Black Holes, Gravity Waves and String Theory

Thesis by

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In Partial Fulfillment of the Requirements
for the Degree of
Doctor of Philosophy

California Institute of Technology
Pasadena, California

1998
(Submitted May 26, 1998)

Acknowledgements

First I would like to thank my husband John Schwarz for his love, support and gentle spirit throughout the years I have been journeying with him through the strange and interesting land of physics. I also want to thank him very much for providing a role model of intellectual and moral courage in his career, by insisting on working on string theory despite the career setbacks that were involved before string theory was accepted by his peers.

I also want to thank my advisor Renata Kallosh for her faith in me and the deep respect she gives to the process of graduate education. I cannot imagine another person in physics with her combination of brilliance, hard work, emotional wisdom and caring for the people around her. I thank my friends Melissa Midzor, Rick Jenet and Pat Wrean for seeing me through some trying times with humor and activism. I want to extend a general appreciation of the new generation of graduate students who are beginning to assert themselves as consumers of graduate education. I also thank my friend and co-conspirator Amanda Peet for her relentless spirit and enthusiasm, which remind me why I embarked upon this wild journey through physics in the first place. My niece Barbara Tagatac has always been there for me and I thank her as well.

There are people in modern culture who have also brought me support and comfort during this process by being role models of courage and lovers of art and knowledge. Elaine Pagels, Evelyn Fox Keller, Patti Smith, Alice Paul, Gertrude Stein, Djuna Barnes, Marija Gimbutas, Jeanette Winterson and Courtney Love are just a few of the women of my culture who have helped build a world that supports the odd and unconventional women of this country and our odd and unconventional dreams. When we don't fit the world, somehow we help the world change to fit us. I also have to thank the great Madame Cliquot for her invention of remouage, without which there would be much less civilization in the world today.

I must acknowledge the powerful impact on my life made by Paul Ginsparg and Tim Berners-Lee, for freeing physics from the bounds of the printed page, making it possible for us to share our thoughts and ideas with people all over the world at nearly the speed of light. We are evolving into a relativistic culture. Information has never travelled so fast before across this planet of ours. I am indebted to this new technology, because it really has made my creative life more exciting and rewarding.

This is the theory.

In the beginning was a perfect ten-dimensional universe that cleaved into two. While ours, of three spatial dimensions and the oddity of time, expanded to fit our grossness, hers, of six dimensions wrapped itself away in tiny solitude.

This sister universe, contemplative, concealed, waits in our future as it has refused our past. It may be the symbol behind all our symbols. It may be the mandala of the East and the Grail of the West. The clouded mirror of lost beauty that human beings have stared into since we learned to become conscious of our own face.

Can anyone deny that we are haunted? What is it that crouches under the myths we have made? Always the physical presence of something split off.

Jeanette Winterson, *Gut Symmetries*

Abstract

This thesis examines the relationships between black holes and gravity waves in string theory. First we review charged black hole solutions in the low-energy limit of string theory, some of which have been shown to be exact conformally invariant backgrounds for string propagation. Then we review the properties of gravitational wave spacetimes known as PP waves, which are related to exact extreme charged black holes through chiral null models on the string worldsheet, and compare particle and string geodesic focusing in a constant plane wave background.

The colliding plane wave metric discovered by Ferrari and Ibañez to be locally isometric to the interior of a Schwarzschild black hole is extended to the case of general axion-dilaton black holes. Because the transformation maps either black hole horizon to the focal plane of the colliding waves, this entire class of colliding plane wave spacetimes only suffers from the formation of spacetime singularities when the inner horizon itself is singular, which occurs in the Schwarzschild and singular dilaton black hole limits. The supersymmetric limit, corresponding to the extreme axion-dilaton black hole, yields the Bertotti-Robinson metric with the axion and dilaton fields flowing to fixed constant values. The maximal analytic extension of this metric across the Cauchy horizon yields a spacetime in which two sandwich waves in a cylindrical universe collide to produce a semi-infinite chain of Reissner-Nordstrom-like wormholes. The "stringy stretched horizon" as developed by Susskind is examined from the point of view of colliding plane waves.

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Chapter 1 Introduction

In this thesis we will investigate and compare relationships between string theory, black holes and gravity waves. These relationships are worth exploring because plane gravity waves are exact classical solutions to string theory, so any of their properties that can be related to black holes deserve to be explored.

In chapter two we review relevant aspects black hole physics, starting with the local definition of a black hole event horizon in terms of the focusing of null geodesics of massless test particles. We review charged black holes coupled to a massless dilaton in low energy string theory, which have in their extreme limit zero entropy at finite temperature, and demonstrate the difference between the canonical spacetime metric and the "string frame" metric that couples to the string world sheet for black holes with electric and magnetic charge. [1, 2, 3] Then we examine axion-dilaton black holes found by Kallosh et al. [4], find out which extreme charged black holes provide exact conformally invariant backgrounds for string propagation, and review the stringy stretched horizon and quantum-corrected string causality as developed by Susskind, Lowe and others. [5, 6, 7]

In chapter three we examine properties of a class of gravitational wave metrics known as PP waves, which have a null Killing vector and so propagate without dispersion. As exact string backgrounds they are related through chiral null models to extreme charged black holes. [8] Plane waves form a subclass of PP waves with translation invariance in the transverse spatial dimensions. We discuss which plane waves are exact solutions to which string theories and compare the geodesic focusing of test particles and test strings in a plane wave background.

Colliding plane wave spacetimes in Einstein relativity are generically singular yet there does exist a class of nonsingular solutions. [9] We examine the Kasner spacetime which controls the asymptotic behavior for collinearly polarized colliding plane waves and the difference between the singular and nonsingular solutions. We look briefly at string-derived perturbations to colliding plane wave spacetimes. [10]

Chapter four begins with a review of the colliding plane wave spacetime found by Ferrari and Ibañez [11] and independently by Yurtsever [13] that is locally isometric to a portion of the trapped region of a Schwarzschild black hole. Then we extend those results to $U(1) \times U(1)$ axion-dilaton black holes and show that there are selection rules on the focal lengths of the incoming waves. [14] In the case of full saturation of the BPS limits, corresponding to the extreme charged $U(1)$ black hole, the incoming plane waves have constant infinite amplitude and zero focal length.

We explore the extreme limit of axion-dilaton colliding plane waves and show that the field strength becomes purely magnetic, with the axion and dilaton fields taking the fixed values they flow to at the horizon of the corresponding black hole, independent of their values at infinity. We perform the maximal analytic extension of the spacetime to a black hole spacetime, which necessitates periodic boundary conditions on one transverse coordinate. We conclude with a discussion of the differences between test string and test particle propagation across the Killing-Cauchy horizon of the axion-dilaton colliding plane wave system. We show that the geodesic focusing that creates the horizon is controlled by the supersymmetric limit, and we relate stringy focusing on the focal plane to the stringy stretched horizon as explored by Susskind and others.

Chapter 2 A review of black holes in string theory

2.1 What defines a black hole

Before we investigate any connections between black holes, string theory and gravitational waves, we will review the definition of a "black hole". In plain language, a black hole is an object whose gravitational field is so strong that light cannot escape.

A more elegant and precise way of making this definition is to explore the global causal properties of a spacetime M endowed with metric $g_{\mu\nu}$. Let's assume that the combination $(M, g_{\mu\nu})$ satisfy a condition known as *strong aymptotic predictability*, which in simple terms means that some kind of initial value problem exists somewhere, if not everywhere, in M.[1] The region of M known as *future null infinity*, or $\mathcal{J}^+$, is where light should wind up in the infinite future, if the light has "escaped". We want to examine the *causal past* of that escaped light, or $J^-(\mathcal{J}^+)$. A discrepancy between M and $J^-(\mathcal{J}^+)$ will tell us whether light somehow has started out in M but never made it to $\mathcal{J}^+$, so the difference between M and $J^-(\mathcal{J}^+)$ tells us about the "light that did not escape" to $\mathcal{J}^+$.

Armed with these concepts we can now define the *black hole region* B of spacetime M as

$$B = [M - J^-(\mathcal{J}^+)]. \tag{2.1}$$

The *black hole event horizon* H is the boundary between the light that escapes and the light that doesn't escape. In technical terms, H is defined to be the part of M that lies on the boundary of the causal past of future null infinity in M, or

$$H = \dot{J}^-(\mathcal{J}^+) \cap M. \tag{2.2}$$

[1] For a lengthy, rigorous treatment of this topic, see [15] and references contained therein.

This set of definitions is concise and elegant, but requires knowledge of the entire spacetime M, in the infinite past and infinite future. No local observer has access to such information. A locally measurable definition of a black hole rests on the concept of the *trapped surface*, which itself relies on the concept of *geodesic focusing*. A trapped surface is formed when null geodesics focus in a manner to be explained later in this section.

Geodesic focusing is the source of nearly all singularities and causal pathologies that occur in classical general relativity [15]. This is where one might expect differences to arise between black holes in string theory and black holes in general relativity. The geodesic equations for test strings and test particles are

$$\text{Particles}: \qquad \frac{d^2x^\mu}{d^2\tau} + ?^{\,\mu}_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0 \tag{2.3}$$

$$\text{Strings}: \qquad \Box X^\mu + ?^{\,\mu}_{\nu\lambda}(X(\tau,\sigma))\partial_a X^\nu \partial^a X^\lambda = 0, \tag{2.4}$$

where the derivatives ∂_a and $\Box$ in the string formula (2.4) are with respect to string worldsheet coordinates (τ,σ) [16]. For non-interacting strings, light cones formed by massless solutions to the string equation (2.4) match solutions to the particle equation (2.3) after integration over σ, because (2.3) is contained in the center-of-mass limit of (2.4) [5]. However, when one adds string interactions, it was shown in [6] that string light cones becomes "leaky" and geodesic focusing becomes altered in a manner that will be discussed in a later section, and this becomes important in stringy black hole physics. For now, we shall concentrate on the focusing of geodesics that are defined by the normal test particle geodesic equation (2.3) in general relativity. Later we will compare geodesic focusing of particles and strings.

The simplest example of the causal consequences of geodesic focusing of light is gravitational lensing around a point mass in $d = 3$, as illustrated in figure 2.1 below. A light flashes at spacetime event $E_1 = (t = 0, x = 1, y = 0)$ and the light leaving E_1 is lensed by the spacetime geometry so that an observer O at spatial location $\vec{x}_O$ sees two images of the flash from E_1. The two images seen by O represent two different null geodesics γ_a and γ_b, both of which leave $(x = 1, y = 0)$ at $t = 0$. The geodesic γ_a

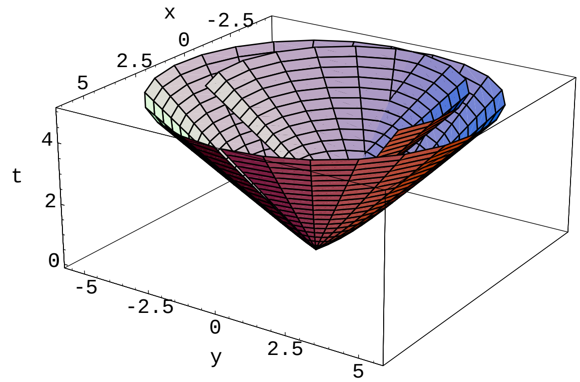

Figure 2.1: Light cone of the event $(0, 1, 0)$ as distorted by a point mass with the worldline $(t, 0, 0)$ in $d = 3$. The inner flap is generated by null geodesics that have left the causal boundary of $(0, 1, 0)$.

crosses $\vec{x}_O$ at $t = t_a$ and the geodesic γ_b crosses $\vec{x}_O$ at $t = t_b$.

If $t_b \neq t_a$ there is a problem. Suppose $t_a < t_b$. The events $(t_a, \vec{x}_O)$ and $(t_b, \vec{x}_O)$ cannot *both* lie on the future light cone of the event E_1, because the timelike observer O experiences both events. Therefore the geodesic γ_a must lie on the future light cone of E_1, while γ_b started out on the future light cone of E_1 and somehow left it. Since the problem goes away only when $t_a = t_b$, it must be true that this is where the problem starts and where null geodesics begin to fail to determine causal boundaries in spacetime.

This is explicitly illustrated in figure 2.1. The future light cone of the event $(0, 1, 0)$ in the presence of the point mass travelling on the worldline $(t, 0, 0)$ grows an inner flap generated by null geodesics that cross a second time along the negative x-axis.

In general, if two null geodesics γ_a and γ_b intersect once at some spacetime event E_1 and then reintersect at a later spacetime event E_2, then both γ_a and γ_b leave the "boundary of the causal future" of E_1 when they cross again at E_2. Therefore any event E_3 at $t_3 > t_2$ along either γ_a or γ_b can be reached by a *timelike* curve from E_1. The null geodesics γ_a and γ_b lose their power to determine causal boundaries for E_1 after they cross at E_2.

Geodesic focusing is not a problem as long as there exists a discrete set of multiple images. Geodesic focusing at the continuum level is more dangerous and hence more interesting. At this level a quantity θ, called the expansion scalar, determines when geodesic focusing is going to interfere with the unique delimitation of causal boundaries. The value of θ also tells us when there is a trapped surface, and hence when there is a black hole, in M. θ can be calculated for null or timelike geodesics. Since null geodesic focusing defines a trapped surface, we will only consider null geodesics in what follows.

