Quantum source of entropy for black holes

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We associate to any quantum field propagating in the background metric of a black hole an effective density matrix whose statistical entropy can be interpreted as a contribution to the total entropy of the black hole. By evaluating this contribution in a simplified case, we show that in general it can be expected to be finite and proportional to the area of the black hole. As a by-product of our calculation we obtain a general expression for the entropy of any real Gaussian density matrix.

I. INTRODUCTION

The horizon[1] of a black hole—insofar as this surface remains well defined in the quantum context—divides spacetime into "interior" and "exterior" regions. With respect to the latter, and in particular with respect to observations conducted entirely outside the horizon, the black hole appears to behave as a thermodynamic object of entropy $2\pi(k/l_P{}^2)A$, where A is the horizon area and $l_P=(8\pi G\hbar/c^3)^{1/2}$.

Possibly the first evidence for this was the discovery that classically an isolated black hole rapidly settles into an equilibrium state characterized by only a handful of parameters, the precise number depending on how many gauge fields exist in nature.[2] With the entropy specified as a function of these parameters, all other thermodynamic quantities can be derived as usual, and one finds in particular the well-known formula

$$T_{\rm bh}=\frac{c^3\hbar}{8\pi Gk}M^{-1}$$

for a spherical hole of mass M.

The interaction of such an equilibrium black hole with various sorts of external matter has been analyzed[3] in the framework of classical metric plus quantum field; and the hole's thermodynamic aspect has been borne out in almost every respect. To our knowledge the most complete result along such lines[4] concerns scattering of massless spin-zero bosons off a black hole, and states that for a given mode the effect of the scattering is to convert a thermal input (Gibbs canonical state) of any temperature into a thermal output of some other temperature, nearer to $T_{\rm bh}$. Such scattering (which of course includes the Hawking radiation as the special case of zero input temperature) will always increase the value of

$$\Psi=S_{\rm ext}-U_{\rm ext}/T_{\rm bh}\ ,$$

where $S_{\rm ext}$ and $U_{\rm ext}$ are the entropy and mean energy of the external quanta. This is exactly the effect that interaction with a heat bath of temperature $T_{\rm bh}$ would have. (At least the increase of Ψ is guaranteed when the input density matrix is diagonal in the particle-number basis.[5] We do not know whether this guarantee can be extended to the case of a general in state, in which phase correlations among states of different particle number may be present.)

Another way to express the above increase of Ψ is to say that the sum $S=S_{\rm ext}+S_{\rm bh}$ increases when the quasistatic "back reaction" $dM=-dU_{\rm ext}$ is taken into account (because then $dS=dS_{\rm ext}+dS_{\rm bh}=dS_{\rm ext}+dM/T_{\rm bh}=dS_{\rm ext}-U_{\rm ext}/T_{\rm bh}=d\Psi$). In still other words, the so-called "generalized second law" is "semiclassically" (and modulo the above caveat) valid for any process of scattering of radiation quanta off a black hole. A number of *Gedanken experiments*[6] involving box-lowering processes not obviously reducible to scattering have corroborated this impossibility of constructing perpetual motion devices by means of interactions with equilibrium black holes.

Finally, there is also evidence for the "second law" even when the black holes involved are far from equilibrium, but unfortunately only in the limit $\hbar\to 0$. In that limit $S_{\rm bh}/S_{\rm ext}\to\infty$ (because $l_P\to 0$), and the increase of $S_{\rm bh}+S_{\rm ext}$ reduces to the statement that the total horizon area cannot decrease, something which, though not fully proven, has been reduced to the so-called cosmic censorship conjecture.[7]

Although the above and similar results establish that a black hole is, with respect to its environment, a thermodynamic object, they neither explain the origin of its entropy nor answer the closely related question of why $S_{\rm bh}+S_{\rm ext}$ always increases. One's first impulse would be to identify $S_{\rm bh}$ with $k\ln N$, N being the number of "internal black-hole states." But such an identification suffers from two drawbacks.

In the first place the number of internal states compatible with a given external appearance is infinite, as exemplified by the Oppenheimer-Snyder solutions in which the "interior" contains a Friedmann universe of arbitrarily large diameter.[8] Perhaps one could exclude such configurations classically by requiring that the black hole "not have a white hole in its past,"[9] but such a condition would seem difficult to formulate for quantum gravity where "tunneling" would probably be possible between any two configurations, and where in particular no unique spacetime metric would represent the past history of any $t=0$

quantum state.

In the second place, no law of entropy increase would follow, even if a finite number of internal states could be associated to a given black-hole exterior. The corresponding deduction for ordinary thermodynamic systems is based on the hypothesis of weak coupling between the system and its environment. This allows one to define separate "phase-space volumes," $N_{\rm sys}$ and $N_{\rm env}$, in such a way that $N_{\rm tot} \approx N_{\rm sys} N_{\rm env}$, which yields the desired additivity $S \approx S_{\rm sys} + S_{\rm env}$ upon the taking of logarithms. Then, if the system is in approximate internal equilibrium, $N_{\rm sys}$ will approximately equal the number of internal states compatible with the values of its thermodynamic parameters. For a black hole, however, neither the assumption of weak coupling nor (probably) that of internal equilibrium is tenable. Instead, the coupling outside$\rightarrow$inside is very strong, whereas the converse coupling inside$\rightarrow$outside is nonexistent.[10] Moreover the interior is far from equilibrium, since the only Killing vector there (say for the Schwarzschild black hole) is spacelike.[12]

The observation that the region outside the horizon is to all intents and purposes an autonomous system suggests we seek the entropy there[13] rather than attempting to overcome the difficulties brought forward above.[14] To that end introduce $\rho_{\rm red}$, the reduced density matrix corresponding to the observables available on a spacelike hypersurface $\mathscr{H}$, extending from (some two-dimensional cross section of) the horizon to spatial infinity. Because the external region is autonomous, $\rho_{\rm red}$ undergoes a well-defined (albeit nonunitary) evolution as $\mathscr{H}$ advances in time. Moreover the entropy $S_{\rm red} = -{\rm tr}(\rho_{\rm red} \ln \rho_{\rm red})$ will in general be nonzero even when the overall quantum state is pure. Entropy of this sort may be said to be purely quantum in origin, since an analogous situation is impossible classically. To solve the "riddle of black-hole entropy" along these lines would be to show first that $S_{\rm red}$ splits into the sum of $2\pi A$ with the usual entropy of external matter, and second that $S_{\rm red}$ increases with time.

In the following we consider only one possible contribution to $S_{\rm red}$, but one of the sort which, semiclassically, carries all the entropy once the hole has evaporated.[16] Specifically, we consider[13] a scalar field propagating in a black-hole background and estimate the contribution to $S_{\rm red}$ arising because $\rho_{\rm red}$ lacks the information contained in the vacuum correlations between points inside and outside the horizon.[17] We do not consider other possible contributions, such as that due to the geometrical variables associated with the horizon itself (its "shape"), which is perhaps more promising as a candidate for the primary source of black-hole entropy.

