.x_1y_1z_1\, x_2y_2z_2\, x_3y_3z_3$$

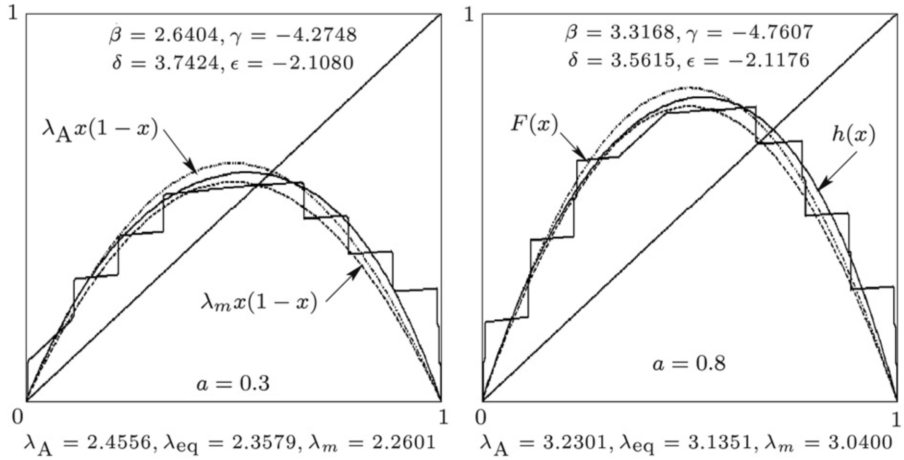


**Fig. 2.** Equivalent one-dimensional Henon maps. $\lambda_A = 6A$ and $\lambda_m = 4h_m$ with $A$ and $h_m$ the area and maximum height of the one dimensional Henon, and $\lambda_{eq} = (\lambda_A + \lambda_m)/2$.

ensures that equivalent functional relations on $\mathbb{R}$ preserve the order of the original on $\mathbb{R}^n$, making Eq. (26) particularly valuable in transforming the dimension of functions. Such an order-preserving relation not only displays the character of the original, but by restraining the transformed version from the freedom of its additional dimensions also ensures the role of non-unique multiplicities inherent in the one-dimensional evolution of the reduced function.

In applying these considerations to the reduction of the two-dimensional Henon map $H_a(x, y)$ Eq. (22) to an equivalent one-dimensional form $h(x)$ on [0, 1], we proceed as follows.

*Step* 1. Consider the domain of $H_a$ to be $[x_M, x_P] \times [bx_M, bx_P]$, where

$$x_P = \frac{1}{2a}\left[b + \sqrt{b^2 - 4a(x_M - 1)}\right]$$

is the solution of $f(x, bx) = x_M := x_-$ that ensures its equivalence to $x_M$ under $H$.

*Step* 2. This domain and the corresponding range of $H$ are mapped onto the respective unit squares with $(x_M, x_P) \mapsto (0, 1)$,

*Step* 3. The unit squares are finally transformed to their one-dimensional equivalents according to Eq. (26), thereby defining the one-dimensional form $h(x)$ of $H_a(x, y)$.

Plots of two such functions are shown in Fig. 2 as the stepped curves $F(x)$, so formed because of the nature of the one-dimensional reduction. As immediately apparent from the figures, the one-dimensional equivalents $F(x)$ are

(i) Non-injective; hence non-invertible,

(ii) Perceptibly non-symmetric, with a positive shift of the maximum to $x_m > 0.5$, and wider because of this shift, as compared with the corresponding logistic map with the same maximum which are shown in the figures as dashed curves.