If n^a is a tangent vector to a null geodesic γ, in spacetime M with metric $G_{\mu\nu}$, then we need to look at $B_{\alpha\beta} = \widehat{D_\alpha n_\beta}$ where $\widehat{\ }$ means restriction to a $d-2$ dimensional subspace orthogonal to n^α.[2] θ is defined by $\theta = \mathrm{Tr}B$, where the trace is only over the afore-mentioned two-dimensional subspace. The rotation ω_{ab} and shear σ_{ab} tensors are the antisymmetric and symmetric traceless parts, respectively, of $B_{\alpha\beta}$. The evolution equation for θ with respect to the affine parameter τ along γ is

$$n^\gamma D_\gamma\, \theta = \frac{d\theta}{d\tau} = -\frac{1}{d-2}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} + \omega_{\alpha\beta}\omega^{\alpha\beta} - R_{\alpha\beta}\;\xi^\alpha\xi^\beta, \tag{2.5}$$

known as the "focusing equation" for null geodesics. Spacetimes with $\omega_{ab} \neq 0$ are not foliatable into spacelike hypersurfaces and hence are not stably causal, so that term is zero if we exclude such spacetimes from consideration. Since $\sigma_{ab}\sigma^{ab} \geq 0$, if $R_{cd}\; n^c n^d \geq 0$, it follows that

$$\frac{d\theta}{d\tau} + \frac{1}{2}\theta^2 \leq 0 \quad \rightarrow \quad \theta^{-1} \geq {\theta_0}^{-1} + \frac{1}{2}\tau. \tag{2.6}$$

Since $\theta \sim \frac{1}{V_\perp}\frac{dV_\perp}{d\tau}$, where V_- is the transverse volume of a bundle of "nearby geodesics," we don't want the RHS of the above inequality to cross through zero. If the expansion $\theta_0 < 0$ at some $\tau = \tau_0$ along some geodesic, then $\theta \rightarrow -\infty$ along that geodesic within affine parameter $\tau \leq 2/|\theta_0|$. So $V_- \rightarrow 0$ in a finite amount of "proper time" after the bundle begins to focus, or converge, at τ_0. When this happens to geodesics that are initially intersecting at some previous $\tau < \tau_0$, either the initial value problem breaks down or the geodesics fail to be extendible past the focal plane.

[2] The subtleties in this projection are discussed in [17].

The latter alternative defines a spacetime singularity and is generally accompanied by the blowing up of curvature invariants in that region. The nonsingular alternative will be described when we discuss plane gravity waves in a later section.

A trapped surface tells us when θ is zero or negative, which by (2.6) means that "nearby" geodesics will focus and $\theta \to -\infty$ in a finite amount of affine parameter. This is the foundation of the Hawking-Penrose singularity theorem for black holes.[15]

To be more precise, a trapped surface T is a compact, $d-2$ dimensional spacelike submanifold of M having the property that both the "ingoing" and "outgoing" sets of null geodesics everywhere on T satisfy $\theta < 0$. A marginally trapped surface T' satisfies $\theta \leq 0$. It can be proven that in a black hole spacetime, both T and T' are contained within the black hole region B. An outer marginally trapped surface S is basically a marginally trapped surface that is the boundary of a $d-1$ dimensional volume called the trapped region C.

In order to describe the evolution properties of black hole spacetimes, we need to consider a black hole at a given time, or equivalently in a foliatable spacetime, at a given spacelike slice or Cauchy surface Σ. The total trapped region $\mathcal{T}$ at a given time Σ is the closure of the union of all trapped regions C at time Σ. The apparent horizon $\mathcal{A}$ is then the boundary of $\mathcal{T}$. On the apparent horizon $\mathcal{A}$, null geodesics satisfy $\theta = 0$. If we define $\mathcal{H} = H \cap \Sigma$ as the "true event horizon" then we can say $\mathcal{A} \subset \mathcal{H}$. This eventually leads to the conclusion that everywhere on H, the null geodesic generators satisfy $\theta \geq 0$, if spacetime in question satisfies the null convergence condition $R_{\alpha\beta}\, n^\alpha n^\beta \geq 0$.

This is how the black hole area theorem [15] is proven. The area of a black hole at time Σ_1 is the area of $\mathcal{H}_1 = H \cap \Sigma_1$. Since the null geodesics that generate H have positive or zero expansion, and hence have V_- expanding or staying the same into the future, it is not possible for the area of $\mathcal{H}_2$ to become smaller than the area of $\mathcal{H}_1$ if $\Sigma_2 > \Sigma_1$.

This is a very important relation because it validates the idea that the area of a black hole can act like entropy, and it gives us cause to seek out a way to explain this entropy-like quantity, the area of a black hole event horizon, using quantum microstates in a quantum theory of gravity. The cornerstone of this analysis is based on the geodesic focusing properties of test particles. Stringy geodesic focusing, as we shall see, is more complicated, and contributes, along with string dualities, to a stringy picture of a black hole, which will be reviewed later in this chapter.

2.2 The charged dilaton black hole

Consider the spacetime effective action for $d = 4$ gravity coupled to electromagnetism plus a massless dilaton with the coupling

$$S_{eff} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left(-R + 2D_\mu \Phi D^\mu \Phi + e^{-2a\Phi} F_{\mu\nu} F^{\mu\nu} \right) \tag{2.7}$$

As was uncovered in [1, 2] a massless dilaton coupling to $\mathrm{Tr} F^2$ alters the charged black hole solution from the familiar the Reissner-Nordstrom solution, and the precise manner in which the dilaton-coupled black hole differs is sensitive to the coefficient a.

If the action (2.7) represents a low-energy limit of string theory, then $a = 1$. The magnetically charged black hole solution with mass M, magnetic monopole P and dilaton Φ_m

$$F = P e^{\Phi_0} \sin\theta \, d\theta \wedge d\phi, \qquad e^{-2\Phi_m} = e^{-2\Phi_0} \left(\frac{\rho - 2|\Sigma|}{\rho} \right), \tag{2.8}$$

has a spacetime metric that can be written

$$ds^2 = -\frac{(\rho - 2M)}{\rho} \, dt^2 + \frac{\rho}{(\rho - 2M)} \, d\rho^2 + \rho(\rho - 2|\Sigma|) \, d\Omega, \tag{2.9}$$

where Σ is the asymptotic dilaton charge

$$\Sigma = -\frac{1}{4\pi} \int_\infty d^2 S^\mu D_\mu \Phi_m = \frac{P^2}{2M}. \tag{2.10}$$

Electric-magnetic duality allows us to find the electrically charged solution by way of the duality transformation $F \to \tilde{F}_{\mu\nu} = \frac{1}{2} e^{-2\Phi_m} \epsilon_{\mu\nu}{}^{\alpha\beta} F_{\alpha\beta}$, $\Phi_m \to \Phi_e = -\Phi_m$ and

$P \to Q$ to give the same metric as (2.9) but with electric field and dilaton

$$\tilde{F} = \frac{Qe^{-\Phi_0}}{\rho}\, dt \wedge d\rho, \qquad e^{-2\Phi_e} = e^{-2\Phi_0}\left(\frac{\rho}{\rho - 2|\Sigma|}\right), \quad \Sigma = -\frac{Q^2}{2M}. \tag{2.11}$$

The metric (2.9) can be written in a form closer to that of the Reissner-Nordstrom metric with the shift $\rho = r + |\Sigma|$, giving

$$ds^2 = -\frac{(r - r_+)(r - r_-)}{R(r)^2}\, dt^2 + \frac{R(r)^2}{(r - r_+)(r - r_-)}\, dr^2 + R(r)^2\, d\Omega, \tag{2.12}$$

where $R(r)^2 = r^2 - |\Sigma|^2$, $r_+ = 2M - |\Sigma|$ and $r_- = |\Sigma|$. The familiar Reissner-Nordstrom metric when written in these coordinates has $r_\pm = M \pm \sqrt{M^2 - Q^2}$ and $R(r) = r^2$. There is an event horizon at $r = r_+$ and and inner Cauchy horizon at $r = r_-$. For Reissner-Nordstrom black holes, the maximal analytic extension of the metric yields an infinite chain of black holes [15], although in [18] it was demonstrated that the inner horizon is most likely physically unstable and never formed in gravitational collapse in Nature. For the charged dilaton black holes, there is an event horizon at $r_+ = 2M - |\Sigma|$. However, the inner horizon at $r_- = |\Sigma|$ has zero area, since $R(|\Sigma|) = 0$. The inner horizon touches the singularity at $r = |\Sigma|$.

An extreme charged black hole is what happens when $r_+ \to r_-$. If we parametrize $r_\pm = M \pm r_0$, then the extremal limit corresponds to $r_0 \to 0$. For the Reissner-Nordstrom metric, $r_0 = \sqrt{M^2 - Q^2}$ so the extreme state is reached for $M = Q^2$. For the charged dilaton black hole, $r_0 = M - |\Sigma|$ so extremality is reached for $M^2 = Q^2/2$, but the area of the extreme Reissner-Nordstrom horizon is finite, whereas the surface $r = |\Sigma| = M$ has zero area.

This difference in the event horizon area at extremality leads us to an interesting point about black hole entropy in the presence of scalar fields, raised in [1, 2] and further explored in [3]. The Reissner-Nordstrom black hole has Hawking temperature

$$T_H = \frac{\sqrt{M^2 - Q^2}}{2\pi(M + \sqrt{M^2 - Q^2})^2} = \frac{r_0}{2\pi r_+^2}, \tag{2.13}$$

which vanishes for the extreme case $r_0 \to 0$, while the entropy $S = A/4 = \pi r_+^2$ is always finite for the extreme case. From the metric in coordinates (2.9) it can be easily deduced that the Hawking temperature for charged dilaton black hole is

$$T_H = \frac{1}{8\pi M e^{\Phi_0}}, \tag{2.14}$$

which is finite and nonzero even for the extreme limit. The black hole area $A = 4\pi R(r_+)^2 = 0$ for $M^2 = Q^2/2$, so the extreme black hole somehow manages to have zero entropy at finite temperature, which means it behaves more like an elementary particle with a mass gap than a thermodynamic ensemble of states. Furthermore, this behavior occurs only for value $a = 1$ in (2.9). Thus was born the idea that string theory somehow contains within it a microscopic basis for understanding the apparent macroscopic thermodynamics of black holes.

2.3 The string metric vs. the canonical metric

In string theory there exists a distinction between the "canonical metric" $g_{\mu\nu}$ appearing as a field in the spacetime action (2.7), and the "string metric" $\hat{g}_{\mu\nu}$ that appears in the worldsheet sigma model action as

$$S = \frac{1}{2\pi\alpha'} \int d\sigma\, d\tau\, \hat{g}_{\mu\nu}(X) \partial^a X^\mu(\sigma,\tau) \partial_a X^\nu(\sigma,\tau) + \frac{1}{2\pi} \int d\sigma\, d\tau\, \mathcal{R}^{(2)}\, \Phi, \tag{2.15}$$

where the index a runs over the worldsheet coordinates (σ, τ).

The effective action as calculated from the vanishing of the worldsheet β function at lowest order is

$$S_{eff} = \frac{1}{16\pi} \int d^4x \sqrt{-\hat{g}}\, e^{-2\Phi} \left(-\hat{R} - 4 D_\mu \Phi D^\mu \Phi + F_{\mu\nu} F^{\mu\nu} \right), \tag{2.16}$$

and one can switch between (2.7) and (2.16) (if $a = 1$ in (2.7)) via the conformal transformation

$$\hat{g}_{\mu\nu} = e^{2\Phi}\, g_{\mu\nu}. \tag{2.17}$$

A conformal transformation can't change the causal structure of a spacetime because it only changes the affine parameter along the geodesic and has no effect on the direction. However, sometimes reparametrization can have a significant effect on the measurable geometry. If a null geodesic for metric $g_{\mu\nu}$ has affine parameter λ and a null geodesic for metric $\hat{g}_{\mu\nu}$ has affine parameter $\hat{\lambda}$ then

$$\frac{d\hat{\lambda}}{d\lambda} = e^{2\Phi}, \tag{2.18}$$

so if $\Phi \to \infty$, spacetime pathologies that may appear within a finite affine parameter as measured by $g_{\mu\nu}$ could be perceived as infinitely far away as measured by the metric $\hat{g}_{\mu\nu}$. (This is the inverse of how Penrose diagrams are able represent timelike, null and spatial infinity using finite distances on a diagram, but in that case the conformally transformed metric is intended to be an unphysical representation of the actual physical metric.) Also, if $\Phi \to \infty$ then volumes measured by metric $g_{\mu\nu}$ to be vanishing could take finite values as measured by $\hat{g}_{\mu\nu}$.

This is precisely what happens with extreme magnetic black holes. The canonical metric $g_{\mu\nu}$ in (2.9) is the same whether the charge is electric or magnetic, but the string metric $\hat{g}_{\mu\nu}$ is not the same for electric and magnetic charge. The difference is most pronounced in the extreme limit. The string metric for the extreme magnetically charged black hole is

$$d\hat{s}_m^2 = e^{2\Phi_0}(-dt^2 + e^{4\Phi_m}\, d\rho^2 + \rho^2 d\Omega), \quad e^{2\Phi_m} = e^{2\Phi_0}\left(\frac{\rho}{\rho - 2|\Sigma|}\right), \tag{2.19}$$

and the amount of affine parameter expended along a radial null geodesic connecting r_0 and r

$$\Delta\lambda = e^{\Phi_0}\left(\rho - \rho_0 + 2|\Sigma|\, \log\frac{\rho - 2|\Sigma|}{\rho_0 - 2|\Sigma|}\right), \tag{2.20}$$

which becomes infinite as $\rho \to 2|\Sigma|$. The extreme electrically charged black hole becomes

$$d\hat{s}_e^2 = e^{2\Phi_0}(-e^{4\Phi_e}\, dt^2 + d\rho^2 + e^{4\Phi_e}\, \rho^2 d\Omega), \quad e^{2\Phi_e} = e^{2\Phi_0}\left(\frac{\rho - 2|\Sigma|}{\rho}\right). \tag{2.21}$$

The amount of affine parameter expended along a null radial geodesic connecting ρ_0 and ρ

$$\Delta\lambda = e^{\Phi_0}\left(\rho - \rho_0 + 2|\Sigma|\log\frac{\rho}{r_0}\right), \tag{2.22}$$

and the amount of affine parameter needed to reach $\rho = 2|\Sigma|$ from ρ_0 is always finite.

It's important to remember that when we're talking about null geodesics in the context of string propagation using the string metric, we're talking about the behavior of the null limit of the string center-of-mass motion described in equation (2.4). As we shall see later, the deviations from this center-of-mass behavior become important when we discuss string propagation near black hole horizons.

The dilaton in string theory also determines the coupling constant g_s in the string loop expansion through $g_s = e^{\Phi_0}$, where Φ_0 is the constant part of the field. In a curved asymptotically spacetime Φ_0 is usually meant to be the value of Φ at spatial infinity. Notice that the electric and magnetic dilaton fields are driven to opposite extremes of coupling, where we might expect string loop corrections to be large near the horizon of the magnetic black hole but vanish near the horizon of the electric black hole. However, it turns out that each of these black hole spacetimes makes an exact string background without quantum corrections under the right conditions, which will be explained in section 2.5.