Nevertheless we are able to show that the contribution we do consider is proportional to the horizon area A, and that the proportionality constant will have the correct order of magnitude if there is an ultraviolet cutoff at around the Planck length l_P. We also argue that the total $S_{\rm red}$ must increase with time in the full quantum theory.

In the following model calculations, we consider a real scalar field satisfying the Klein-Gordon equation on a fixed background. We use units in which $\hbar = c = k = 1$.

Consider the scalar field to be in a pure state to start with. Then the density matrix ρ is trivial and the entropy vanishes:

$$S = -{\rm tr}(\rho \ln \rho) = 0 .$$

However, we will calculate the quantity ${\rm tr}(\rho_{\rm red} \ln \rho_{\rm red})$, and we will find that it does not vanish. We will see, moreover, that the major contribution to the entropy comes from the high-frequency modes of the field, with a wavelength much smaller than the radius of curvature of the background manifold. Our simplified model will be based therefore on a scalar field on a flat background in its vacuum state. In forming $\rho_{\rm red}$ we will trace out the variables associated to a spatial region Ω representing the interior of the black hole at a given time.

We remark here that nowhere in our calculations will we use directly the fact that Ω is the interior of a black hole, and one may wonder to what extent the procedure can be applied to some other region: after all, in any calculation of entropy one chooses to perform some coarse graining on the observables, which could consist in ignoring all observables associated with measurements in some region of space. What distinguishes a black-hole interior in this sense is, on one hand, the objective limitations to any attempt by observers outside a black hole to refine their coarse graining (if they want to keep communicating the results of their measurements to each other) and, on the other, the fact that most other choices for the region Ω would not lead to a useful notion of entropy, since the external region would not be autonomous and one would not be able to show, using the argument in Sec. IV, that the quantity one calculates is nondecreasing with time.

II. ENTROPY OF A COLLECTION OF COUPLED HARMONIC OSCILLATORS

We model the scalar field on $\mathbb{R}^3$ as a collection of coupled oscillators on a lattice of space points, labeled by capital latin indices, the displacement at each point giving the value of the scalar field there. In this case the Lagrangian can be given by

$$L = \tfrac{1}{2} G_{MN} \dot{q}^M \dot{q}^N - \tfrac{1}{2} V_{MN} q^M q^N , \qquad (1)$$

where q^M gives the displacement of the Mth oscillator and $\dot{q}_M$ its generalized velocity. The symmetric tensor G_{MN} is positive definite, and therefore invertible, i.e., there exists a G^{MN} such that

$$G^{MP} G_{PN} = \delta^M{}_N ,$$

and we can thus consider G_{MN} as a metric on the configuration space of the coupled harmonic oscillators. The tensor V_{MN} is also symmetric and positive definite. Introducing the conjugate momentum to q^M,

$$P_M = G_{MN} \dot{q}^N ,$$

we can write the Hamiltonian for our system as

$$H = \tfrac{1}{2} G^{MN} P_M P_N + \tfrac{1}{2} V_{MN} q^M q^N . \qquad (2)$$

Next, consider the symmetric matrix W_{MN} defined by $W \geq 0$ and

$$W_{MA} W^A{}_N = V_{MN} ,$$

where we have used the metric to raise indices (in other words, the matrix W is the square root of V in the scalar product G). Using this expression for V_{MN} in terms of W_{MN}, we obtain the usual expression for the Hamiltonian of a system of coupled harmonic oscillators,

$$H=\tfrac{1}{2}G^{MN}(P_M-iW_{MA}q^A)^*(P_N-iW_{NB}q^B)+\tfrac{1}{2}\mathrm{tr}W\ , \tag{3}$$

where the first term is positive definite, as can be seen by going to a basis that diagonalizes G^{MN}, and the term with $\mathrm{tr}\,W$ is the zero-point energy.

Notice that $a_M^*=(P_M-iW_{MA}q^A)^*=(P_M+iW_{MA}q^A)$ and $a_M=(P_M-iW_{MA}q^A)$ are creation and annihilation operators, but they do not correspond to normal modes, since they satisfy the commutation relation

$$[a_M,a_N^*]=2W_{MN}\ . \tag{4}$$

We can use (3) to obtain the Schrödinger representation for the ground state. If we call $|\psi_0\rangle$ the ground state for the system of coupled harmonic oscillators, then, in order to minimize H, $|\psi_0\rangle$ must satisfy the following equation:

$$(P_M-iW_{MA}q^A)|\psi_0\rangle=0 \quad \text{for all } M\ . \tag{5}$$

But, in the Schrödinger representation, $P_M=-i\partial/\partial q^M$; therefore we have

$$\left[\frac{\partial}{\partial q^M}+W_{MB}q^B\right]\psi_0(\{q^A\})=0\ , \tag{6}$$

where $\{q^A\}$ denotes the collection of all q^A's, one for each oscillator. The solution of Eq. (6) is given by

$$\psi_0(\{q^A\})\equiv\langle\{q^A\}|\psi_0\rangle = \left[\det\frac{W}{\pi}\right]^{1/4}\exp(-\tfrac{1}{2}W_{AB}q^Aq^B)\ . \tag{7}$$

The density matrix for the vacuum state is given by $\rho=|\psi_0\rangle\langle\psi_0|$. In the Schrödinger representation this density matrix is

$$\begin{aligned}\rho(\{q^A\},\{q'^B\})&\equiv\langle\{q^A\}|\psi_0\rangle\langle\psi_0|\{q'^B\}\rangle\\ &=\psi_0(\{q^A\})\psi_0(\{q'^B\})^*\\ &=\left[\det\frac{W}{\pi}\right]^{1/2}\\ &\quad\times\exp[-\tfrac{1}{2}W_{AB}(q^Aq^B+q'^Aq'^B)]\ .\end{aligned}$$

Now consider a region Ω of $\mathbb{R}^3$. The oscillators in this region will be specified by greek letters, so that, e.g., the displacement of the αth oscillator will be q^α. If we consider the information on the displacement of the oscillators inside Ω as unavailable, we can obtain a reduced density matrix $\rho_{\rm red}$ for the oscillators outside Ω, integrating out over $\mathbb{R}$ for each of the oscillators in the region Ω. If lower case latin letters denote oscillators in the complement of Ω, then we have

$$\begin{aligned}\rho_{\rm red}(\{q^a\},\{q'^b\})&\equiv\langle\{q^a\}|\rho_{\rm red}|(q'^b\}\rangle\\ &=\int\prod_\alpha dq^\alpha\langle\{q^a,q^\alpha\}|\rho|\{q'^b,q^\alpha\}\rangle\\ &=\left[\det\frac{W_{AB}}{\pi}\right]^{1/2}\exp[-\tfrac{1}{2}W_{ab}(q^aq^b+q'^aq'^b)]\int\prod_\alpha dq^\alpha\exp[-W_{\alpha\beta}q^\alpha q^\beta-W_{a\alpha}(q^a+q'^a)q^\alpha]\ ,\end{aligned} \tag{8}$$

where we have set

$$W_{AB}=\begin{bmatrix}W_{ab} & W_{a\alpha}\\ W_{ab} & W_{\alpha\beta}\end{bmatrix}.$$