It was found that the quadratic logistic cannot represent this one-dimensional form adequately, but a fourth order polynomial

$$h(x) = \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4, \quad x \in [0, 1]$$

satisfying

(a) $h(1) = 0$: $\beta + \gamma + \delta + \epsilon = 0$; (b) The area $\beta/2 + \gamma/3 + \delta/4 + \epsilon/5$ of $h(x)$ is the same as the area $A$ of $F(x) : A = \beta/2+\gamma/3+\delta/4+\epsilon/5, x \in [x_M, x_P]$; (c) $h$ attains the maximum of $F(x_m)$ at $x_{eq} = 0.5+c_x(x_m-0.5)$, with $c_x$ a positive constant taken in these calculations to be 0.2, to simulate the "shift" of the maximum: $\beta+2\gamma x_{eq}+3\delta x_{eq}^2+4\epsilon x_{eq}^3 = 0$; (d) $h(x_{eq}) = y_{eq}$, where $y_{eq} = 0.25c_y(\lambda_m+\lambda_A) := 0.25\lambda_{eq}$ is an equivalent symmetric logistic maximum of $h$ where $\lambda_m = 4F(x_m)$ and $\lambda_A = 6A$ are the $\lambda$ values of $F$ based on $y_m = F(x_m)$ and $A$. Here $c_y$ was taken to be 0.5: $\beta x_{eq} + \gamma x_{eq}^2 + \delta x_{eq}^3 + \epsilon x_{eq}^4 = y_{eq}$.
was considered acceptable. The resulting $h(x)$ is shown in the figure as solid curves; also shown is a chain-dash logistic of the same area $A$ as that of $F(x)$. These results clearly demonstrate that the one-dimensional equivalent of the Henon map is non-injective and logistic-like, with inherent characteristics that reflect the higher dimensionality of $H(x, y)$. Fig. 3 further demonstrates the effects of higher dimensionality on the relative dynamics of $h(x)$ and the symmetric $\lambda_{eq}x(1-x)$, and by implication that of the original Henon $H_a(x, y)$ for parameters $a = 1.1$, 1.4, 1.5, and 1.6 that are chosen to be greater than the chaotic threshhold of $a = 1.058049$. For $a = 1.1$, all the three manifestations $\lambda_A$, $\lambda_{eq}$ and $\lambda_m$ of logisticity are well below the chaos threshold of $\lambda = 3.5699456$ [29], while for the other three values of $a$, the magnitude of $\lambda_{eq}$ that defines $h(x)$ is in the chaotic range. Nonetheless it is significant that one-dimensionally the Henon map does not display chaotic behavior until after $a > 1.5$ and that the "shifted logistics" defined by $\lambda_m$ and $\lambda_A$ are significantly more stable than the corresponding $\lambda_{eq}$-logistics which are in turn stabler than $H_a(x, y)$. In fact, the shifted logistic $h(x)$ show the familiar period doubling sequence even when the equivalent $\lambda_{eq}$ is well entrenched in the chaotic region showing the remarkable stability of non-symmetric logistic-type interactions compared with their symmetric counterparts.