2.4 Axion-dilaton black holes

Dimensionally-reduced superstring theory in $d = 4$ can be described as an $N = 4$ supergravity theory. One subset of the field content of this theory is $U(1) \times U(1)$, with a vector and an axial vector. In this context, the charged dilaton black hole of section 2.2 can be generalized to include dyonic and axionic charge. The low energy effective action with these solutions can be written in the form

$$\begin{aligned} S_{eff} = \frac{1}{16\pi}\int d^4x\,\sqrt{-g}\,(-R + \frac{1}{2}\frac{\partial_\mu\lambda\partial^\mu\bar{\lambda}}{(\mathrm{Im}\lambda)^2} \quad &- \quad e^{-2\Phi}\sum_{n=1}^{2} F^{(n)}_{\mu\nu}F^{(n)\mu\nu} \\ &+ \quad i\psi\sum_{n=1}^{2} F^{(n)\star}_{\mu\nu}F^{(n)\mu\nu}). \end{aligned} \tag{2.23}$$

The complex scalar field $\lambda = \psi + ie^{-2\Phi}$, where ψ is the axion.

Black hole solutions to the equations of motion of (2.23) can be written [4, 19, 20]

$$ds^2 = -\frac{(r_+ - r)(r - r_-)}{(r^2 - |\Upsilon|^2)}\, dt^2 + \frac{(r^2 - |\Upsilon|^2)}{(r - r_+)(r - r_-)}\, dr^2 + (r^2 - |\Upsilon|^2)\, d\Omega, \qquad (2.24)$$

where

$$r_\pm = M \pm r_0, \quad r_0^2 = M^2 + |\Upsilon|^2 - 4\sum_{n=1}^{2} |?^{(n)}|^2, \quad ?^{(n)} = \frac{1}{2}(Q_n + iP_n), \qquad (2.25)$$

where (n) refers to the n^{th} $U(1)$ field, and $\{Q_n, P_n\}$ are the U(1) electric and magnetic charges. The dilaton charge Σ of the previous section is generalized to

$$\Upsilon = \Sigma - i\Delta = -\frac{2}{M}\sum_{n=1}^{2} (\overline{?}^{(n)})^2, \quad \Sigma = \frac{\vec{P}^2 - \vec{Q}^2}{2M}, \quad \Delta = -\frac{\vec{P}\cdot\vec{Q}}{M}. \qquad (2.26)$$

The metric for the most general charge combinations of $\vec{Q} = \{Q_1, Q_2\}, \vec{P} = \{P_1, P_2\}$ has the global structure of the Reissner-Nordstrom solution, with an event horizon at $r = r_+$, an inner Cauchy horizon at $r = r_-$. The singularity in these coordinates is at $r = |\Upsilon|$. The magnetically charged dilaton black hole of the previous section corresponds to the case $\vec{P} = \{P, 0\}, \vec{Q} = \{0, 0\}$ and the electrically charged black hole corresponds to $\vec{Q} = \{Q, 0\}, \vec{P} = \{0, 0\}$. The Reissner-Nordstrom metric with $\Sigma = \Delta = 0$ can be obtained using $\vec{Q} = \{Q/\sqrt{2}, 0\}, \vec{P} = \{0, Q/\sqrt{2}\}$ to give $r_0 = \sqrt{M^2 - Q^2}$.

The parameter r_0 measures how far the black hole is from the extremal state $r_+ = r_-$, where the trapped region threatens to vanish and reveal a naked singularity to the universe. However, it has also long been known that there exist supersymmetric embeddings of extreme charged black holes. [21] The parameter r_0 then measures the breaking of those supersymmetries. If we label the two complex central charges of $N = 4$ supergravity as (z_1, z_2), where $z_1 = \sqrt{2}(?^{(1)} + i?^{(2)})$ and $z_2 = \sqrt{2}(?^{(1)} - i?^{(2)})$, then r_0 can be written

$$r_0^2 = \frac{(M^2 - |z_1|^2)\,(M^2 - |z_2|^2)}{M^2}. \qquad (2.27)$$

The Bogomolny bound on the positivity of the mass in this system requires $M \geq |z_{1,2}|$ for the largest of the two central charge eigenvalues. So in this sense supersymmetry and cosmic censorship seem related.

The entropy and temperature of axion-dilaton black holes can be written in terms of the central charges as

$$T = \frac{1}{2\pi M} \frac{\sqrt{(M^2 - |z_1|^2)\,(M^2 - |z_2|^2)}}{\left(\sqrt{M^2 - |z_1|^2} + \sqrt{M^2 - |z_2|^2}\right)^2}, \qquad (2.28)$$

and

$$S = \pi \left(\sqrt{M^2 - |z_1|^2} + \sqrt{M^2 - |z_2|^2}\right)^2. \qquad (2.29)$$

Suppose we start with $z_1 \geq z_2$. The condition $r_0 = 0$ then corresponds to the saturation of one SUSY bound by $M = |z_1|$. This restores one quarter of the broken supersymmetries in that theory at the extreme horizon, so that we get an unbroken $N = 1$ supersymmetry for the general extreme axion-dilaton black hole. In this limit the black hole temperature $T = 0$ while the entropy becomes $S = \pi(M^2 - |z_2|^2) = \pi(|z_1|^2 - |z_2|^2)$, which is finite and nonzero.

There is some subtlety in saturating both Bogomolny bounds at once, which can only happen for $M = |z_1| = |z_2|$. If we first set $M \to |z_1|$ in the $U(1) \times U(1)$ theory and then take the limit $|z_2| \to |z_1|$, then the temperature $T \to 0$ and the entropy $S \to 0$. If we start with a single $U(1)$ by setting $|z_1| = |z_2|$ and then take the limit $M \to |z_1|$, the temperature $T \to (8\pi M)^{-1}$ and the entropy $S \to 0$. This is the limit of the extreme charged black hole of the previous section, with $M = |\Upsilon|$, finite temperature and zero entropy. But with either limit, $N = 2$ supersymmetry is restored at the event horizon.

The dilaton field is given by

$$e^{-2\Phi} = e^{-2\Phi_0} \left(\frac{r^2 - (\Sigma^2 + \Delta^2)}{(r+\Sigma)^2 + \Delta^2} \right), \tag{2.30}$$

and the axion field is

$$\psi = \frac{\psi_0 \left((r+\Sigma)^2 + \Delta^2 \right) - 2 e^{-2\Phi_0} \Delta\, r}{(r+\Sigma)^2 + \Delta^2}. \tag{2.31}$$

The axion and dilaton fields add to this interesting behavior at the horizon in the $r_0 \to 0$ limit. At the extreme horizon these fields lose all dependence on the asymptotic values (ψ_0, $e^{-2\Phi_0}$) and depend only on the values of Dirac-quantized conserved charges. The charges $(\vec{Q}, \vec{P})$ are not those charges, but they are related to the Dirac-quantized conserved charges by $Q_j + iP_j = e^{\phi_0}\,(n_j - \bar{\lambda}_0 m_j)$, with $(n_j, m_j) \in Z$. As $r \to r_+ = r_-$ the axion and dilaton fields at the horizon flow to the critical values

$$\psi_f = \frac{\sum n_i m_i}{\sum m_i^2}, \quad e^{-2\Phi_f} = \frac{\left(\sum\limits_{i<j} (n_i m_j - n_j m_i)^2 \right)^{1/2}}{\sum m_i^2}. \tag{2.32}$$

This behavior was plotted in figure 2.2 using *Mathematica*. In the plot the event horizon is at $r = 0$; the coordinate r is not the same one used in the metric (2.24).

One consequence of (2.32) is that for extreme $U(1) \times U(1)$ axion-dilaton black holes, the string metric $\hat{g}_{\mu\nu} \to e^{2\Phi_f} g_{\mu\nu}$ and $r_0 \to 0$, so the string metric and the canonical metric are related by a fixed constant factor at the horizon. As long as this constant is finite, extreme axion-dilaton black holes look basically the same measured by either the string or canonical metric.

Near the horizon of an extreme axion-dilaton black hole, shifting $\rho = r - M$, the metric (4.9) can be written

$$ds^2 = -\frac{\rho^2}{M^2 - |\Upsilon|^2}\, dt^2 + \frac{M^2 - |\Upsilon|^2}{\rho^2}\, d\rho^2 + (M^2 - |\Upsilon|^2)\, d\Omega. \tag{2.33}$$

This spacetime is the product of anti-de Sitter spacetime in two dimensions times a two-sphere, or $adS_2 \times S^2$. It is also known as Bertotti-Robinson spacetime [22], and

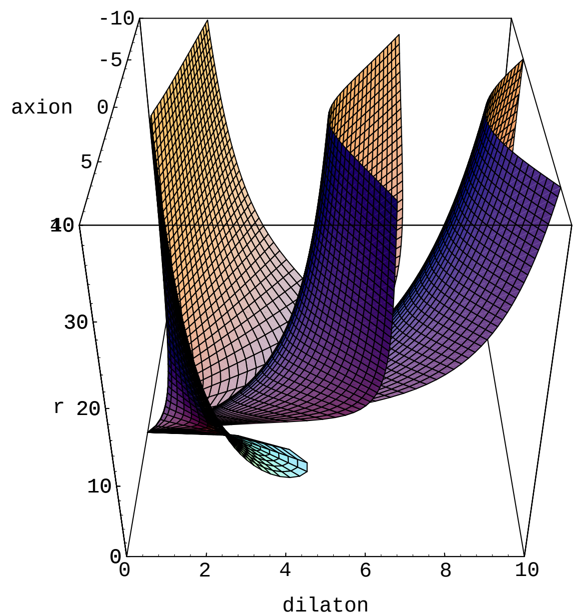

Figure 2.2: These contours of constant ϕ_0 show how the axion and dilaton fields lose their dependence on ϕ_0 and a_0 and flow to fixed values on the horizon of an extreme axion-dilaton black hole. Here the coordinate r measures distance from the extreme horizon.

will mentioned again in the next two chapters because this metric can also describe the spacetime of colliding plane gravitational waves.

2.5 Black holes as exact string backgrounds

The supersymmetry properties of extreme charged black holes in supergravity theories [23, 21, 20] suggested that these black holes might be protected from higher derivative corrections to the low-energy effective action. In the context of string theory, it is now known that certain types of extreme charged black holes are exact conformally invariant backgrounds for superstring sigma models. [8, 24] They belong to a general class of models called chiral null models, named so because a null Killing vector in the spacetime metric translates into a conserved chiral current in the world-sheet sigma model. A single chiral current in a sigma model is sufficient to establish conformal invariance to all orders in the α' expansion, provided the conditions at one loop are satisfied.

The general chiral null model Lagrangian can be written

$$L = F(x)\partial u\left(\bar{\partial} v + K(u,x)\bar{\partial} u + 2A_i(x,u)\bar{\partial} x^i\right) + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R}\Phi, \qquad (2.34)$$

where the chirally-coupled vector field A_i comes from setting $B_{\mu i}$, the off-diagonal components of the antisymmetric tensor, equal to $G_{\mu i}$, the off-diagonal components of the metric tensor (in the string frame). Solutions to the equations of motion for (2.34) as given in [8] were proven to be conformally invariant to all orders. These solutions include extreme electrically charged black holes in $d = 4$, obtained by dimensional reduction of chiral null models in $d = 5$.

The Kaluza-Klein extreme electric black hole which is a solution to (2.7) with the parameter $a = \sqrt{3}$ can be obtained from the dimensional reduction of a $d = 5$ fundamental string [25], which has the chiral null Lagrangian

$$L_{FS} = F(r)\partial u \bar{\partial} v + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R}\Phi, \qquad (2.35)$$

where $r = \sqrt{x_i x^i}$, and

$$F(r)^{-1} = 1 + \frac{M}{r}, \quad \Phi = \Phi_0 + \frac{1}{2}\ln F. \qquad (2.36)$$

The extreme electrically charged black hole (2.21) can be obtained through the dimensional reduction of a generalized version of (2.35)

$$L_{GFS} = F(r)\partial u \bar{\partial} v + \partial u \bar{\partial} u + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R}\Phi, \qquad (2.37)$$

where F and Φ are still given by (2.36).

In order to get conformally invariant string backgrounds with magentic charge, it was shown in [24] that one must generalize the chiral null model (2.34) to include curvature in the transverse coordinates so that the term $\partial x_i \bar{\partial} x^i$ becomes $(G_{ij} + Bij)\partial x_i \bar{\partial} x^i$. In addition, the dimensional reduction must be from $d = 6$ using the internal symmetry group $U(1)_E \times U(1)_M$, so that the electric and magnetic fields come from compactifying in different transverse directions. The conformal invariance in this case stems from the combination of the conserved chiral current with $N = 4$ worldsheet supersymmetry in the four transverse dimensions, meaning that extreme black holes with magnetic charge are only exact backgrounds for heterotic or Type II superstring theories, unlike the extreme black holes with electric charge, which are also exact for bosonic strings. The extreme black dyonic holes which were proved to be exact string backgrounds in this manner include a subset of axion-dilaton black holes with zero axion, hence $\vec{Q} \cdot \vec{P} = 0$.

2.6 String propagation near a black hole

A discussion of the impact of string theory and supersymmetry in the microscopic derivation of black hole entropy is beyond the scope of this thesis. However there are some crucial developments in that direction that are relevant to the colliding plane wave systems disucssed in later chapters.

One of the big problems with black hole event horizons and inner horizons is the enhanced sensitivity to small scales of distance. If the coordinate ρ is defined to measure the proper distance from the event horizon of a Schwarzschild black hole, then near the horizon the Schwarzschild metric can be approximated by

$$ds^2 = -\rho^2 d\omega^2 + d\rho^2 + d\xi^2 + d\eta^2. \tag{2.38}$$

The proper time in the frame of a falling object is related to the proper time as measured by the Rindler observer at constant ρ for large ω by

$$\frac{d\tau}{d\omega} \sim \rho e^{-\omega}, \tag{2.39}$$

so that the Rindler observer sees the object with a time resolution that becomes infinite. This is where the difference between the propagation of test particles (2.3) and test strings (2.4) is important. Looking at a particle geodesic with infinite time resolution leads to problems because the Rindler observer can presumably see Planck scale physics if she waits long enough, even if the black hole is very large. This is one reason why semi-classical field theory has not wound up the winner in the race to explain black hole entropy.