We will also use

$$W^{AB}=\begin{bmatrix}W^{ab} & W^{a\alpha}\\ W^{ab} & W^{\alpha\beta}\end{bmatrix}$$

for the inverse of W_{AB} (W^{AB} is *not* obtained by raising indices with G^{AB}), and the following notation for the various square W matrices: $\widetilde{W}^{ab}$ is the inverse of W_{ab}; $\widetilde{W}_{ab}$ is the inverse of W^{ab}; $\widetilde{W}^{\alpha\beta}$ is the inverse of $W_{\alpha\beta}$; and $\widetilde{W}_{\alpha\beta}$ is the inverse of $W^{\alpha\beta}$. Completing the squares and integrating out in (8), we have

$$\rho_{\rm red}(\{q^a\},\{q'^b\})=[\det(\widetilde{W}_{ab}/\pi)]^{1/2}\exp[-\tfrac{1}{2}W_{ab}(q^aq^b+q'^aq'^b)]\exp[\tfrac{1}{4}\widetilde{W}^{\alpha\beta}W_{a\alpha}W_{\beta b}(q+q')^a(q+q')^b]\ , \tag{9}$$

where we have used the identity[19]

$$\det W_{AB}\equiv\det\widetilde{W}_{ab}\det W_{\alpha\beta}\ .$$

Notice that, in Eq. (8), only the last term in the exponent under the integral couples oscillators inside Ω to oscillators outside Ω. It is this term which, on integration, contributes to the formation of the mixed state from the pure state.

To rewrite the density matrix in a convenient form, it is useful to prove the following identity:

$$\widetilde{W}_{ab}\equiv W_{ab}-W_{a\alpha}\widetilde{W}^{\alpha\beta}W_{\beta b}\ . \tag{10}$$

We just need to show that the right-hand side is the inverse of W^{ab}:

$$W^{ca}(W_{ab}-W_{a\alpha}\widetilde{W}^{\alpha\beta}W_{\beta b})=W^{cA}W_{Ab}-W^{c\gamma}W_{\gamma b}-(W^{cA}W_{A\alpha}-W^{c\gamma}W_{\gamma\alpha})\widetilde{W}^{\alpha\beta}W_{\beta b}=\delta^c{}_b-W^{c\gamma}W_{\gamma b}+W^{c\gamma}\delta_\gamma{}^\beta W_{\beta b}=\delta^c{}_b\ .$$

Then, defining

$$M_{ab}\equiv\widetilde{W}_{ab}\quad\text{and}\quad N_{ab}\equiv W_{a\alpha}\widetilde{W}^{\alpha\beta}W_{\beta b}\ ,$$

substituting these in (9), and using the identity (10), we obtain

$$\rho_{\rm red}(\{q^a\},\{q'^b\})=\left[\det\frac{M_{ab}}{\pi}\right]^{1/2}\exp[-\tfrac{1}{2}M_{ab}(q^aq^b+q'^aq'^b)]\exp[-\tfrac{1}{4}N_{ab}(q-q')^a(q-q')^b]\ . \tag{11}$$

We want to find now the entropy of such a density matrix. To this end we will first study the entropy of a Gaussian density matrix obtained for a simpler system of two oscillators, and then extend this to a general Gaussian density matrix.

A. Entropy of a Gaussian density matrix: one degree of freedom

Consider a system of two oscillators, each with one degree of freedom. Let a and b be the annihilation operators for the two oscillators. Consider the state vector

$$\begin{aligned}|\psi\rangle&=Ce^{\gamma a^*b^*}|0\rangle_a\otimes|0\rangle_b\\&=C\sum_{n=0}^{\infty}\gamma^n|n\rangle_a\otimes|n\rangle_b\ ,\end{aligned} \tag{12}$$

where γ is some real number and the normalization constant $C=(1-\gamma^2)^{1/2}$. Forming $\rho\equiv|\psi\rangle\langle\psi|$ and tracing out over oscillator b we get for the reduced density matrix, the "Gibbsian" state

$$\begin{aligned}\rho_{\rm red}&=\sum_{m=0}^{\infty}{}_b\langle m|\psi\rangle\langle\psi|m\rangle_b\\&=\sum_{m=0}^{\infty}C^2\gamma^{2m}|m\rangle_a\langle m|\ .\end{aligned} \tag{13}$$

The entropy associated with this density matrix is given by

$$\begin{aligned}S&=-\mathrm{tr}(\rho_{\rm red}\ln\rho_{\rm red})\\&=-\ln(1-\gamma^2)-\frac{\gamma^2}{1-\gamma^2}\ln\gamma^2\ .\end{aligned} \tag{14}$$

In order to relate this entropy to that of a Gaussian density matrix, we express (13) in the Schrödinger representation. From (12), and using straightforward algebra,

$$a|\psi\rangle=\gamma b^*|\psi\rangle,\quad b|\psi\rangle=\gamma a^*|\psi\rangle\ , \tag{15}$$

which just expresses the fact that $|\psi\rangle$ is obtained from the vacuum by a Bogoliubov transformation. Let x and y be the displacement of the two oscillators from equilibrium. Let p and q be the corresponding conjugate momenta. Then the creation and annihilation operators for the two oscillators can be written (in suitable units) as

$$\begin{aligned}&a=2^{-1/2}(p-ix),\quad a^*=2^{-1/2}(p+ix)\ ,\\&b=2^{-1/2}(q-iy),\quad b^*=2^{-1/2}(q+iy)\ .\end{aligned} \tag{16}$$

Thus, from (15) and (16),

$$\begin{aligned}&[(p-\gamma q)-i(x+\gamma y)]|\psi\rangle=0\ ,\\&[(q-\gamma p)-i(y+\gamma x)]|\psi\rangle=0\ .\end{aligned} \tag{17}$$

If we now change variables to $u\equiv x+\gamma y$ and $v\equiv y+\gamma x$, with

$$P_u=(1-\gamma^2)^{-1}(p-\gamma q)$$

and

$$P_v=(1-\gamma^2)^{-1}(q-\gamma p)\ ,$$

we can rewrite (17) as

$$\begin{aligned}&[(1-\gamma^2)P_u-iu]|\psi\rangle=0\ ,\\&[(1-\gamma^2)P_v-iv]|\psi\rangle=0\ .\end{aligned} \tag{18}$$

In the Schrödinger representation, (18) becomes a pair of differential equations. Solving these, we obtain

$$\psi(u,v)=K\exp\left[-\frac{1}{2}\frac{1}{(1-\gamma^2)}(u^2+v^2)\right]\ , \tag{19}$$

where K is some constant, and

$$\begin{aligned}u^2+v^2&=(x+\gamma y)^2+(y+\gamma x)^2\\&=(x^2+y^2)(1+\gamma^2)+4\gamma xy\ .\end{aligned}$$

Hence,

$$\psi(x,y)=K\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+y^2}{2}-\frac{2\gamma}{1-\gamma^2}xy\right]\ . \tag{20}$$