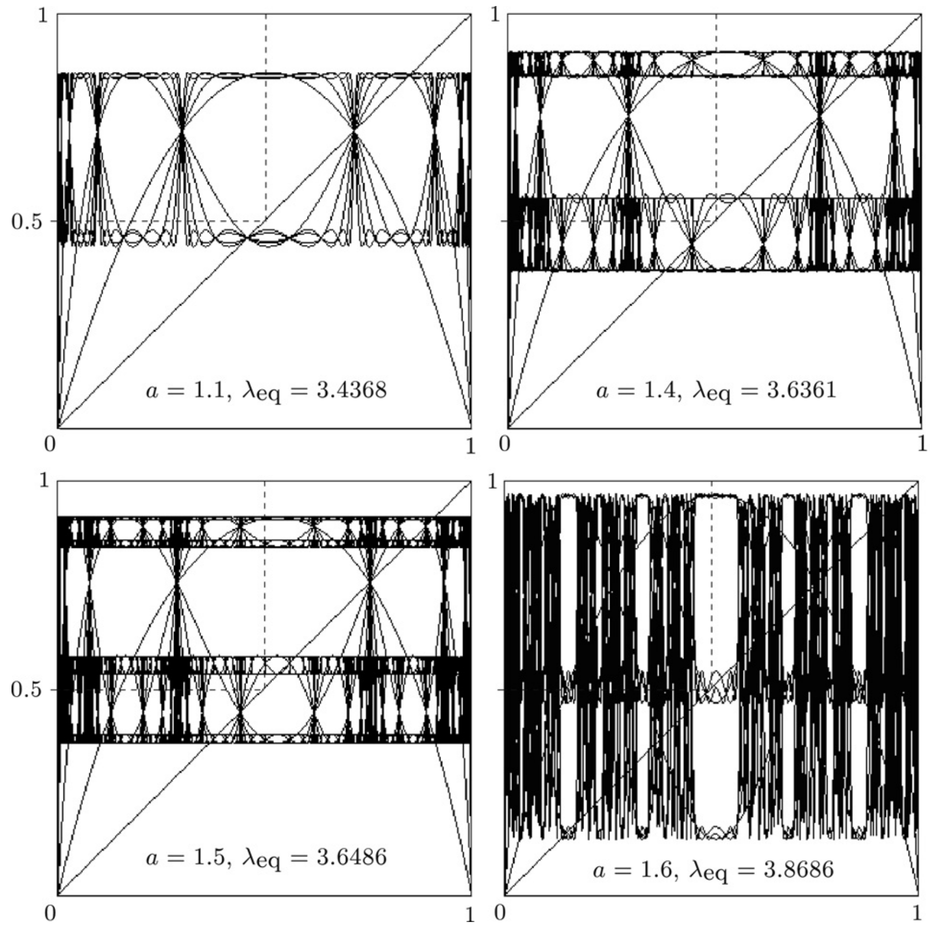


**Fig. 3.** Comparison of the one-dimensional representation $h$ of $H$ with the symmetric logistic $\lambda_{\mathrm{eq}}x(1-x)$ for various values of $\lambda = \lambda_{\mathrm{eq}}$. Recall that $3 < \lambda \leq 3.4495$, $3.4495 < \lambda \leq 3.5441$, $3.5441 < \lambda \leq 3.5644$ for the 2, $2^2$, $2^3$-cycles of the logistic map, and $\lambda_* = 3.5699$ marks the transition to chaos.

The special significance of one-dimensional dynamics relative to any other finds an intuitive interpretation from the Sarkovskii ordering (note the upper and lower bounds)

$$\begin{array}{ccccccccc}
3 & \rhd & 5 & \rhd & 7 & \rhd & \cdots & & \\
2\cdot 3 & \rhd & 2\cdot 5 & \rhd & 2\cdot 7 & \rhd & \cdots & & \\
 & \vdots & & \vdots & & \vdots & & & \\
2^n\cdot 3 & \rhd & 2^n\cdot 5 & \rhd & 2^n\cdot 7 & \rhd & \cdots & & \\
 & \vdots & & \vdots & & \vdots & & & \\
\cdots & \rhd & 2^m & \rhd & \cdots & \rhd & 2 & \rhd & 1
\end{array}$$

of positive integers of the Sharkovskii Theorem which states that if $f : [a, b] \to \mathbb{R}$ is a continuous function having a $n$-periodic point, and if $n \rhd m$, then $f$ also has a $m$-periodic point. Noting that the periodicity of an $f$-interaction between two spaces essentially denotes the number of independent degrees of freedom required to completely quantify the dynamics of $f$, it is inferred that while a fixed point of "dimension" 1 contains the basic information of all other periods, a period-3 embodies every other dimension in itself. Hence, it is surmised that dynamics on 1-dimension, by being maximally restrained compared with any other, allows for the greatest emergence of structures as multifunctional graphical limits, while dimension 3 by being least restrained is ideally suited for an outward, well-defined, and aesthetically appealing, simultaneous expression of the multitude of eventualities that the graphical limits entail.