The story for strings falling into the black hole is very different, and leads to the "stringy stretched horizon" uncovered by Susskind [7]. If we try to describe a string using a time resolution $d\tau/d\omega = \varepsilon$ then we need to cut off the mode expansion for the string length at $N \sim 1/\varepsilon$. But the mean squared size of a string cut off at mode N grows like $\log N$, *i.e.* strings fill more space as we try to measure them with greater time resolution. The total length and longitudinal spread both grow like $1/\varepsilon$ as we increase the resolution.

Since the resolution $\varepsilon \sim 1/N$, the Rindler observer would see the passing string begin to grow in longitudinal size and transverse spread like e^{ω}, and grow in mean squared radius like w, until a natural cutoff is reached when appears to reach the size of the black hole and thermalizes around the horizon [26]. This seems contrary to causality because the string has ostensibly fallen into the black hole, so how could the quantum information in the string thermalize around the outside of the horizon?

The answer was provided by Lowe [5, 6] who showed that in string field theory, the equal-time commutator of string fields does not vanish outside the light cone of the string but is nonzero and spreads as one narrows the time resolution of the measurement. The delimitation of causal boundaries in string theory follows a quantum uncertainty principle, and so the notion of a trapped surface is not precise in string theory but resolution-dependent.

Chapter 3 Plane gravity waves and string theory

3.1 PP wave spacetimes

3.1.1 Classification and basic properties

It was realized at the end of the last decade that there exists a large class of time-dependent solutions to the Einstein equations that serve as exact nonperturbative backgrounds for string propagation. [27, 28] These solutions represent gravitational wave spacetimes characterized by the existence of a covariantly conserved null Killing vector. The general form of the metric in harmonic coordinates is

$$ds^2 = -dU\, dV + h(U,X)\, dU^2 + \sum_{i=1}^{d-2} (dX^i)^2, \tag{3.1}$$

where d is the dimension of spacetime and i labels spacelike directions transverse to the wave propagation direction. [29] The null Killing vector ∂_V is a maximally degenerate principal null vector, so these spacetimes, called "PP waves" (plane-fronted waves with parallel rays), are type N in the Petrov classification of solutions to the vacuum Einstein equations. A principal null direction is a direction in which the gravitational analog of energy can flow — here it is only going in the direction of ∂_V, hence the maximal degeneracy.

The only non-vanishing components of the Riemann tensor in the above coordinate system are $R_{UiUj} = -\frac{1}{2}\partial_i\partial_j h(U,X)$, so the curvature can be a discontinuous function of U. The only nonvanishing component of the Ricci tensor is then

$$R_{UU} = -\frac{1}{2}\sum_i \partial_i^2\, h(U,X). \tag{3.2}$$

For vacuum solutions, $h(U, X)$ is a harmonic function in the transverse hyperplane with completely arbitrary U dependence. But harmonic functions satisfy a superposition principle, and hence so do vacuum PP wave spacetimes. So it appears that in the case of vacuum PP waves, the gravitational field has the properties of a *linear* field theory.

From above we can see that if the function $h(U, X)$ is quadratic in the X^i then the curvature becomes a function of U alone. To be more precise, a PP wave becomes a plane wave with $d-2$ commuting spacelike Killing vectors when

$$h(U, X) = \sum_{i,j} h_{ij}(U)\, X^i X^j, \tag{3.3}$$

where the functions $h_{ij}(U)$ encode the polarization and amplitude of the wave.

The nonvanishing components of the Weyl and Ricci tensors are then given by

$$C_{UiUi} = \frac{\mathrm{Tr}\, h}{d-2} - h_{ii}(U), \qquad C_{UiUj} = -h_{ij}(U), \qquad R_{UU} = -\mathrm{Tr}\, h \tag{3.4}$$

where $\mathrm{Tr}\, h = \sum_i h_{ii}(U)$. Note that a vacuum solution satisfies $\mathrm{Tr}\, h = 0$, so a plane wave with $\mathrm{Tr}\, h \neq 0$ is not a pure gravity wave. Note also that the Weyl tensor can be made to vanish if the polarization is purely diagonal and if $h_{ii}(U) = \mathrm{Tr}\, h/(d-2)$ for all i. Since the Weyl tensor measures the part of the Riemann tensor that doesn't come from the Ricci tensor, this signifies a wave that is pure scalar and/or electromagnetic. We'll see an example of this later.

We can make the plane symmetry manifest by switching to Rosen coordinates

$$ds^2 = -du dv + \sum_{ij} g_{ij}(u)(dx^i)^2, \tag{3.5}$$

using the coordinate transformation [30]

$$U = u, \tag{3.6}$$

$$V = v + \frac{1}{2}\frac{dg_{ij}}{du}\, x^i x^j, \tag{3.7}$$

$$X^i \;=\; F^i_j(u)\, x^j. \tag{3.8}$$

where

$$g_{ij} = F^k_i F^k_j, \quad \frac{d^2 F^i_j}{d^2 u} = h_{ik}\, F^k_j \quad \text{and} \quad F^k_i \frac{dF^k_j}{du} = \frac{dF^k_i}{du} F^k_j. \tag{3.9}$$

In the cases we'll be looking at, the polarization is diagonal ($h_{ij} = 0$ for $i \neq j$) and the metric in Rosen coordinates is also diagonal. In Rosen coordinates it is obvious that the spacelike Killing vectors $\frac{\partial}{\partial x^i}$ become null wherever $g_{ii}(u) = 0$. As we shall see later, this breakdown of plane symmetry is a manifestation of geodesic focusing and the consequent breakdown in the initial value problem for plane wave spacetimes in general.

A concrete example of a plane wave that will be important later is the sandwich wave with constant parallel polarization. In harmonic coordinates in four spacetime dimensions the metric is

$$\begin{aligned} ds^2 &= -dU\, dV - h(U)\,(X^2 \pm Y^2)\, dU^2 + dX^2 + dY^2, \quad 0 \leq U \leq \Delta U \quad (3.10) \\ &= -dU\, dV + dX^2 + dY^2, \quad U < 0,\; U > \Delta U \end{aligned}$$

The metric with $X^2 - Y^2$ gives a pure gravity wave with vanishing Ricci tensor. The metric with $X^2 + Y^2$ yields a vanishing Weyl tensor, and couples to a stress tensor whose only nonvanishing component is T_{UU}. (We've chosen conventions so that $h(U) > 0$ corresponds to $T_{UU} > 0$.)

In Rosen coordinates, using the transformation (3.9), the metric becomes

$$\begin{aligned} ds^2 &= -du\, dv + dx^2 + dy^2, \quad u \leq 0 \quad (3.11) \\ &= -du\, dv + F_x(u)^2\, dx^2 + F_y(u)^2\, dy^2, \quad 0 \leq u \leq \Delta U, \\ &= -du\, dv + \left(F_x(\Delta U)\frac{u - f_x}{\Delta U - f_x}\right)^2 dx^2 + \left(F_y(\Delta U)\frac{u - f_y}{\Delta U - f_y}\right)^2 dy^2, \quad u \geq \Delta U, \end{aligned}$$

where $F_x(0) = F_y(0) = 1$ and

$$\frac{\ddot{F}_x}{F_x} = \pm\frac{\ddot{F}_y}{F_y} = h(U), \quad f_x = \Delta U - \frac{F_x(\Delta U)}{\dot{F}_x(\Delta U)}, \quad f_y = \Delta U - \frac{F_y(\Delta U)}{\dot{F}_y(\Delta U)}. \tag{3.12}$$

The constants f_x, f_y are the focal lengths of the wave. When $u = f_x$ or f_y, the corresponding spacelike Killing vector becomes null, breaking the spatial translation symmetry and upsetting the existence of a global initial value problem. For $h(U) > 0$, in other words for source fields that satisfy the strong energy condition, the equations (3.12) guarantee that at least one of the focal lengths will satisfy $f_i > \Delta U$, so that at least one of the spacelike translation symmetries will be broken by propagation through the wave. This is why exact plane waves are not globally hyperbolic spacetimes, meaning that there does not exist a global foliation of a plane wave spacetime into space and time. This was first noted by Penrose. [31] This will be discussed in much greater detail later.

In the simplest case where the function $h(U) = h_0$, the metric in Rosen coordinates for the (+) polarization (zero Weyl tensor) is given by (3.11) with

$$F_x(u) = F_y(u) = \cos(\sqrt{h_0}\, u), \quad f_x = f_y = \Delta U + \cot(\sqrt{h_0}\, \Delta U). \tag{3.13}$$

Notice that for $\Delta U = \frac{\pi}{2\sqrt{h_0}}$, the focal length of the wave is $f_x = f_y = \Delta U$, so that parallel null geodesics focus exactly on the trailing edge of the wave. This relationship between the width and the amplitude will come up again in relation to black holes. We'll compare test particle and string propagation through this plane wave in later sections.

3.1.2 PP waves as exact string backgrounds

PP wave spacetimes represent a geometrical limit of Einstein gravity in which gravity behaves like a free field theory. Therefore these spacetimes should be exact solutions to any generally covariant higher-derivative theory of gravity. The simplest way to anticipate this is to try to form any coordinate invariant combination of two or more curvature tensors from the PP wave metric (3.1). One will soon find that there are no non-vanishing curvature invariants available for the construction of quantum correction terms to a coordinate-invariant low energy effective spacetime action. [32, 33]

Therefore, in *any* generally covariant theory of gravity polynomial in derivatives of the metric, the equations of motion for metrics restricted to this type are always $R_{\mu\nu} = 0$ and $h(U, X)$ is always a harmonic function of the transverse coordinates X^i.

Since PP waves obey the flat massless wave equation, they also give us an exact classical identification of the graviton. Usually the correspondence between curvature fluctuations and free gravitons is demonstrated by using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and neglecting terms in the action which are more than quadratic in the small fluctuation h. For PP waves this is valid even when h is arbitrarily large. The plane symmetric PP wave travels at the speed of light, and has a transverse, traceless polarization tensor, no matter how strong the wave is. Therefore the assignment of the mass and spin of the graviton from general relativity is not dependent on the notion of a small fluctuation in the metric, and the mass and polarization states of the graviton are protected from any generally covariant higher order curvature terms in the equations of motion, at least in perturbation theory.

We should therefore hope that PP wave backgrounds are *nonperturbatively* exact string backgrounds. If they weren't, string theory would not be such a good candidate for a quantum theory of gravity.

Amati and Klimčik [28] showed by explicit calculation that this was true for a plane gravity wave of the form

$$ds^2 = -dU\,dV + h(U) \sum_{i=1}^{d-2} A_i\,(X^i)^2\,dU^2 + \sum_{i=1}^{d-2} (dX^i)^2 \tag{3.14}$$

where the A_i are constants. Assuming a constant dilaton background, the authors inserted this metric into the worldsheet action (2.15) for a bosonic string. They were able to calculate the conformal anomaly exactly for the plane wave metric, because the V integration in the generating functional can be done trivially to get a delta function in U_0 (the classical value of the field U) and the X^i integrals are Gaussian. Using the gauge $g_{ab} = e^{\phi}\eta_{ab}$, the goal is for the effective action to be independent of ϕ. The ϕ-dependent part of the effective action was calculated to be

$$?_{\phi} = \frac{26-d}{48\pi}\int d\sigma\, d\tau\, \left(\frac{1}{2}\phi\Box\phi - \mu^2 e^{\phi}\right) - \frac{1}{8\pi}(\sum_i A_i)\int d\sigma\, d\tau\, h(U_0)(\partial U_0)^2\,\phi, \quad (3.15)$$

so conformal invariance requires $d = 26$ and $\sum A_i = 0$. Their analysis yielded the same results for superstrings with $d = 10$. Therefore we know that the spin of the graviton is protected nonperturbatively at the level of conformal field theory in string theory. A tadpole power-counting argument in [28] showed that this nonperturbative vanishing of the conformal anomaly can be maintained even after extending the form of the metric to the general PP wave (3.1) satisfying $\Box h(X,U) = 0$, so one can say that even at the level of nonperturbative string theory, PP waves represent the free-field limit of gravity.

Plane gravity waves with $\mathrm{Tr}h \neq 0$ aren't pure gravity waves; the non-traceless part of such a wave must come from other massless fields in the theory. Güven [34] found plane wave solutions of the $d = 10$ $N = 1$ supersymmetric Einstein-Yang-Mills effective action for heterotic string theory with vector and antisymmetric tensor fields linear in the transverse coordinate X^i were not corrected at higher order in α' and preserve the $N = 1$ supersymmetry of the action.

A PP wave metric, having a null Killing vector, gives rise to a conserved chiral current on the worldsheet, but the most general PP wave metric spoils the chiral symmetry with an extra off-diagonal component $\bar{A}_i$. The Lagrangian is [8]

$$L = \partial u \bar{\partial} v + K(u,x)\partial u \bar{\partial} u + 2\left(A_i(x,u)\partial u \bar{\partial} x^i + \bar{A}_i(x,u)\bar{\partial} u \partial x^i\right) + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R}\Phi. \quad (3.16)$$

Using $\mathcal{F}_{ij} = \partial_i A_j - \partial_j A_j$ and $\bar{\mathcal{F}}_{ij} = \partial_i \bar{A}_j - \partial_j \bar{A}_j$, the conditions for conformal invariance at lowest order in α' are

$$\partial_i \mathcal{F}^{ij} = 0, \quad \partial_i \bar{\mathcal{F}}^{ij} = 0, \quad \Phi = \phi(u), \quad (3.17)$$

$$-\frac{1}{2}\,\partial^2 K(u,x) + \partial^i\partial_u(A_i + \bar{A}_i) + 2\partial_u^2\,\phi + O(\alpha'^{s+k}\partial^s\mathcal{F}\partial^k\bar{\mathcal{F}}) = 0. \qquad (3.18)$$

The higher order corrections depend only on the transverse spatial derivatives of the field strengths, therefore if the equations (3.17) and (3.18) are satisfied at lowest order in α' and the field strengths are independent of the transverse coordinates, then the PP wave metric is an exact conformally invariant background for string theory.