Notice that this $\psi(x,y)$ is of the form (7). From (20), forming

$$\rho[(x,y),(x',y')]=\psi(x,y)\psi^*(x',y')\ ,$$

and tracing out over the second oscillator, we have

$$\begin{aligned}\rho_{\rm red}(x,x')&=\int dy\,K^2\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+y^2}{2}-\frac{2\gamma}{1-\gamma^2}xy-\frac{1+\gamma^2}{1-\gamma^2}\frac{x'^2+y^2}{2}-\frac{2\gamma}{1-\gamma^2}x'y\right]\\&=K^2\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}\right]\left[\frac{\pi(1-\gamma^2)}{1+\gamma^2}\right]^{1/2}\exp\left[\frac{\gamma^2(x+x')^2}{(1-\gamma^2)(1+\gamma^2)}\right]\ .\end{aligned} \tag{21}$$

Using the normalization of $\psi(x,y)$ to determine the value of K, we have

$$\rho_{\rm red}(x,x')=\left[\frac{1-\gamma^2}{\pi(1+\gamma^2)}\right]^{1/2}\exp\left[-\frac{1+\gamma^2}{1-\gamma^2}\frac{x^2+x'^2}{2}+\frac{\gamma^2}{1-\gamma^4}(x+x')^2\right].$$

If we now let

$$\mu\equiv\gamma^2,\quad M\equiv\frac{1-\mu}{1+\mu},\quad\text{and}\quad N\equiv\frac{4\mu}{1-\mu^2}\ , \tag{22}$$

we have

$$\rho_{\rm red}(x,x')=\left[\frac{M}{\pi}\right]^{1/2}\exp[-\tfrac{1}{2}M(x^2+x'^2)-\tfrac{1}{4}N(x-x')^2]\ . \tag{23}$$

This is a Gaussian density matrix of the form (11). We have thus obtained the result that the entropy of the one-parameter family of Gaussian density matrices for one degree of freedom (23), where M and N depend on the parameter γ, is the expression (14), or, in terms of μ,

$$S=-\frac{\mu\ln\mu+(1-\mu)\ln(1-\mu)}{1-\mu}\ . \tag{24}$$

Notice that Eqs. (22), with the condition $\mu<1$, can be inverted to get

$$\mu=1+2M/N-2[M/N(1+M/N)]^{1/2}\ .$$

We will need, however, the entropy of (23) for arbitrary M and N. To see how this is related to the result above, consider a density matrix of the same form (23), but with M and N freely specified. The entropy S, being dimensionless, can depend only on the ratio

$$\lambda\equiv N/M$$

of the dimensional parameters in the density matrix. But this means that, if we define

$$\mu\equiv1+2\lambda^{-1}-2[\lambda^{-1}(1+\lambda^{-1})]^{1/2}\ , \tag{25}$$

we can generalize our result to state that the entropy of any density matrix of the form (23), with M and N freely specified, and μ given by (25), is the expression (24). In particular, for comparison with what follows, this is the entropy of the density matrix[20]

$$\rho_{\rm red}(x,x')=\pi^{-1/2}\exp[-\tfrac{1}{2}(x^2+x'^2)-\tfrac{1}{4}\lambda(x-x')^2]\ . \tag{26}$$

B. Entropy of a Gaussian density matrix: general case

In the more general case of our model for the scalar field, the direct generalization of Eq. (23) is (11). In order to calculate its entropy, we want to write it in a form to which we can apply the previous result, namely, as a product of density matrices of the form (26). To achieve this, we will construct a basis (not necessarily orthonormal) in which both M and N are represented by diagonal matrices.[21] To that end, consider M itself as a metric on configuration space, and choose as a basis a complete orthonormal set of vectors with respect to M. This fixes the basis up to an M-orthogonal transformation, which can be used to diagonalize any other symmetric tensor, and in particular N_{ab}. If we do this we can rewrite (11) as

$$\rho_{\rm red}(\{q^a\},\{q'^b\})=\prod_n\{\pi^{-1/2}\exp[-\tfrac{1}{2}(q_nq^n+q'_nq'^n)-\tfrac{1}{4}\lambda_n(q-q')_n(q-q')^n]\}$$

(just for this equation, we are not using summation conventions), where the λ_n's are the diagonal elements of N_{cb} in our basis. Obviously, these diagonal elements are the eigenvalues of the operator

$$\Lambda^a{}_b\equiv(M^{-1})^{ac}N_{cb}\ ,$$

and, as such, they can be calculated in any basis. We now have $\rho_{\rm red}$ in the form

$$\rho_{\rm red}=\bigotimes_n\rho(\lambda_n)\ ,$$

which means that the entropy is given by

$$S=\sum_n S[\rho(\lambda_n)]\ . \tag{27}$$

a sum of terms corresponding to the different eigenvalues of Λ, each one giving a contribution of the form (24) seen for the two oscillator case.

Summarizing, we have thus shown the following.

Theorem. The entropy associated with a Gaussian density matrix of the form (11) for a system with many degrees of freedom is given by

$$S=-\sum_n\frac{\mu_n\ln\mu_n+(1-\mu_n)\ln(1-\mu_n)}{1-\mu_n}\ ,$$

where μ_n is the unique positive solution of $\lambda_n=4\mu_n/(1-\mu_n)^2$, given by (25), and $\{\lambda_n\}$ are the eigenvalues of $\Lambda^a{}_b\equiv(M^{-1})^{ac}N_{cb}$. The relationship between S and the λ_n's can also be expressed directly as

$$S=\sum_n\{\ln(\tfrac{1}{2}\lambda_n{}^{1/2})+(1+\lambda_n)^{1/2}\ln[(1+\lambda_n{}^{-1})^{1/2}+\lambda_n{}^{-1/2}]\}\ .$$

We now give some further identities satisfied by the various W's. From $W^{aA}W_{A\alpha}=0$ and $W_{aA}W^{A\alpha}=0$ we have, respectively,

$$W^{ac}W_{ca} = -W^{a\gamma}W_{\gamma a} \quad \text{and} \quad W_{ac}W^{ca} = -W_{a\gamma}W^{\gamma a} , \tag{28}$$

[and, if W_{AB} were not a symmetric tensor, we could have written down similar but inequivalent identities by interchanging latin and greek indices in (28)]. From these, one can easily derive several identities of the form

$$W^{a\alpha} = -W^{ac}W_{c\gamma}\widetilde{W}^{\gamma\alpha} .$$

For later use, notice that, using the definitions of the tensors M and N, and the first identity in (28), we can write equivalently Λ as

$$\Lambda^a{}_b = -W^{a\alpha}W_{\alpha b} , \tag{29}$$

which shows clearly how Λ depends on the choice of unavailable oscillators. Also, all of its eigenvalues can be seen to be positive, by the following argument. If we consider W_{ab} as a metric for the configuration space of the oscillators outside Ω, we can use it to lower the index a in $\Lambda^a{}_b$, which gives

$$W_{ab}\Lambda^m{}_b = -W_{am}W^{m\alpha}W_{\alpha b} = W_{a\gamma}W^{\gamma\alpha}W_{\alpha b'} ,$$

where we have used the second identity in (28). But, since W_{AB}, and hence $W^{\alpha\beta}$, are positive definite, this shows explicitly that $\Lambda^a{}_b$ is a positive (semidefinite) operator, and that its eigensubspaces are orthogonal in the metric W_{ab}.