This argument of the fundamental structure-generating smallest dimension can also be viewed in the following perspective. A *space-filling curve* is a continuous function $\mathcal{P} : I = [0, 1] \to I^k$ that fills up the entire unit cube $I^k$, that is which passes through all points of the cube. The actual construction of $\mathcal{P}$ which proceeds iteratively [30], is distinguished as compared to the structures generated by maps on $I$, *by the iterates being multifunctional with respect to every factor space in the image of* $\mathcal{P} : I \to I^k$. Thus the iterative system takes advantage of the extra dimensions of the image space available to it to generate additional multifunctional relations that eventually leads to $\mathcal{P}$ visiting every point of $I^k$; something that is

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002

impossible when $k = 1$, [29]. This restriction forces the dynamics for $k = 1$ to be structurally richer and more varied than the corresponding multi-dimensional counterparts.

This simulation of multidimensional maps to equivalent one-dimensional forms justifies the perceived [8] pre-eminent role of one-dimensional iterative systems in defining non-equilibrium, steady states of complex systems, and allows us to draw the following conclusions. Lexicographic reduction is an efficient technique for generating order-preserving basic one-dimensional mappings that clearly bring out the non-injective ill-posedness in the dynamics of the Henon map. The significant differences in the nature of the equivalent $h(x)$ as compared to the one-dimensional quadratic map for the equivalent $\lambda_{eq}$ denote the nontrivial effects of higher dimensions on the dynamics of evolution. Thus the increased stability of $h(x)$ as compared with $H(x, y)$ is most probably attributable to the period-doubling-pitchfork-bifurcation criterion that must be ensured on the route to chaos for $h(x)$ that does not appear to be necessary in higher dimensions. As with the Lorenz differential system [8], this has the effect that non-reductionist emergence of patterns necessary for the complex evolution of a dynamical system – so fundamental for the rich diversity of Nature – is inhibited in multidimensional systems and the three-dimensional mode of expression is an effectively efficient mechanism for the outward expression of the interrelated and interdependent complex dynamics in a colorful and more appealing fashion.