There are two other limits in which (3.16) yields an exact string background. The solution $A_i = -\bar{A}_i = -\frac{1}{2}\epsilon_{ij}x^j$ with $K = -x_i x^i$ and ϕ constant in $d = 4$ gives the non-semisimple WZW model found in [35]. Since $A_i = G_{ui} + B_{ui}$ and $\bar{A}_i = G_{ui} - B_{ui}$, these solutions correspond to the vector field coming entirely from the antisymmetric tensor field. Supersymmetric string wave [36, 37] solutions have $\bar{A}_i = 0$ so that the vector field comes equally from the string-frame metric and antisymmetric tensor field. It was shown in [8] that supersymmetric string waves provide exact backgrounds for bosonic and heterotic string theories.

3.1.3 How Do Test Particles Propagate Through the Focal Plane?

As was discussed in the first chapter, geodesic focusing is the phenomenon of interest behind black hole physics, including the entropy-like behavior of the area of the event horizon. Penrose [31] showed that plane gravity waves focus null geodesics in a manner that results not in an event horizon, but a Killing-Cauchy horizon beyond which the initial value problem breaks down. Because plane waves are exact string backgrounds, and exhibit geodesic focusing, they provide a good laboratory for comparing test particle and test string behavior.

Truncating the metric (3.10) to $d = 3$ yields

$$ds^2 = -dU\,dV - h(U)X^2\,dU^2 + dX^2, \qquad \begin{aligned} h(U) &= \left(\frac{\pi}{2\Delta U}\right)^2 \quad 0 < U < \Delta U, \\ &= 0 \quad U < 0,\ U > \Delta U, \end{aligned} \qquad (3.19)$$

where $\Delta U = \pi a/2$. In $d = 3$ if $\mathrm{Tr}h = 0$ then $h(U) = 0$, so this metric cannot be a vacuum solution, it must be coupled to other massless fields. (As will be shown later, the metric (3.10) with $X^2 + Y^2$ is related to the trapped region of an extreme axion-dilaton black hole, so this simple example is not unrelated to black hole physics in string theory.) We want to examine geodesic focusing in this spacetime, so that we may compare it with the focusing of test strings in the next section.

The geodesic equations are

$$\ddot{V} + \frac{\partial h}{\partial U} X^2 \dot{U}^2 + 4h(U)\, \dot{U}\, X\dot{X} = 0, \quad \ddot{U} = 0, \quad \ddot{X} + h(U)\, \dot{U}^2\, X = 0, \tag{3.20}$$

and the null condition gives

$$\dot{U}\,\dot{V} + h(U)X^2\,\dot{U}^2 - \dot{X}^2 = 0. \tag{3.21}$$

The above equations are invariant under rescaling the affine parameter by $\tau \to \alpha\tau$, so the paths of massless test particles are the same for particles of all energies, a general feature of Einstein relativity. Therefore it is convenient and proper to choose for the above spacetime $U = \tau$, after which the equations are easily solved. Null geodesics passing through this wave take the form

$$\begin{array}{rrcl}
U < 0 & X(\tau) & = & p_0\,\tau, \quad V(\tau) = p_0^2\,\tau, \\
0 < U < \Delta U & X(\tau) & = & c_0\, \sin(\omega_0\,\tau) + d_0\, \cos(\omega_0\,\tau), \\
 & V(\tau) & = & \int \dot{X}(\tau)^2\, d\tau + v_{02}, \\
U > \Delta U & X(\tau) & = & p_f\,\tau + x_f, \quad V(\tau) = p_f^2\,\tau + v_f,
\end{array} \tag{3.22}$$

where $\omega_0 = \frac{\pi}{2\Delta U}$ and the parameter p_0 represents the test particle momentum in the x-direction. The six constants above are determined by the continuity of $X(\tau)$, $\dot{X}(\tau)$, and $V(\tau)$ (but not $\dot{V}(\tau)$) across the surfaces $U = 0$ and $U = \Delta U$. The geodesics were plotted below using *Mathematica*.

In figures 3.1 and 3.2 the plane wave passes between $U = 0$ and $U = 200$. After the null geodesics focus at $U \sim 216$, they fail to determine the boundary of the causal

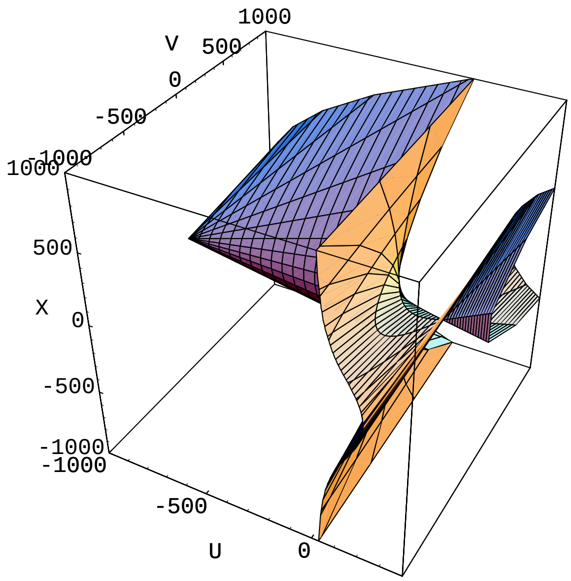

Figure 3.1: Null geodesics from $(-1000, 0, 0)$ pass through the wave between $U = 0$ and $U = 200$ and are focused to a point. A similar picture was shown by Penrose in [31].

future of the initial event, and the light cone is expanded out along the direction *parallel* to the wave. Null geodesics from an event at $U = -\infty$ would focus exactly at $f = \Delta U = 200$.

Because of the extreme distortion of the light cone by the plane wave, every spacelike hypersurface in this spacetime intersects at least one null geodesic more than once. A global Cauchy surface cannot be defined, but for local calculations one can define a partial Cauchy surface and compute field theory Bogolyubov coefficients. Gibbons [38] showed that although the quantum theory of a scalar field in a single plane wave background is easily calculable and yields no particle creation, the theory itself becomes ill-defined at the Cauchy horizon.

3.1.4 How Do Test Strings Propagate Through the Focal Plane?

String propagation through gravity waves has been fruitfully explored in the past in the context of scattering amplitudes. A notion of "stringy singularity" based on

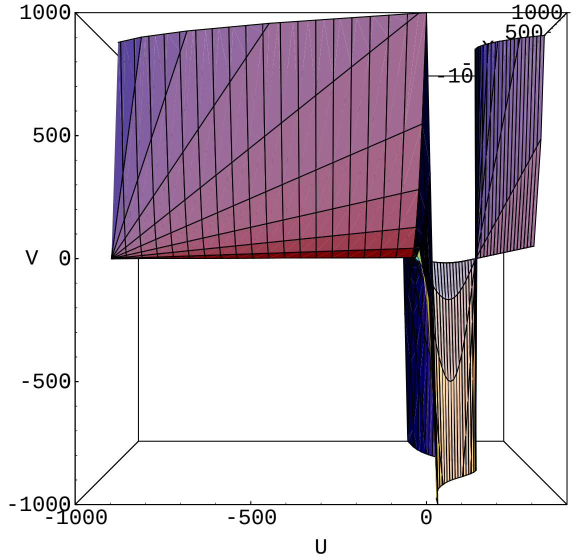

Figure 3.2: The same as figure 3.1, looking down the x-axis.

infinite string excitation was examined by Horowitz and Steif [32, 33], Sanchez and de Vega [39] and others. While this looks like a good operational definition of singular string propagation, it doesn't shed light on the nature of causal volume delimitation in string theory and the potential physically-relevant pathologies that could occur when causal volumes are delimited by solutions to worldsheet rather than wordline mathematics. For this reason, we step back to that earlier work and re-examine it from a geodesic rather than an S-matrix point of view.

In extending the geodesic picture to string theory, the test particle geodesics that define the boundary of the test particle light cone are represented by the zero mode of the string. This is the center-of-mass coordinate that obeys that standard geodesic equation. If we only look at the geodesics of test string zero modes, then the singularities and causal pathologies of general relativity remain with minor modifications (in the cases where we trust the background spacetime approximation, at least.)

This is basically telling us that test particles propagate in "stringy general relativity" rather similarly to how they propagate in ordinary general relativity. The biggest difference comes from the rescaling of the stringy affine parameter relative to the test particle affine parameter by $e^{2\Phi}$. This has a noticeable effect mainly in the case of a dilaton black hole with purely magnetic charge. [32]

If we take all string modes into account, the counterpart to a geodesic equation in string theory becomes

$$\Box X^\mu + ?^{\,\mu}_{\nu\lambda}(X(\tau,\sigma))\partial_a X^\nu \partial^a X^\lambda = 0. \qquad (3.23)$$

In the single plane wave metric (3.39) the equations reduce to

$$\begin{aligned} \ddot{V} - V'' + \frac{\partial h}{\partial U} X^2 (\dot{U}^2 - U'^2) + 4h(U)\, X\, (\dot{U}\dot{X} - U'X') = 0, \\ \ddot{U} - U'' = 0, \quad \ddot{X} - X'' + h(U)\, (\dot{U}^2 - U'^2)\, X = 0. \end{aligned} \qquad (3.24)$$

The mass shell constraints come from the vanishing of the worldsheet stress tensor and automatically satisfy the first equation above. If we choose the gauge $U = U(\tau)$ we get

$$\dot{U}\,\dot{V} = -h(U)X^2\dot{U}^2 + (\dot{X}^2 + X'^2), \quad \dot{U}V' = 2\,\dot{X}\,X' \qquad (3.25)$$

and the remaining second order equation reduces to

$$\ddot{X} - X'' + h(U)\,\dot{U}^2\,X = 0. \qquad (3.26)$$

These equations don't allow the rescaling of string proper affine parameter, so if we further fix the gauge by $U = p\tau$ and try to rescale p out of the equations through $\tau' = p\tau$, factors of p end up in the X' terms. Setting $L_{string} = 1$ and expanding in open string modes using $X(\tau,\sigma) = \sum X_n(\tau)\cos(n\sigma)$, we get

$$\ddot{X}_n(\tau) = -\left(\frac{n^2}{p^2} + h(U)\right) X_n(\tau), \qquad (3.27)$$

with $V(\tau,\sigma) = \sum V_n(\tau)\cos(n\sigma)$ obtainable by straightforward integration of (3.25).

Assigning $\omega_n = \sqrt{\frac{n^2}{p2} + h_0}$ and $\omega_0 = \sqrt{h_0} = \pi/2\Delta U$, it is convenient to expand in the basis:

$$\begin{aligned} U < 0 \quad X_0(\tau) &= p_0\,\tau, \quad X_n(\tau) = a_n\cos(n\tau/p) + b_n\sin(n\tau/p) \qquad (3.28) \\ 0 < U < \Delta U \quad X_0(\tau) &= c_0\,\sin(\omega_0\,\tau) + d_0\,\cos(\omega_0\,\tau), \end{aligned}$$

$$X_n(\tau) \;=\; c_n\,\sin(\omega_n\,\tau) + d_n\,\cos(\omega_n\,\tau)$$

$$U > \Delta U \quad X_0(\tau) \;=\; p_f\,\tau + x_f, \quad X_n(\tau) = e_n \cos(n\tau/p) + f_n \sin(n\tau/p)$$

with $U = \tau$ and the $V_n(\tau)$ obtained by integrating (3.25).

This is related to the more common expansion for strings in flat spacetime

$$U = p\tau, \quad X(\tau, \sigma) = X_0(\tau) + i \sum_n \frac{\alpha_n}{n}\, e^{-in\tau}\, cos(n\sigma) \tag{3.29}$$

through

$$a_n = -\frac{2p}{n}\mathrm{Im}\,\alpha_n, \quad b_n = \frac{2p}{n}\mathrm{Re}\,\alpha_n. \tag{3.30}$$

Applying continuity equations across the waves boundaries at $U = 0$ and $U = \Delta U$ gives the linear transformation between incoming and outgoing mode constants (a_n, b_n) and (e_n, f_n):

$$\begin{aligned} e_n \;&=\; a_n \{\cos(n\Delta U/p)\cos(\omega_n\,\Delta U) + (\frac{\omega_n\,p}{n})\sin(n\Delta U/p)\sin(\omega_n\,\Delta U)\} \\ &+\; b_n \{-\sin(n\Delta U/p)\cos(\omega_n\,\Delta U) + (\frac{n}{\omega_n\,p})\cos(n\Delta U/p)\sin(\omega_n\,\Delta U)\}, \\ f_n \;&=\; a_n \{\sin(n\Delta U/p)\cos(\omega_n\,\Delta U) - (\frac{\omega_n\,p}{n})\cos(n\Delta U/p)\sin(\omega_n\,\Delta U)\} \\ &+\; b_n \{\cos(n\Delta U/p)\cos(\omega_n\,\Delta U) + (\frac{n}{\omega_n\,p})\sin(n\Delta U/p)\sin(\omega_n\,\Delta U)\}. \end{aligned} \tag{3.31}$$

Transforming back to the basis (α_n, α_{-n}) by undoing (3.30), the Bogolyubov coefficients obtained match those obtained for $d = 4$ in [30], which according to the conventions used here is

$$|B_n|^2 = \frac{1}{4}\left(\frac{p}{n\omega_n}\right)^2 \omega_0^4\, \sin^2(\omega_n \Delta U). \tag{3.32}$$

It is significant that this coefficient is zero in scalar quantum field theory [38]. As Gibbons explained, there is no mixing between in and out bases in that case because there is a global null Killing vector guaranteeing that frequencies can be measured in the same way before and after the wave's passage. Strings are excited because they have extended structure and oscillate time and space. String in and out bases are

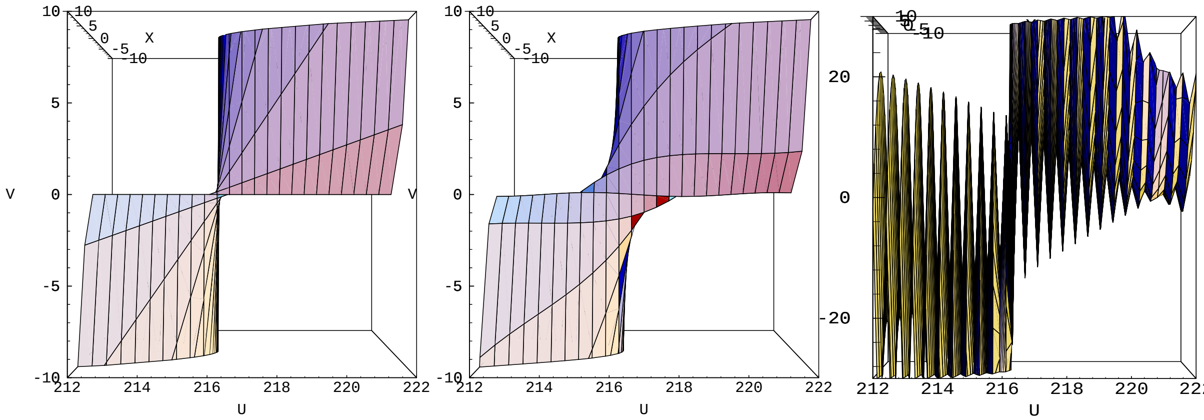

Figure 3.3: The surfaces swept out by $X(\tau, 0)$ plotted for $p = 1000, p = 2$ and $p = .01$ display the scale dependence of the string-defined light cone, behavior not present in the test particle limit.

getting mixed in outright defiance of this target space Killing vector that has such a powerful restrictive effect on quantum fields.