What we have to do now is to write down the form of the operator $\Lambda^a{}_b$, taking into account the dynamics of the specific field we want to study and the specific set of oscillators that will be ignored, determine its eigenvalues, and, using the theorem above, its entropy.

III. THE KLEIN-GORDON CONTINUUM LIMIT

Consider now a real scalar field satisfying the Klein-Gordon equation, and the problem of calculating the entropy associated with this field in the presence of a black hole. As discussed earlier, the latter will be simulated as some region Ω of flat space. The entropy will be calculated with the procedure described in the preceding section, but, for convenience, we will take the continuum limit of that formalism. In this limit, we have

$$\tfrac{1}{2}V_{AB}q^Aq^B \to \tfrac{1}{2}\langle\phi\,|\,\nabla^2+m^2\,|\,\phi\rangle = \int[\tfrac{1}{2}(\nabla\phi)^2+\tfrac{1}{2}m^2\phi^2]d^3x . \tag{30}$$

Our goal is to construct the operator corresponding to the $\Lambda^a{}_b$ of the preceding section, evaluate its eigenvalues, and calculate the entropy (27) for some appropriate region Ω. We start with the momentum representation of the operators involved. From

$$V(x,y) = \int \frac{d^3k}{(2\pi)^3}(k^2+m^2)e^{ik\cdot(x-y)} , \tag{31}$$

we get

$$W(x,y) = \int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{1/2}e^{ik\cdot(x-y)} \tag{32}$$

and

$$W^{-1}(x,y) = \int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{-1/2}e^{ik\cdot(x-y)} , \tag{33}$$

where continuous indices over $\mathbb{R}^3$ have replaced the matrix indices, and the above expressions should be thought of as (kernels of) integral operators. These integrals are well defined as Fourier transforms of distributions. In particular, e.g., for the massless case, $W(x,y)$ is proportional to the so-called "finite part" of r^{-4}, where $r = ||x-y||$ consistently with a dimensional analysis of (32). From (29), the operator Λ is obtained as a sum over the oscillators in the region Ω:

$$\Lambda(x,y) = -\int_\Omega d^3z\, W^{-1}(x,z)W(z,y) , \tag{34}$$

i.e.,

$$\Lambda(x,y) = -\int_\Omega d^3z \left[\int \frac{d^3k}{(2\pi)^3}(k^2+m^2)^{-1/2}e^{ik\cdot(x-z)}\right]\left[\int \frac{d^3p}{(2\pi)^3}(p^2+m^2)^{1/2}e^{ip\cdot(z-y)}\right] . \tag{35}$$

We now have to solve the eigenvalue equation

$$\int d^3y\, \Lambda(x,y)f(y) = \lambda f(x) , \tag{36}$$

and use the eigenvalues in the expression for the entropy (27).

A. The need for a cutoff

Consider first the case $m=0$. Since the entropy is a dimensionless quantity, S has to be invariant under a rescaling of the region Ω, and the only answer we can expect to get for it is ∞ (unless it vanishes). In the $m\neq 0$ case, S could be a function of mR, where R is some characteristic size of Ω. If the entropy was not infinite, we would expect it to vanish in the limit as the size $R\to 0$, but this limit is equivalent to $m\to 0$, which gives an infinite entropy from the above argument. We therefore get again $S=\infty$. Physically, thus, the divergence of the entropy seems to be of ultraviolet origin, since it is not removed by a nonvanishing mass: it comes about from including modes of arbitrarily small wavelength. But the only way for entropy to be physically meaningful is for it to be finite. This means that there has to be a fundamental length in the theory. We will therefore introduce a new dimensional parameter l, that will act as a cutoff. Now the entropy can be a function of R/l.

The question that remains, however, is that of how to take into account this cutoff in our calculations, without knowing exactly how it arises physically. Some possibilities are the following.

(1) We do the whole problem on a lattice of spacing l. This would be the most physically reasonable option, but we would lose the calculational advantages of the continuum. (2) We use a momentum cutoff l^{-1}: integrals like those in (31)—(33) have a finite domain of integration. This seems to make sense physically, but it has some drawbacks related to properties of the operators we want to use, like the sign of their eigenvalues. (3) we use a position cutoff near the boundary of Ω: the integration over z in (35) is restricted to points at a distance l at least from the boundary. In this way we do not allow correlations between points inside and outside Ω whose distance is less than l. Notice also that the high-frequency modes that we expect to contribute most to S are localized near the boundary.

In the calculations that follow, as we will see, we assume our geometry to have certain symmetries, that allow us to reduce the three-dimensional problem to an effective one-dimensional one. In this reduced problem we will use the third possibility listed above for the cutoff implementation. As for the value of the cutoff, we can think of it as being of the order of the Planck length. It certainly has to be much smaller than the size R of the region Ω and the radius of curvature of spacetime.

B. The half-space case

To begin with, we will consider the simple example in which the unavailable region Ω is a whole half-space in $\mathbb{R}^3$. Then the total entropy we will compute will be infinite, but we are interested in showing that the area of the boundary surface factors out, and calculating the entropy per unit area. The latter result will agree to a very good approximation with a more realistic calculation taking into account the actual shape of a black-hole horizon and spacetime curvature. More precisely, in our example, for dimensional reasons, we will obtain, e.g., for the $m=0$ case, an entropy per unit area

$$S/A=Cl^{-2} ,$$

where C is some constant and l the cutoff. If the model had included other dimensional parameters, like the mass m or a characteristic size R of Ω, we would expect to obtain

$$S/A=C(R/l,ml)l^{-2} ,$$

where $C(x,y)$ is a slowly varying function of its arguments.

Consider then a scalar field initially in the vacuum state in $\mathbb{R}^3$, and carry out the tracing-out procedure for the region $\Omega\equiv\{x \mid x_2<0\}$. If we decompose all three-vectors v into their component normal to the boundary, $v_\perp$ and a two-vector along the boundary, $v_{\|}$, Eqs. (32) and (33) become, respectively,

$$W(x,y)=\int\frac{dk_\perp}{2\pi}\int\frac{d^2k_{\|}}{(2\pi)^2}({k_\perp}^2+{k_{\|}}^2+m^2)^{1/2}e^{ik_\perp(x_\perp-y_\perp)}e^{ik_{\|}\cdot(x_{\|}-y_{\|})} \tag{37}$$

and

$$W^{-1}(x,y)=\int\frac{dk_\perp}{2\pi}\int\frac{d^2k_{\|}}{(2\pi)^2}({k_\perp}^2+{k_{\|}}^2+m^2)^{-1/2}e^{ik_\perp(x_\perp-y_\perp)}e^{ik_{\|}\cdot(x_{\|}-y_{\|})} . \tag{38}$$