## References

[1] G. 't Hooft, Quantum gravity as a dissipative deterministic system, Class. Quantum Grav. 19 (1999) 3263–3279.
[2] A. Sengupta, Is nature quantum non-local, complex holistic, or what? I — Theory & analysis. Nonlinear Anal.: RWA (2009) (in press).
[3] R.E. Nisbett, K. Peng, I. Choi, A. Norenzayan, Culture and systems of thought: Holistic versus analytic cognition, Psychol. Rev. 108 (2001) 291–310.
[4] J.A. Wheeler, Information, physics, quantum: The search for links, in: W.H. Zurek (Ed.), Complexity, Entropy and the Physics of Information, in: SFI Studies in the Sciences of Complexity, vol. VIII, Addison-Wesley, 1990.
[5] C. Fuchs, A. Peres, Quantum theory needs no "interpretation", Phys. Today 53 (3) (2000) 70–71.
[6] S. Goldstein, Bohmian mechanics, in: E.N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, 2006. http://plato.stanford.edu/archives/sum2006/entries/qm-bohm/.
[7] D. Bohm, A suggested interpretation of quantum theory in terms of "hidden" variables I, II, Phys. Rev. 85 (1952) 166–179; 180–193.
[8] A. Sengupta, Chaos, nonlinearity, complexity: A unified perspective, in: A. Sengupta (Ed.), Chaos, Nonlinearity, and Complexity: The Dynamical Paradigm of Nature, in: StudFuzz, vol. 206, Springer-Verlag, Berlin, 2006, pp. 270–352.
[9] G. Strang, Linear Algebra and its Applications, Harcourt Brace Jovanovich, 1988.
[10] G. Preparata, An Introduction to a Realistic Quantum Physics, World Scientific Publishing Co. Pte. Ltd., Singapore, 2002.
[11] P. Horodecki, R. Horodecki, M. Horodecki, Entanglement and thermodynamical analogies, Acta Physica Slovaca 48 (1998) 141–156. arXiv:quant-ph/9805072.
[12] M. Schlosshauer, Decoherence, the measurment problem, and interpretations of quantum mechanics, Rev. Modern Phys. 76 (2004) 1267–1305.
[13] P. Grangier, Contextual objectivity: A realistic interpretation of quantum mechanics; contextual objectivity and quantum holism. arXiv:quant-ph/0012122; arXiv:quant-ph/0301001, 2001; 2003, pp. 1–5; 1–4.
[14] D. Durr, S. Goldstein, N. Zanghi, Quantum equilibrium and the origin of absolute uncertainty, J. Stat. Phys. 67 (1992) 843–907.
[15] J. Bricmont, Science of chaos or chaos in science? Phys. Mag. 17 (1995) 159–208.
[16] R.C. Bishop, Resonances, unstable systems, and irreversibility: Matter meets mind, Inter. J. Theor. Phys. 44 (2005) 1879–1888.
[17] J.P. Antoine, Quantum mechanics beyond hilbert spaces, in: A. Bohm, H.D. Doebner, P. Kielanoski (Eds.), Irreversibility and Causality, Semi-Groups and Rigged Hilbert Spaces, in: Lecture Notes in Physics, vol. 504, Springer-Verlag, Berlin, 1998.
[18] M.G. Murdeshwar, General Topology, Wiley Eastern Limited, New Delhi, 1990.
[19] A. Bohm, N.L. Harshman, Quantum theory in the rigged Hilbert space — Irreversibility from causality, in: A. Bohm, H.D. Doebner, P. Kielanowski (Eds.), Irreversability and Causality, Semigroups and Rigged Hilbert Spaces, in: Lecture Notes in Physics, vol. 504, Springer-Verlag, Berlin, 1998.
[20] I.E. Antoniou, I. Prigogine, Intrinsic irreversibility and integrability of dynamics, Physica A 192 (1993) 443–464.
[21] M. Castagnino, M. Gadella, O. Lombardi, Time reversal, irreversibility and arrow of time in quantum mechanics, Found. Phys. 36 (2006) 407–426.
[22] L.M. Burko, R.H. Price, Ballistic trajectory: Parabola, ellipse, or what? Amer. J. Phys. 73 (2005) 516–520.
[23] R.M. Wald, General Relativity, The University of Chicago Press, Chicago, 1984.
[24] R.M. Wald, The thermodynamics of black holes, Living Rev. Relativity 4 (2001). http://www.livingreviews.org/Articles/Volume4/2001-6wald/.
[25] A. Ashtekar, Quantum nature of the big bang: An analytic and numerical investigation. arXiv:gr-qc/0604013, 2006.
[26] M. Bojowald, Elements of loop quantum cosmology, arXiv:gr-qc/0505057. Also: Loop Quantum Cosmology, arXiv:gr-qc/0601085, 2005.
[27] R. Buccheri, M. Buccheri, Evolution of human knowledge and the endophysical perspective, in: R. Buccheri, A.C. Elitzur, M. Saniga (Eds.), Endophysics, Time, Quantum and the Subjective, World Scientific, Singapore, 2005.
[28] P.W. Anderson, D.L. Stein, Broken symmetry, emergent properites, dissipative structures, life: Are they Related? in: P.W. Anderson (Ed.), Basic Notions of Condensed Matter Physics, The Benjamin/Cummings Publishing Company, Inc., 1984.
[29] H.O. Peitgen, H. Jurgens, D. Saupe, Chaos and Fractals: New Frontiers of Science, Springer-Verlag, New York, 1992.
[30] J.R. Munkres, Topology: A First Course, Prentice-Hall of India Private Limited, 1992.

Please cite this article in press as: A. Sengupta, Is nature quantum non-local, complex holistic, or what?: II — Applications, Nonlinear Analysis: Real World Applications (2008), doi:10.1016/j.nonrwa.2008.09.002