String motion through the wave represented by (3.39) looks the same globally as the particle motion when plotted at the same scale as in figures 3.1 and 3.2. The main difference becomes visible in the focusing region when the momentum is varied, as shown below:

Figure 3.3 shows that the focal region as determined by strings becomes smeared by strings as the momentum decreases. This does not mean that string trajectories are no longer leaving the boundary of the causal future after they cross. This still has to be true at large distances. String effects obscure the location of the focal plane but not the global effects of geodesic focusing itself.

This calculation was done using free classical strings. Integrating over the string coordinate σ reproduces the center-of-mass limit in pictured in Figure 3.2. However, strings interact, and as shown in [6], interacting strings spread information outside the center-of-mass light cone, therefore the qualitative picture in figure 3.3 agrees with the results of interacting string field theory.

To resolve the focal plane of the wave, we could examine strings at equal time with spacelike separation (according to the center-of-mass) and let the spacelike separation shrink to zero. But the commutator of string fields has a spread in distance that grows as the time scale used to resolve it shrinks. So we can't know reliably when the spacelike separation has shrunk to zero if we want to pin that knowledge down with very high time resolution. So we can't resolve the focal plane of the gravity wave using string probes.

This insight has already led to progress in understanding string thermalization near a black hole event horizon. Since geodesic focusing is also at the heart of singularity theorems, this might suggest that a spacetime singularity cannot be resolved with a string probe.

3.2 Colliding plane wave spacetimes

3.2.1 Classification and basic properties

Colliding plane wave spacetimes in $d = 4$ whose metrics satisfy the vacuum Einstein equations $R_{\mu\nu} = 0$ have been classified and understood by Yurtsever [9]. His general solution for the collision region for incoming waves with constant collinear polarization can be extended to arbitrary d in [10]. The metric can be written in the form

$$ds^2 = -e^{-N(u,v)} du\, dv + e^{-U(u,v)} \sum_{i=1}^{d-2} e^{\lambda_i V(u,v)} (dx^i)^2 \tag{3.33}$$

with the conditions

$$\sum_{i=1}^{d-2} \lambda_i = 0, \qquad \sum_{i=1}^{d-2} \lambda_i^2 = D = d - 2. \tag{3.34}$$

The equations of motion for this system are most illuminating if we make a coordinate transformation from (u, v) to (α, β) defined by

$$\alpha = e^{-\frac{D}{2} U(u,v)}, \quad \beta_u = -\alpha_u,\ \beta_v = \alpha_v. \tag{3.35}$$

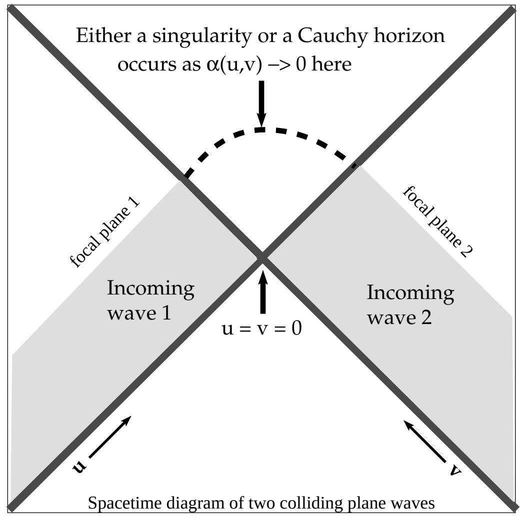

Figure 3.4: Spacetime diagram of two colliding plane waves, with the collision region shown to the future of the null surfaces $u = 0$, $v = 0$.

In the collision region $(u > 0, v > 0)$ the metric can then be written

$$ds^2 = l_1 l_2 \, \alpha^{\frac{1+D}{D}} \, e^{Q(\alpha,\beta)/2} \, (-d\alpha^2 + d\beta^2) + \alpha^{\frac{2}{D}} \sum_i^D e^{\lambda_i V(\alpha,\beta)} \, (dx^i)^2, \tag{3.36}$$

where $l_{1,2}$ are length scales determined by the incoming waves. The vacuum Einstein equations reduce to

$$V_{\alpha\alpha} + \frac{V_\alpha}{\alpha} - V_{\beta\beta} = 0, \tag{3.37}$$

$$Q_\alpha = -\frac{\alpha D}{2}({V_\alpha}^2 + {V_\beta}^2), \quad Q_\beta = -\alpha D \, V_\alpha \, V_\beta, \tag{3.38}$$

plus constraints for the initial data along $(u = 0, v)$ and $(u, v = 0)$. So the pure gravitational collinearly polarized colliding plane wave system of (3.33,3.36) in d spacetime dimensions reduces to solving the two-dimensional wave equation (3.37). The equations (3.38) are then integrable to yield $Q(\alpha, \beta)$, as shown in the appendix. Using knowledge about solutions to (3.37), the limit $\alpha \to 0$ of (3.33,3.36) can be shown to result in either an all-encompassing spacetime singularity or a nonsingular Killing-Cauchy horizon [9, 10], which will be examined in more detail ahead.

Non-vacuum colliding plane wave systems have been discovered in $d = 4$ for scalar and electromagnetic sources by Chandrasekhar and others. [40, 11, 12] Two classes of general solutions for a Maxwell-dilaton system were discovered by Breton [41]. The most interesting non-vacuum colliding plane wave metric in four spacetime dimensions (for the purposes of this dissertation) is the Bertotti-Robinson spacetime. [22] The metric in the collision region can be written

$$ds^2 = -du\,dv + \cos^2(\frac{u}{a} + \frac{v}{b})\,dx^2 + \cos^2(\frac{u}{a} - \frac{v}{b})\,dy^2, \tag{3.39}$$

with the length scales a and b determined by the incoming waves before the collision. We will be seeing more of this metric when we discuss the connection between black holes and colliding plane waves in chapter four.

3.2.2 Obtaining incoming waves from collision region solutions

The Bertotti-Robinson metric (3.39) covers the collision region of a colliding plane wave system. Incoming single wave solutions are obtained through the Khan-Penrose prescription [42]

$$\frac{u}{a} \to \frac{u}{a} H(\frac{u}{a}), \quad \frac{v}{b} \to \frac{v}{b} H(\frac{v}{b}), \tag{3.40}$$

where $H(x)$ is the Heaviside step function. Following the Khan-Penrose prescription, the incoming wave for the region $(u > 0,\ v < 0)$ gives the metric

$$ds^2 = -du\,dv + \cos^2(\frac{u}{a})\,(dx^2 + dy^2), \tag{3.41}$$

which from (3.13) we can see has the very simple form in harmonic coordinates

$$\begin{aligned} ds^2 &= -dU\,dV - \frac{1}{a^2}\,(X^2 + Y^2)\,dU^2 + dX^2 + dY^2, \quad 0 \le U \le \Delta U \qquad (3.42) \\ &= -dU\,dV + dX^2 + dY^2, \quad U < 0, U > \Delta U, \end{aligned}$$

where $\Delta U = \pi a/2$. Note that in the limit $\Delta U \to 0$, the wave amplitude $h_0 = 1/a^2 =$

$(\pi/2\Delta U)^2$ becomes infinite while the product $\sqrt{h_0}\Delta U = \pi/2$ remains constant.

3.2.3 Asymptotic structure of colliding plane wave spacetimes

Given the pathological null geodesic focusing behavior shown for single plane waves shown in figure 3.1, one might expect that when two of these waves collide, something even more nasty might happen. Penrose and Khan gave the first explicit example of a spacetime with two exact plane waves that collide to form a spacetime singularity. [43] Tipler proved a singularity theorem [44] for colliding plane waves, and gave some explicit examples of colliding plane wave spacetimes both with and without singularities. [45, 46, 47] Finally the general asymptotic structure of singular and nonsingular pure gravity plane wave collisions were elucidated by Yurtsever [9, 48], who also extended the applicability of the singularity theorem for exact plane waves to the more physical case where the incoming waves are only nearly plane symmetric across some finite transverse region with size L_T. Yurtsever's result was extended to arbitrary spacetime dimension in [10].

As shown in the appendix, the asymptotic behavior of metric (3.33) hinges on the fact that solutions to the equation (3.37) behave like $V(\alpha,\beta) \sim \epsilon(\beta)\ln\alpha + \delta(\beta)$ as $\alpha \to 0$, which with (3.38) gives the leading behavior

$$V(\alpha,\beta) \sim \epsilon(\beta)\ln\alpha + \delta(\beta), \quad Q(\alpha,\beta) \sim \frac{D}{2}\epsilon^2(\beta)\ln\alpha + \mu(\beta). \tag{3.43}$$

If we put this into (3.36) and change time coordinates via $t = \alpha^{q/2}$ we get

$$\begin{aligned} ds^2 &= e^{-\mu(\beta)/2} l_1 l_2 (-dt^2 + t^{2p_\beta} d\beta^2) + \sum_{i=1}^{D} e^{-\lambda_i \delta(\beta)}\, t^{2p_i}\,(dx^i)^2 \\ p_\beta &\equiv (q-2)/q, \qquad q \equiv \tfrac{D}{4}\epsilon^2(\beta) + \tfrac{D+1}{D} \\ p_i &\equiv q_i/q, \qquad\quad q_i \equiv \tfrac{2}{D} - \lambda_i \epsilon(\beta), \end{aligned} \tag{3.44}$$

in the limit of small time $t \to 0$.

The parameters $l_{1,2}$ are length scales on the order of the focal lengths of the incoming waves, and the functions $\epsilon(\beta)$, $\mu(\beta)$, and $\delta(\beta)$ are determined by amplitudes of the incoming waves. The functions $\mu(\beta)$ and $\delta(\beta)$ are irrelevant to the structure of the singularity but the value of $\epsilon(\beta)$ is important. The formula relating $\epsilon(\beta)$ to the incoming wave amplitudes is given in the appendix.

In relation to the incoming plane waves, the new coordinate t represents the time from the initial collision to the eventual geodesic focusing on the focal plane of the colliding wave system, which is continuous with the focal planes of the incoming waves. Whether the geodesic focusing on the collision region focal plane leads to an all-encompassing spacetime singularity or the formation of a Killing-Cauchy horison and the breakdown of the initial value problem depends on the value of $\epsilon(\beta)$.

To see this more clearly, let's examine the Kasner metric [49], which controls the asymptotic limit of colliding plane gravity waves. The Kasner metric

$$ds^2 = -dt^2 + \sum_j^{d-1} t^{2p_j}(dx^j)^2 \tag{3.45}$$

is a time-dependent, spatially homogeneous solution to the vacuum Einstein equations typically used to model anisotropic cosmologies. The Kasner exponents obey the relations

$$\sum_j^{d-1} p_j = \sum_j^{d-1} p_j^2 = 1. \tag{3.46}$$

The Kasner metric has curvature

$$R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} = \frac{-8(2P_3 + P_4)}{t^4}, \tag{3.47}$$

with a spacetime singularity at $t \to 0$. The constants P_3 and P_4 are the permutation-invariant combinations of products of three and four distinct Kasner exponents, respectively. For example, in four spacetime dimensions, $P_3 = p_x\, p_y\, p_z,\ P_4 \equiv 0$.

The Kasner universe is singular as the time $t \to 0$, when space dimensions with a positive Kasner exponent are infinitely compressed and those with negative exponents are infinitely expanded. The only nonsingular solution occurs when a single Kasner exponent $p_k = 1$ and, by virtue of the relations (3.46), all the other Kasner exponents $p_i = 0$ for $i \neq k$. In this limit $P_3, P_4 = 0$ and the spacetime is flat, with the metric

$$ds^2 = -dt^2 + t^2\,(dx^k)^2 + \sum_{i \neq k} (dx^i)^2. \tag{3.48}$$

Note the similarity to the Rindler metric

$$ds^2 = -(\xi^k)^2\,d\tau^2 + (d\xi^k)^2 + \sum_{i \neq k} (d\xi^i)^2, \tag{3.49}$$

if we take $t \to i\xi^k, x^k \to \tau$. The Rindler metric (3.49) is a wedge of Minkowski spacetime as seen by an observer uniformly accelerating in the x^k direction. The flat Kasner metric (3.48) is a wedge of Minkowski spacetime *behind the event horizon* of a Rindler observer. What appears to the accelerating observer (3.49) as the Rindler event horizon where $\frac{\partial}{\partial\tau}$ becomes null, looks from the point of view of the metric (3.48) like a Killing-Cauchy horizon where the spacelike Killing vector $\frac{\partial}{\partial x^k}$ becomes null, spoiling the initial value problem. This relationship is one aspect of the general relationship between colliding plane wave spacetimes and black holes that will be explored in detail in chapter four.

Going back to the colliding plane wave asymptotic metric (3.44), we see that the nonsingular asymptotic behavior is only possible if the longitudinal exponent $p_\beta = 0$, one of the transverse exponents $p_k = 1$, and all of the other $p_j = 1$ for $k \neq j$, which requires

$$\epsilon(\beta) = \pm 2\sqrt{D-1}/D, \quad \lambda_k = \mp\sqrt{D-1}, \quad \lambda_j = \pm 1/\sqrt{D-1}. \tag{3.50}$$

Almost satisying these conditions will not prevent the singularity. These conditions on the incoming waves must be satisfied precisely. The function $\epsilon(\beta)$ is determined from the incoming waves data according to the integral (6.12). Yurtsever [9] showed that the nonsingular solutions ensured by (3.50) are unstable to arbitrarily small plane-symmetric perturbations in the incoming waves, and therefore a spacetime singularity is the generic outcome of the collision.