Next, if we make the ansatz

$$f(x)=e^{iv_{\|}\cdot x_{\|}}f(x_\perp) \tag{39}$$

for the eigenfunctions of Λ, the eigenvalue equation (36) reduces to

$$\lambda f(x_\perp)=-\int_0^\infty dy_\perp\left[\int_{-\infty}^{-l}dz_\perp\left[\int_{-\infty}^{\infty}\frac{dk_\perp}{2\pi}({k_\perp}^2+{v_{\|}}^2+m^2)^{-1/2}e^{ik_\perp(x_\perp-z_\perp)}\right]\right.$$
$$\left.\times\left[\int_{-\infty}^{\infty}\frac{dp_\perp}{2\pi}({p_\perp}^2+{v_{\|}}^2+m^2)^{1/2}e^{ip_\perp(z_\perp-y_\perp)}\right]\right]f(y_\perp) , \tag{40}$$

where the cutoff l has been introduced, and the integration along the boundary has been performed. Equation (40) gives, for each $v_{\|}$, an effective one-dimensional problem for a massive scalar field with effective mass

$$m_e=({v_{\|}}^2+m^2)^{1/2} . \tag{41}$$

The solution of this one-dimensional problem obtained for a fixed $v_{\|}$ gives the spectrum of eigenvalues $\{\lambda_n(m_el), n\in Z\}$ of the corresponding integral operator. Each λ_n gives a contribution to the entropy that, from the theorem proved in Sec. II, can be written as

$$\sigma_n(m_el)=-\frac{\mu_n\ln\mu_n+(1-\mu_n)\ln(1-\mu_n)}{1-\mu_n} , \tag{42}$$

where

$$\mu_n(m_el)=1+\frac{2}{\lambda_n}-2\left[\frac{1}{\lambda_n}\left[1+\frac{1}{\lambda_n}\right]\right]^{1/2} . \tag{43}$$

Summing over the contributions of all eigenvalues we get

$$\sigma(m_el)=\sum_{n=0}^{\infty}\sigma_n(m_el) , \tag{44}$$

and the total entropy is obtained by integrating over all $\nu_{\|}$, using the fact that, for a surface area A in configuration space, the density of modes in momentum space is $A/(2\pi)^2$:

$$\begin{aligned} S &= \frac{A}{(2\pi)^2}\int d^2\nu_{\|}\sigma(m_e l) \\ &= \frac{A}{2\pi}\int_{R^{-1}}^{l^{-1}} dk\, k\sigma[(k^2+m^2)^{1/2}l] \\ &= \frac{A}{2\pi l^2}\int_{l(R^{-2}+m^2)^{1/2}}^{(1+l^2m^2)^{1/2}} d\xi\,\xi\sigma(\xi)\ . \end{aligned} \tag{45}$$

In (45) we have introduced two cutoffs in the momentum along the boundary of Ω. For the ultraviolet cutoff we used possibility (2) mentioned above. It is physically reasonable to use this cutoff, although mathematically the integral would have been well defined even for an infinite upper limit of integration. As regards the infrared cutoff, we would expect that it also is not needed for the integral to be finite [even though $\sigma(0)$ is infinite], since otherwise the coefficient $C(R/l,ml)$ would not be a slowly varying function of its arguments, as required physically when $ml\to 0$, $R/l\to\infty$. Nevertheless in the physical situations we want to approximate, this cutoff enters in a natural way because of the finite size of the region Ω.

C. Finiteness and estimate of the entropy

What we want to show now is that this expression for the entropy is finite. First we will establish finiteness for $\sigma(\xi)$. For an integral operator like ours, the eigenvalues accumulate at 0 (see the Appendix). Thus, from

$$\mu_n \sim \lambda_n/4 \quad \text{for } \lambda_n\to 0\ ,$$

and

$$\sigma_n \sim -\mu_n\ln\mu_n \quad \text{for } \mu_n\to 0\ ,$$

we see that $\sigma(\xi)$ is finite if $\sum_n \lambda_n\ln\lambda_n$ is finite. In order to find the fall-off behavior of λ_n, we have to study the properties of the integral operator in (40). This equation can be written (changing $z\to -z$, which takes Ω into its complement) as

$$\pi^{-2}\int_0^\infty dy\left[\int_\epsilon^\infty dz\, K_0(x+z)\frac{K_1(z+y)}{(z+y)}\right]f(y)=\lambda f(x)\ , \tag{46}$$

where K_0 and K_1 are the modified Bessel functions of zeroth and first order, respectively; we have renamed $(m_e x, m_e y, m_e z)\mapsto(x,y,z)$ to obtain dimensionless variables, and $\epsilon\equiv m_e l$. In operator terms, Λ is a convolution, $\Lambda=\Lambda_0 * \Lambda_1$, or its kernel is a composite kernel. More precisely, it can be shown (see the Appendix) that Λ has a kernel in $\mathscr{L}^2([0,\infty)\times[0,\infty))$, convolution of two kernels in C_1, which are continuously differentiable and Hermitian. From these properties it follows that its eigenvalues satisfy Eq. (A6):

$$\sum_{n=0}^{\infty}\lambda_n^{1/2}<\infty\ , \tag{47}$$

which obviously bounds $\sum_n \lambda_n\ln\lambda_n$. This shows that $\sigma(\xi)$ is finite.

For the entropy to be finite, the integral in (45) must also converge. Since we know that $\sigma(\xi)$ blows up for $\xi\to 0$, this means that, unless we introduce, at least in the massless case, the lower momentum cutoff in (45), we need a fall-off like $\sigma(\xi)=O(\xi^{-2+\delta})$ as $\xi\to 0$, for some $\delta>0$.

An upper bound on the entropy could be obtained from an estimate of $\sigma(\xi)$. We have

$$\sigma(\xi)\equiv\sum_n \sigma_n(\xi)\le c\sum_n \lambda_n{}^{1/2}\ ,$$

for some constant $c\simeq 0.56$, as can be estimated numerically by using (42) and (43). But then, if $\{\nu_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are the singular values of Λ, Λ_0, and Λ_1, respectively,

$$\begin{aligned} \sigma(\xi) &\le c\sum_n \nu_n{}^{1/2}\le c\sum_n(\alpha_n{}^{1/2}\beta_n{}^{1/2}) \\ &\le c\left[\sum_n\alpha_n\right]^{1/2}\left[\sum_m\beta_m\right]^{1/2} \\ &= c(\mathrm{tr}\,|\Lambda_0|)^{1/2}(\mathrm{tr}\,|\Lambda_1|)^{1/2}\ , \end{aligned} \tag{48}$$

where the first inequality comes from (A5) in the Appendix, with $p=\frac{1}{2}$, the second one from (A7), with $p=\frac{1}{2}$, the third one is Holder's inequality, and the fourth one is the definition of the trace of an operator. The problem of giving an upper bound to $\sigma(\xi)$ is thus reduced to that of finding the trace of two operators (this would be very simple—numerically—if it was known that Λ_0 and Λ_1 are positive operators).