The colliding exact plane wave problem is of course unphysical, because the waves are infinite in transverse extent. However, Yurtsever also proved that two gravitational waves that are approximately plane symmetric across some finite transverse region with size $\sim L_T$, with the curvature falling off in some arbitrary way outside of that region, will collide to form a singularity if their focal lengths are much smaller than this transverse size. [48] This argument is based on causality — there is not enough proper time between the collision and the formation of the singularity for finite-size effects to propagate to the center of the wave. It is not known whether an event horizon forms around the singularity.

This analysis was all based on pure gravity; there are singular and nonsingular Einstein-Maxwell-dilaton colliding plane wave solutions known [41] and their asymptotic behavior was investigated in [14] and shown to have Kasnerian singular and nonsingular limits. Nonsingular solutions to Einstein's equations in the presence of axion, dilaton and N $U(1)$ fields were explored in [14] and will be explained in depth in chapter four, when we explore the relationship between black holes and colliding plane waves.

3.2.4 String perturbations of colliding plane wave spacetimes

Unlike the single plane wave spacetimes disucssed in the previous section, colliding plane wave metrics do not in general provide exact backgrounds for test string propagation, although we will see exceptions to this rule in the next chapter. In the collision region of the generic colliding plane wave spacetime, there are no null Killing vectors at all (except at the focal plane where one or more spacelike Killing vector can become null), and all of the potential covariant higher-derivative terms in the effective action become nonzero. One way to probe higher- order string effects on a colliding plane wave background geometry is to solve the string perturbed equations of motion in the Kasner metric which represents the leading behavior of a colliding plane wave spacetime (3.44) near the singularity.

To lowest order in string tension α', the spacetime effective action for the field $g_{\mu\nu}(X)$ from (2.15) is

$$S_{eff} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-g}\, e^{-2\Phi} \left(R + 4D_\mu \Phi D^\mu \Phi + \frac{\lambda}{2} R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} + O(\lambda^2) \right), \quad (3.51)$$

where $\lambda = \alpha'/2\, , \alpha'/4$ for the bosonic and heterotic string respectively. [1]

The conformal transformation that gives the standard gravitational kinetic term is

$$g_{\mu\nu} \to \hat{g}_{\mu\nu} = e^{-\frac{4}{d-2}\Phi} g_{\mu\nu}, \quad (3.52)$$

and the resulting diagonalized action is

$$S_{eff} = \frac{1}{16\pi G_d} \int d^d x \sqrt{-\hat{g}}\, (\hat{R} - \frac{4}{d-2} D_\mu \Phi D^\mu \Phi + \frac{\lambda}{2}\, e^{-\frac{4}{d-2}\Phi}\, \hat{R}_{\mu\alpha\beta\gamma} \hat{R}^{\mu\alpha\beta\gamma} + O(\lambda^2)). \quad (3.53)$$

In the following, we will assume that $\Phi = \phi_0 + \lambda\phi(x)$ and set $\phi_0 = 0$ since the constant mode can be absorbed into the ratio of α'/G_d. We will solve the equations for $g_{\mu\nu}$ and ϕ to $O(\lambda)$ in the colliding plane wave geometry, and then rescale to get the physical metric $\hat{g}_{\mu\nu}$. The equations of motion for $g_{\mu\nu}$ and ϕ at $O(\lambda)$ are

$$\begin{aligned} R_{\mu\nu} + 2D_\mu D_\nu \Phi + \lambda R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} &= O(\lambda^2), \quad (3.54) \\ D_\mu D^\mu \Phi - 2D_\mu \Phi D^\mu \Phi - \frac{\lambda}{4} R_{\mu\alpha\beta\gamma} R^{\mu\alpha\beta\gamma} &= O(\lambda^2), \end{aligned}$$

We'll seek solutions of the form

$$ds^2 = -e^{2N(t)}\, dt^2 + \sum_{i=1}^{\hat{D}} e^{2X_i(t)}\, (dx^i)^2, \quad \hat{D} = d - 1, \quad (3.55)$$

where the perturbations are parametrized by

$$N(t) = \lambda m_1/t^2, \quad X_i(t) = p_i \ln(t/t_0) + \lambda c_{i1}/t^2, \quad \phi(t) = \lambda f_1/t^2, \quad (3.56)$$

[1] The lowest order correction for the type II superstring is $\sim \lambda^3 R^4$.

and the $d-1$ exponents $\{p_i\}$ satisfy

$$\sum_{i=1}^{\hat{D}} p_i = \sum_{i=1}^{\hat{D}} {p_i}^2 = 1. \tag{3.57}$$

The quantities that give the most information about the perturbative "approach" to the singularity (how the lowest order string effects begin to "turn on" as $t \to 0$) are the sign of the dilaton coefficient f_1 and sign of the Ricci tensor $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu$ along null geodesics with tangent vector ξ^μ.

The equations of motion can be solved to give

$$f_1 = P_3 + \frac{1}{2}\,P_4. \tag{3.58}$$

The quantities P_3, P_4 are the permutation invariant products of three and four distinct (in index, not value) Kasner exponents. They vanish identically when the exponents $\{p_i|\, i = 1, \ldots,\ \hat{D}\}$ are any permutation of $(1, 0, \ldots,\ 0)$.

The dimensionless parameter that measures the relative importance of classical string compared with quantum loop effects on some background geometry is $(\alpha'/{G_d}^{2/d-2})\, e^{-4\Phi/d-2}$. This suggests that the relationship between the string scale and the Planck scale depends on the spacetime curvature and any other background fields which act as sources for the dilaton. In the appendix it is shown that f_1 is negative for all combinations of Kasner exponents in all dimensions. This negative sign is in agreement with the analogous calculation performed in Schwarzschild spacetime. [50]

As for the sign of $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu$, let's look at null geodesics with momentum k_1 in the direction we'll call x^1. The tangent vector ξ heading towards $t = 0$ has nonzero components $\xi = (-|k_1|(t_0/t)^{p_1}, k_1(t_0/t)^{2p_1})$. From (3.54), (3.52) and the relation between $\hat{R}_{\mu\nu}\hat{\xi}^\mu\hat{\xi}^\nu$ and $R_{\mu\nu}\xi^\mu\xi^\nu$ for $\xi^\mu\xi_\mu = 0$ one can show that to $O(\lambda)$

$$\begin{aligned} \hat{R}_{\mu\nu}, \hat{\xi}^\mu\hat{\xi}^\nu &= -\lambda R_{\mu\alpha\beta\gamma} {R_\nu}^{\alpha\beta\gamma}\, \xi^\mu\xi^\nu \\ &= -\frac{4\lambda k_1^2}{t^4}(\frac{t_0}{t})^{2p_1}\left(p_1^2\,(1-p_1) + (P_3 + 2P_4)\right). \end{aligned} \tag{3.59}$$

This vanishes for $d = 4$ because in three spatial dimensions $\hat{p}^2 - \hat{p}^3 = -P_3$ for any choice of the longitudinal direction $x^{\hat{d}}$, and P_4 is zero for $d < 5$. It can be shown that $\hat{R}_{\mu\nu}\hat{\xi}^\mu\hat{\xi}^\nu$ can be either negative or positive, because the first term above is always positive while the second one is always negative, and the two terms are comparable in magnitude. So perturbation theory doesn't show any unique focusing or defocusing behavior for the test particle limit of string theory.

Chapter 4 Plane gravity waves and black holes

4.1 Schwarzschild colliding plane waves

4.1.1 The collision region metric

The Schwarzschild metric is only static in the regions where $\frac{\partial}{\partial t}$ is timelike. Inside the trapped region of a Schwarzschild black hole the metric can be written:

$$ds^2 = -\frac{r}{(2M - r)} \, dr^2 + r^2 \, d\theta^2 + \frac{(2M - r)}{r} \, dt^2 + r^2 \sin^2 \theta \, d\phi^2. \qquad (4.1)$$

In the trapped region, the metric is quite violently dependent on the timelike radial coordinate, while $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$ act as a pair of spacelike commuting Killing vectors. A violently time-dependent spacetime with two commuting spacelike Killing vectors is also a potential description of the spacetime of two colliding plane symmetric gravitational waves. This idea was first recognized and explored in the in the Einstein-Maxwell limit by Chandrasekhar [40]. A colliding plane wave metric locally isometric to the interior of a Schwarzschild black hole was obtained by Ferrari, Ibañez and Bruni [11, 12]. The direct transformation from a Schwarzschild black hole to a colliding plane wave spacetime was described by Yurtsever [13]

The metric (3.33) for the interaction region of two colliding, collinearly polarized, plane symmetric gravity waves becomes for $d = 4$

$$ds^2 = -e^{-N(u,v)} \, du \, dv + e^{-U(u,v)} \left(e^{V(u,v)} \, dx^2 + e^{-V(u,v)} \, dy^2 \right). \qquad (4.2)$$

The metric (4.1) can be put in the form of the metric (4.2) using either of the two coordinate transformations [13]

$$\begin{aligned} r &\rightarrow r_\pm(u,v) = M(1 \pm \sin(\frac{u}{a} + \frac{v}{b})), \quad \theta \rightarrow \frac{\pi}{2} \pm (\frac{u}{a} - \frac{v}{b}), \\ t &\rightarrow x, \quad \phi \rightarrow \frac{y}{M}, \end{aligned} \tag{4.3}$$

yielding the metrics

$$\begin{aligned} ds^2 = &- \frac{4M^2}{ab}\left(1 \pm \sin(\frac{u}{a} + \frac{v}{b})\right)^2 du\, dv + \frac{\cos^2(\frac{u}{a} + \frac{v}{b})}{(1 \pm \sin(\frac{u}{a} + \frac{v}{b}))^2} dx^2 \\ &+ \cos^2(\frac{u}{a} - \frac{v}{b})\left(1 \pm \sin(\frac{u}{a} + \frac{v}{b})\right)^2 dy^2. \end{aligned} \tag{4.4}$$

The coordinate transformation $r \rightarrow r_\pm(u,v)$ allows a choice of two boundary conditions for the value of r on the collision region focal plane at $u/a + v/b = \pi/2$. The $(+)$ choice gives $r = 2M$ at the focal plane, with the curvature invariant $R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta} = 3/4M^4$, matching the value at the event horizon of the black hole. In other words, the $(+)$ choice matches the colliding wave focal plane to the black hole event horizon. The $(-)$ choice leads to $r = 0$ at the focal plane, and there is a curvature singularity there as well. So the $(-)$ choice in the coordinate transformation (4.3) maps the colliding plane wave focal plane to the singularity at $r = 0$.

As for the values of the length parameters (a, b), if we want the metric (4.4) to reduce to Minkowski spacetime at $u = v = 0$, as it should if it represents the collision of two plane waves propagating against a flat background, then we introduce the constraint $ab = 4M^2$ on the incoming waves. The average focal length of the collision region is then $f = \sqrt{f_1 f_2} = \pi\sqrt{ab}/2 = \pi M$.

4.1.2 Obtaining the incoming waves

The metric (4.4) applies to the $u > 0, v > 0$ region of a colliding plane wave spacetime. The Khan-Penrose prescription (3.40) applied to the metric (4.4) gives the incoming waves. The incoming wave in the region $u>0, v<0$ can be written

$$ds^2 = -\left(1 \pm \sin(\frac{u}{a})\right)^2 du\, dv + \frac{\cos^2(\frac{u}{a})}{(1 \pm \sin(\frac{u}{a}))^2}\, dx^2 + \cos^2(\frac{u}{a}) \left(1 \pm \sin(\frac{u}{a})\right)^2 dy^2, \quad (4.5)$$

which can be put in the Rosen form (3.5) by a coordinate transformation $u \to \omega(u)$.

This metric has $R_{\mu\alpha\nu\beta}R^{\mu\alpha\nu\beta} = 0$ as is true for all PP waves. But the Weyl tensor component

$$C_{uxux} = \frac{-3\cos^2(\frac{u}{a})}{a^2\,(1 \pm \sin(\frac{u}{a}))^3} \quad (4.6)$$

blows up as $U \to \pi a/2$ for the $(-)$ waves, showing that the $(-)$ incoming waves are singular in some sense before they collide.

In almost-harmonic coordinates the metric becomes

$$ds^2 = -(1 \pm \sin\left(\frac{U}{a}\right))^2\, dU dV - \frac{3}{a^2\,(1 \pm \sin\left(\frac{U}{a}\right))}\,(X^2 - Y^2)\, dU^2 + dX^2 + dY^2, \quad (4.7)$$

which shows that these plane waves have constant traceless polarization, as we would expect from a pure gravity wave.

4.2 Axion-dilaton colliding plane waves

4.2.1 The collision region metric

The Schwarzschild metric can be arrived at from the axion-dilaton metric (2.24) by taking the charge to zero, which sends $r_0 \to M$ and $\Upsilon \to 0$. Comparing (2.25) and (4.3), it is easy to deduce that the the coordinate transformation from (r, θ, t, ϕ) to (u, v, x, y) that transforms the trapped region of an axion-dilaton black hole to a colliding plane wave system is given by [1]

$$\begin{aligned} r &\to M \pm r_0 \sin(\frac{u}{a} + \frac{v}{b}), \qquad \theta \to \frac{\pi}{2} \pm \left(\frac{u}{a} - \frac{v}{b}\right), \\ t &\to x\, r_0/(M^2 - |\Upsilon|^2)^{\frac{1}{2}}, \qquad \phi \to y/(M^2 - |\Upsilon|^2)^{\frac{1}{2}}. \end{aligned} \quad (4.8)$$

[1] We're breaking the cyclic boundary conditions for ϕ again but we'll restore them later.