Although we have not been able so far to obtain a numerical value for the upper bound on the entropy using analytical methods, work is currently in progress on numerical calculations that will give us an estimate of the value of $\sigma(\xi)$, and an indication on its behavior for small ξ. This would also tell us how crucial it is to introduce the lower momentum cutoff in the integration over ξ for the massless case.

IV. CONCLUDING REMARKS

The quantity S defined in the previous sections does not claim to be the sole contribution to the entropy of a black hole. One of the drawbacks it has is that it depends on the number of fields present, although this could be cured by the presence of large couplings at high energies between fields, or between the fields and the shape of the horizon (back reaction). What we propose is a framework in which the contribution to the black-hole entropy from nongravitational degrees of freedom (including gravitons) can in principle be calculated.

But one thing that should be shown, if we are to call it entropy, is that $-\mathrm{tr}(\rho_{\mathrm{red}}\ln\rho_{\mathrm{red}})$ is a nondecreasing function of time. This can be seen as follows.

The external region of the black hole (the complement of Ω, in our notation), is an autonomous region, whose evolution does not depend on the interior (this is obviously true if there is a clear distinction between these two regions, but we expect our argument to hold even in the presence of quantum gravitational effects like the blurring of the light cones), and whose energy we expect to be conserved since it can be defined in terms of observables in the asymptotic region far from the hole. Consider the state (density matrix) which maximizes the entropy for a given value of the energy. This density matrix is the microcanonical ensemble, a function of the energy only, and is thus conserved. But it is known classically, and is also true quantum mechanically,[22] that, if the maximum entropy state is conserved, then for any other state the entropy will always increase.

To apply the framework we have proposed to a more realistic situation, one would first try to use, for the region Ω, not the half-space $z<0$, but a sphere $(x^2+y^2+z^2)^{1/2}<R$. To reduce this case to an equivalent one-dimensional problem, in analogy with our calculation, one would make the ansatz, for the eigenfunctions of the operator Λ,

$$f(x)=Y_{lm}(\theta,\phi)f(r) .$$

In this case, though, the technical difficulties involved in the calculation become greater. To use some of the theorems on integral operators that we have made use of, one would like to write Λ as a convolution of two operators. This can be done by suitably changing coordinates so that the interior of Ω becomes characterized by the same range of values of the new radial coordinate as the exterior, but the resulting operators are not as simple as for the half-space case.

Finally, it would be useful to extend the whole formalism to the case of a curved background.

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APPENDIX: PROPERTIES OF THE SPECTRUM OF THE INTEGRAL OPERATOR Λ

In this section we want to prove that the integral equation (46) has a discrete spectrum with eigenvalues converging sufficiently fast to zero.

The equation (redefining for simplicity the eigenvalues λ to absorb a factor π^2) is

$$\int_0^\infty dy \int_\epsilon^\infty dz\, K_0(x+z)\frac{K_1(z+y)}{z+y}f(y)=\lambda f(x) . \qquad \text{(A1)}$$

Our kernel is therefore

$$\begin{aligned}\Lambda(x,y)&=\int_\epsilon^\infty dz\, K_0(x+z)\frac{K_1(z+y)}{z+y}\\ &=\int_0^\infty dz\, K_0(x+z+\epsilon)\frac{K_1(z+y+\epsilon)}{z+y+\epsilon}\\ &=\int_0^\infty dz\, \Lambda_0(x,z)\Lambda_1(z,y) ,\end{aligned} \qquad \text{(A2)}$$

where

$$\Lambda_0(x,y)\equiv K_0(x+y+\epsilon) , \qquad \text{(A3)}$$

$$\Lambda_1(x,y)\equiv\frac{K_1(x+y+\epsilon)}{x+y+\epsilon} . \qquad \text{(A4)}$$

The kernel $\Lambda(x,y)$ is thus composite and the operator Λ a convolution, so we can write formally the eigenvalue equation as

$$\Lambda f=\Lambda_0 * \Lambda_1 f=\lambda f .$$

The questions we wish to address are the following. (a) What is the character of the spectrum of the integral operator Λ (i.e., are the eigenvalues discrete or continuous)? (b) If they are discrete, does the sequence of eigenvalues converge to zero or not? (c) If it converges to zero, what is the rate of convergence? Specifically, does the series $\sum_n \lambda_n \ln\lambda_n$ converge?

We recall here some nomenclature that will be used in this analysis. We give the definitions for integral operators in the interval $[0,\infty)$, but their extension to other cases is straightforward.[23]

A useful notion in dealing with non-self-adjoint operators, like our Λ, is that of singular value. Given an operator K, its singular values are defined to be the square roots of the eigenvalues of K^*K (or KK^*, since these two operators have the same spectrum). The singular values are therefore positive by definition, and, if $K=K^*$, they are the absolute values of its eigenvalues.

The kernel $K(x,y)$ of an integral operator K is said to be in $\mathscr{L}^2$ if

$$||K||_2{}^2\equiv \int_0^\infty dx \int_0^\infty dy\, |K(x,y)|^2<\infty .$$

The operator K is said to be of class C_r if its singular values $\{\nu_n\}$ satisfy

$$\sum_n \nu_n{}^r<\infty .$$

In the special case $r=1$, the operator is called nuclear. It turns out that a kernel $K(x,y)$ is nuclear if it is the composite of two $\mathscr{L}^2$ kernels K_1 and K_2 (Ref. 24), i.e.,

$$K(x,y)=\int dz\, K_1(x,z)K_2(z,y) .$$

Then the following facts are true.

(1) $\Lambda(x,y)$ is an $\mathscr{L}^2$ kernel, when $\epsilon>0$. For it is a composite kernel of two $\mathscr{L}^2$ kernels, Λ_0 and Λ_1 (Ref. 25) (the set of all square-integrable kernels forms a complex vector space, which is closed under composition). We now check that Λ_0 and Λ_1 are $\mathscr{L}^2$. If we define $u\equiv x+y$ and $v\equiv x-y$, then

$$\begin{aligned}||\Lambda_0||_2{}^2&=\int_0^\infty dx\int_0^\infty dy\, |\Lambda_0(x,y)|^2\\ &=\tfrac{1}{2}\int_0^\infty du\int_{-u}^{u} dv\, |K_0(u+\epsilon)|^2\\ &=\int_0^\infty du\, u\, |K_0(u+\epsilon)|^2 ,\end{aligned}$$

which is finite, since $K_0(u)\sim\exp(-u)$ for $u\to\infty$, and $K_0(u)\sim-\frac{1}{2}\ln u$ for $u\to 0$; and

$$||\Lambda_1||_2{}^2 = \int_0^\infty dx \int_0^\infty dy\, |\Lambda_1(x,y)|^2$$
$$= \tfrac{1}{2}\int_0^\infty du \int_{-u}^{u} dv\, u^{-2} |K_1(u+\epsilon)|^2$$
$$= \int_0^\infty du\, u^{-1} |K_1(u+\epsilon)|^2 ,$$

which is finite, but only for $\epsilon > 0$, since $K_1(u) \sim \exp(-u)$ for $u \to \infty$, and $k_1(u) \sim u^{-1}$ for $u \to 0$. From the fact that $\Lambda \in \mathscr{L}^2$, we have[23] that (for $\epsilon > 0$) the spectrum of Λ is discrete and accumulates at 0, which answers questions (a) and (b).

(2) $\Lambda(x,y)$ is a nuclear kernel when $\epsilon > 0$. This follows immediately from the condition stated above for an $\mathscr{L}^2$ kernel to be nuclear, and from the fact that Λ, Λ_0, and Λ_1 are all $\mathscr{L}^2$ kernels. But from this we have that the singular values of Λ satisfy

$$\sum_{n=1}^{\infty} \nu_n < \infty .$$

We have also in general, from a theorem by Weyl,[26] that, for $0 < p < 2$,

$$\sum_{n=1}^{\infty} |\lambda_n|^p \le \sum_{n=1}^{\infty} \nu_n{}^p . \qquad \text{(A5)}$$

Hence the eigenvalues of Λ satisfy

$$\sum_{n=1}^{\infty} |\lambda_n| < \infty ,$$

but this does not yet settle question (c).

(3) $\Lambda(x,y)$ is a $C_{1/2}$ kernel. It can be shown that, if K_1 is C_p and K_2 is C_q, then the composite kernel $K = K_1 * K_2$ is C_r, where $1/r = 1/p + 1/q$ (Ref. 27). Hence, all we have to show is that Λ_0 and Λ_1 are both C_1. But, from another theorem by Weyl,[28] we know that, if K is an $\mathscr{L}^2$, continuously differentiable Hermitian kernel, then its eigenvalues α_n satisfy

$$\lim_{n\to\infty} |\alpha_n| n^{3/2} = 0 .$$

Since both Λ_0 and Λ_1 satisfy these conditions, and for Hermitian operators the singular values are the absolute values of the eigenvalues, it follows that Λ_0 and Λ_1 are C_1.

(4) The eigenvalues of $\Lambda(x,y)$ satisfy

$$\sum_{n=1}^{\infty} \lambda_n{}^{1/2} < \infty , \qquad \text{(A6)}$$

from result (3), Eq. (A5), and the fact that the eigenvalues of Λ are positive. This answers question (c).

(5) The eigenvalues of Λ also satisfy the inequality

$$\sum_{n=1}^{\infty} (\lambda_n)^p < \sum_{n=1}^{\infty} (\alpha_n)^p (\beta_n)^p , \qquad \text{(A7)}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the singular values of Λ_0 and Λ_1, respectively, and $p > 0$ (Ref. 27).

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[1] See, e.g., R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[2] B. Carter, in *General Relativity,* edited by S. W. Hawking and W. Israel (Cambridge University Press, London, 1979).

[3] See, e.g., N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, London, 1984).

[4] J. Bekenstein and A. Meisels, Phys. Rev. D **15**, 2775 (1977); P. Panangaden and R. M. Wald, *ibid.* **16**, 929 (1977); R. D. Sorkin, Institute for Theoretical Physics, University of California, Report No. NSF-ITP-86-51 (unpublished).

[5] This increase of Ψ holds whenever the scattering preserves the thermal state of temperature $T_{\rm bh}$, as will be described elsewhere in more detail; see also the conclusion of this paper.

[6] W. G. Unruh and R. M. Wald, Phys. Rev. D **25**, 942 (1982); J. D. Bekenstein, *ibid.* **9**, 3293 (1974).

[7] S. W.Hawking, Commun. Math. Phys. **25**, 152 (1972).

[8] R. D. Sorkin, R. M. Wald, and Zhang Zhen Jin, Gen. Relativ. Gravit. **13**, 1127 (1981).

[9] R. M. Wald (private communication).

[10] Presumably this coupling is actually exponentially small but not strictly zero, due to the quantum "smearing of the horizon." If a black hole were somehow "the same thing as a white hole" (see Ref. 11) then this conclusion might change, but not obviously in such a way as to make a weak-coupling approximation any more viable.

[11] S. W. Hawking, Phys. Rev. D **13**, 191 (1976).

[12] Perhaps the Killing vector could still set up a kind of equivalence among successive hypersurfaces if one admitted hypersurfaces "ending" at the final singularity.

[13] R. D. Sorkin, in General Relativity and Gravitation, proceedings of the GR10 Conference, Padova, 1983, edited by B. Bertotti, F. de Felice, and A. Pascolini (Consiglio Nazionale delle Ricerche, Roma, 1983), Vol. 2.

[14] The one computation (see Ref. 15) of $S_{\rm bh}$ which may be called fully quantum mechanical also suggests that the entropy is found outside the horizon. Thus, the "Schwarzschild instanton" from which the entropy is there estimated manifestly corresponds under analytic continuation only to the exterior region of the corresponding Lorentzian black hole.

[15] G. W. Gibbons and S. W Hawking, Phys. Rev. D **15**, 2752 (1977).

[16] D. N. Page, Phys. Rev. D **12**, 3260 (1976).

[17] If computed in a "Rindler representation" (Rindler vacuum as fundamental state) this entropy would probably be describable as that carried by the fictitious "acceleration radiation" (see Ref. 6) surrounding the black hole. It may therefore also be closely related to the entropy described by Zurek and Thorne (see Ref. 18), but the relation is not clear to us, since they say they are omitting precisely the entropy of the Rindler particles.

[18] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. **54**, 2171 (1985).

[19] This identity can be derived from the following easily verifiable identity:

$$\begin{vmatrix} A & B \\ C & D \end{vmatrix} \equiv \begin{vmatrix} 1 & B \\ 0 & D \end{vmatrix} \begin{vmatrix} A - BD^{-1}C & 0 \\ D^{-1}C & 1 \end{vmatrix} ,$$

by taking its determinant.

[20] An equivalent way of deriving this result is to employ the scale transformations $x \mapsto Ax$, under which the entropy is invariant; in this case the correct transformation for the density matrix (necessary to preserve its normalization) is obtained by treating it as a scalar density—of weight 1— with M playing

the role of a metric. In the case treated below of a general Gaussian density matrix we will use more explicitly the fact that M can be regarded as a metric in the configuration space for the outside oscillators, and it can be shown that the weight-one transformation law for the density matrix is in fact necessary to guarantee that the entropy be well defined, independent of the choice of basis.

[21]We will merely be proving the general fact that any two symmetric tensors, M_{ab} and N_{ab}, one of which is positive definite, can be diagonalized simultaneously.

[22]R. D. Sorkin, Phys. Rev. Lett. **56**, 1885 (1986).

[23]See, e.g., J. A. Cochran, *The Analysis of Linear Integral Equations* (McGraw-Hill, New York, 1972) (notice that what we call eigenvalues and singular values are the inverses of what in the literature on integral operators are usually called eigenvalues and singular values—and of what Cochran calls characteristic values and singular values).

[24]Cochran (Ref. 23), p. 236.

[25]Cochran (Ref. 23), p. 11.

[26]Cochran (Ref. 23), p. 244.

[27]Cochran (Ref. 23), p. 248.

[28]Cochran (Ref. 23), p. 262.