The axion-dilaton colliding plane wave metric now takes the form

$$\begin{aligned} g_{uv} &= \frac{-2}{ab}\left(\left(M \pm r_0\, \sin(\frac{u}{a}+\frac{v}{b})\right)^2 - |\Upsilon|^2\right), \\ g_{xx} &= \frac{(M^2-|\Upsilon|^2)\, \cos(\frac{u}{a}+\frac{v}{b})^2}{\left(M \pm r_0\, \sin(\frac{u}{a}+\frac{v}{b})\right)^2 - |\Upsilon|^2} \\ g_{yy} &= \cos^2(\frac{u}{a}-\frac{v}{b})\, \frac{\left(\left(M \pm r_0\, \sin(\frac{u}{a}+\frac{v}{b})\right)^2 - |\Upsilon|^2\right)}{M^2-|\Upsilon|^2}. \end{aligned} \tag{4.9}$$

If we want $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$, we have to constrain the length parameters a and b to satisfy $ab = 4(M^2 - |\Upsilon|^2)$. This constraint is significant because a and b measure the focal lengths of the incoming waves, and $f = \pi\sqrt{ab}/2$ is the effective focal length of the colliding wave system. The condition $M \geq |\Upsilon|$ is a supersymmetry bound that helps enforce cosmic censorship in the black hole system. This bound in the colliding plane wave system tells us that the effective focal length of the colliding axion-dilaton plane wave system $f \geq 0$ and only approaches zero in the singular extreme dilaton limit $M = |\Upsilon|$. This looks and acts like a supersymmetric enforcement of cosmic censorship, although it was derived by asking for spacetime to be exactly flat before the arrival of each incoming wave. This is one of the interesting parallels between colliding plane wave systems and black holes.

Abbreviating $r_\pm(u,v) = M \pm r_0\, \sin(u/a + v/b)$, the axion and dilaton fields become

$$\psi(u,v) = \frac{\psi_0\, (\Delta^2 + (\Sigma + r_\pm(u,v))^2) - 2e^{-2\phi_0}\, \Delta\, r_\pm(u,v)}{\Delta^2 + (\Sigma + r_\pm(u,v))^2} \tag{4.10}$$

and

$$e^{-2\phi(u,v)} = e^{-2\phi_0}\, \frac{r_\pm(u,v)^2 - (\Sigma^2 + \Delta^2)}{\Delta^2 + (\Sigma + r_\pm(u,v))^2}. \tag{4.11}$$

Transforming from (t,ϕ) to (x,y) by (4.8), the N $U(1)$ potentials with electric and magnetic charges $(Q^{(n)}, P^{(n)})$ are transformed from $(A_t^{(n)}, A_\phi^{(n)})$ to

$$A_x^{(n)} = \frac{e^{\phi_0}\, r_0\, \left(P^{(n)}\, \Delta + Q^{(n)}\, (\Sigma + r_\pm(u,v))\right)}{\sqrt{M^2 - |\Upsilon|^2}\, \left(r_\pm(u,v)^2 - \Delta^2 - \Sigma^2\right)}, \tag{4.12}$$

$$A_y^{(n)} = -\frac{e^{\phi_0}\, P^{(n)}\, \cos(\frac{u}{a} - \frac{v}{b})}{\sqrt{M^2 - |\Upsilon|^2}}. \tag{4.13}$$

The value of $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ evaluated at the $(\pm)$ focal planes at $u/a + v/b \to \pi/2$ is equal to value of $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda}$ for the equivalent axion-dilaton black hole, evaluated at the horizons $r_\pm = M \pm r_0$

$$R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} = \frac{8\;(M^4 + 4\,M^2\,{r_0}^2 + 12\,M\,{r_0}^3 + 7\,{r_0}^4 - 2\,M^2\,|\Upsilon|^2 + 2\,{r_0}^2\,|\Upsilon|^2 + |\Upsilon|^4)}{(r_\pm^2 - |\Upsilon|^2)^4}. \tag{4.14}$$

This quantity is finite except for the $(-)$ solution in two limits: the Schwarzschild limit ($\Upsilon = 0$, $r_- = 0$), and the singular dilaton black hole limit ($r_- = |\Upsilon| = M$), which is also the zero focal length limit of the system.

4.2.2 Spacetime structure near the focal plane

Axion-dilaton colliding plane waves don't obey the vacuum Einstein equations because the matter fields contribute to a nonzero stress tensor. However, the metric obtained through the transformation (4.8) fits the form of the metric (3.36) for $d = 4$ and the coordinate transformation (3.35) is still valid. (This transformation determines the existence of a foliation of the interaction region into spacelike hypersurfaces $\alpha =$ constant and works for $R_{\mu\nu} \neq 0$ as long as the plane waves are collinearly polarized.)

After the transformsation we end up in (α, β) coordinates with the metric

$$ds^2 = l_1 l_2\, \alpha^{-\frac{1}{2}}\, e^{Q(\alpha,\beta)/2}\, (-d\alpha^2 + d\beta^2) + \alpha(e^{V(\alpha,\beta)}\, dx^2 + e^{-V(\alpha,\beta)}\, dy^2\,). \tag{4.15}$$

Remarkably enough (but not so remarkable once one recalls that this is still essentially a two-dimensional problem), the functions $V(\alpha, \beta)$ and $Q(\alpha, \beta)$ still satisfy (3.43) in the limit $\alpha \to 0$. Therefore, the Kasner asymptotic limit also applies to axion-dilaton colliding plane waves.

The coordinate transformation (3.35) for the metric under consideration can be solved exactly, giving

$$\alpha(u,v) = \frac{1}{2}\left(\cos\frac{2u}{a} + \cos\frac{2v}{b}\right), \quad \beta(u,v) = \frac{1}{2}\left(-\cos\frac{2u}{a} + \cos\frac{2v}{b}\right), \tag{4.16}$$

and is invertible to give $(u(\alpha,\beta), v(\alpha,\beta))$. If we substitute this back into (4.9) and take the limit $\alpha \to 0$ we get

$$\begin{aligned} ds^2 &= \frac{(M \pm r_0)^2 - |\Upsilon|^2}{M^2 - |\Upsilon|^2}\left(\frac{-d\alpha^2 + d\beta^2}{4(1-\beta^2)}\right) \\ &+ \frac{\alpha^2}{(1-\beta^2)}\left(\frac{M^2 - |\Upsilon|^2}{(M \pm r_0)^2 - |\Upsilon|^2}\right) dx^2 + (1-\beta^2)\frac{(M \pm r_0)^2 - |\Upsilon|^2}{M^2 - |\Upsilon|^2} dy^2. \end{aligned} \tag{4.17}$$

Combining (3.35) and (3.43), we can calculate $\epsilon(\beta)$

$$\epsilon(\beta) = -\lim_{\alpha \to 0} V(\alpha,\beta)/U = \lim_{\alpha \to 0} \frac{\ln(g_{xx}/g_{yy})}{\ln(g_{xx}g_{yy})} = 1, \tag{4.18}$$

which means that these metrics are in general nonsingular. However, as $\alpha \to 0$ we have $V(\alpha,\beta) \to \epsilon(\beta)\ln\alpha + \delta(\beta)$, which for this metric is

$$\delta(\beta) = \ln\frac{(1-\beta^2)(M^2 - |\Upsilon|^2)}{(M \pm r_0)^2 - |\Upsilon|^2}. \tag{4.19}$$

This term results in a curvature singularity in the Schwarzshild limit ($r_0 = M, \Upsilon = 0$). The singularity for the extreme dilaton limit where $r_0 = 0$ and $M = |\Upsilon|$ isn't obvious in these coordinates. In the supersymmetric limit $r_0 \to 0$, $R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} \sim (ab)^{-2}$, where $ab = 4(M^2 - |\Upsilon|^2)$ measures the effective focal length of the collision region. The limit $M = |\Upsilon|$ means that either a or $b \to 0$ with the other remaining finite, so that one of the incoming waves has zero focal length, as we shall see in the next subsection, infinite amplitude, and hence the collision region has zero focal length and infinite R^2. [2]

It is important to note that the singular limits of axion-dilaton colliding plane waves do not come from a singular Kasner limit. The effective Kasner exponents

[2] The singularity at $\beta = \pm 1$ is related to the breaking of cyclic boundary conditions in (4.8) and isn't important for this analysis.

of these waves are always the flat combination ($p_x = 1, p_y = p_\beta = 0$). This metric become singular when normally benign constants in subleading terms blow up in two specific and precise limits, the Schwarzschild and extreme dilaton limits.

4.2.3 Obtaining the incoming waves

The Khan-Penrose prescription (3.40) applied to axion-dilaton colliding plane waves yields for the region ($u > 0, v < 0$)

$$\begin{aligned} g_{uv} &= -\frac{\left(M \pm r_0\,\sin(\frac{u}{a})\right)^2 - |\Upsilon|^2}{2(M^2 - |\Upsilon|^2)} \\ g_{xx} &= \cos^2(\frac{u}{a})\,\frac{(M^2 - |\Upsilon|^2)}{\left(M \pm r_0\,\sin(\frac{u}{a})\right)^2 - |\Upsilon|^2} \\ g_{yy} &= \cos^2(\frac{u}{a})\,\frac{\left(M \pm r_0\,\sin(\frac{u}{a})\right)^2 - |\Upsilon|^2}{M^2 - |\Upsilon|^2}. \end{aligned} \tag{4.20}$$

This metric can be put into harmonic coordinates but the solution for general U is too messy to be illuminating. The salient features can be seen as $U \to 0$. The metric in Rosen coordinates satisfies $g_{\mu\nu}(0) = \eta_{\mu\nu}$. In harmonic coordinates the metric is discontinuous, flat for $U \to 0^-$ and for $U \to 0^+$

$$ds^2 = -dU\,dV \quad + \quad (\frac{1}{a^2}\,(-1 + \frac{r_0^2\,(4M^2 + |\Upsilon|^2)}{(M^2 - |\Upsilon|^2)^2})\,X^2 \tag{4.21}$$

$$+ \quad \frac{1}{a^2}\,(-1 - \frac{r_0^2\,(2M^2 + |\Upsilon|^2)}{(M^2 - |\Upsilon|^2)^2})\,Y^2)\,dU^2 + dX^2 + dY^2. \tag{4.22}$$

The polarization tensor is in general not constant, but becomes constant in two limits: the Schwarzschild limit ($r_0 \to M, \Upsilon \to 0$) and the extreme black hole limit ($r_0 \to 0$). In the Schwarzschild limit the polarization becomes constant and traceless, signifying a pure gravity wave. In the extreme limit the polarization becomes constant but the trace is nonzero, so the Ricci tensor is also nonzero. However, the Weyl tensor vanishes for $r_0 \to 0$, signifying a scalar-electromagnetic wave with no gravitational component.

4.3 Extreme axion-dilaton colliding plane waves

On the face of it, it seems ludicrous to discuss the trapped region of an extreme black hole, because this region lives only at $r = r_+ = r_- = M$. However, if we take the colliding plane wave version of the trapped region of an axion-dilaton black hole in (4.9) and examine the $r_0 \to 0$ limit, we get a very nontrivial spacetime, the Bertotti-Robinson colliding plane wave spacetime (3.39) described previously

$$ds^2 = -\frac{4}{ab}(M^2 - |\Upsilon|^2)\, du\, dv + \cos^2(\frac{u}{a} + \frac{v}{b})\, dx^2 + \cos^2(\frac{u}{a} - \frac{v}{b})\, dy^2. \tag{4.23}$$

For axion-dilaton black holes the limit $r_0 \to 0$ corresponds to the saturation of SUSY bounds for mass and central charge in the background supergravity theory and fixed values for the axion and dilaton fields at the extreme horizon. In order to have $g_{\mu\nu}(0,0) = \eta_{\mu\nu}$ we still need the constraint $ab = 4(M^2 - |\Upsilon|^2)$.

The proper time $\Delta\tau$ for a freely-falling observer with no transverse momentum to cross from the point of collision at $u = v = 0$ along a path with $u/a = v/b$ to the focal plane at $u/a + v/b = \pi/2$ is

$$\Delta\tau = \frac{\pi}{2\sqrt{2}}\sqrt{M^2 - |\Upsilon|^2} \tag{4.24}$$

so we can see that the the proper time from the collision to the focal plane only vanishes in the limit $M = |\Upsilon|$, the limit of the dilaton charged black hole with a horizon with zero area described section 2.2.

In the extreme limit the axion and dilaton fields reduce to the constant fixed values (2.32) that they take at the extreme horizon of an axion-dilaton black hole (2.24). The axion and dilaton are constant and take these fixed values over the entire Bertotti-Robinson spacetime, even in the flat region before either wave has passed. Note that the axion and dilaton fields for $r_0 \neq 0$ also take their fixed constant values in the flat region before the waves have arrived, but evolve to their values at $r_\pm$ on the focal planes of the incoming and colliding waves.

As shown in (3.42), an incoming wave obtained from the above Bertotti-Robinson metric via the Khan-Penrose prescription can be rewritten in harmonic coordinates as

$$
\begin{aligned}
ds^2 &= -dU\,dV - \left(\frac{\pi}{2\Delta U}\right)^2 (X^2+Y^2)\,dU^2 + dX^2 + dY^2, \quad 0 \le U \le \Delta U \\
ds^2 &= -dU\,dV + dX^2 + dY^2, \quad U < 0, U > \Delta U,
\end{aligned} \tag{4.25}
$$

where $\Delta U \equiv \pi a/2$.

The constraint $ab = 4(M^2 - |\Upsilon|^2)$ would require that in the limit $M = |\Upsilon|$, at least one of the incoming waves has zero width and infinite amplitude. This is yet another aspect of the pathological nature of the singular charged dilaton black hole given by this limit.

4.4 Maximal analytic extension

We have shown that the asymptotic causal structure of the axion- dilaton colliding plane wave spacetime near the Killing-Cauchy horizon at $u/a + v/b = \pi/2$ is like that of the Kasner metric (3.45)

$$
s^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^z + t^{2p_3}dz^2 \tag{4.26}
$$

in the limit $p_1 = 1$, $p_2 = p_3 = 0$. This corresponds to the wedges of Minkowski spacetime in Rindler coordinates that are "behind the horizon" for the usual constantly accelerating observer. This insight was derived using the general asymptotic structure of colliding plane graviational waves in [9], but it is more easily derived using black hole coordinates. The proper time from $r = r_\pm$ as measured by a nearby freely falling observer is to first order in $|r - r_\pm|$

$$
\tau_+^2(r) \sim 2(r_+ - r)\left(\frac{r_+^2 - |\Upsilon|^2}{r_0}\right), \quad \tau_-^2(r) \sim 2(r - r_-)\left(\frac{r_-^2 - |\Upsilon|^2}{r_0}\right). \tag{4.27}
$$

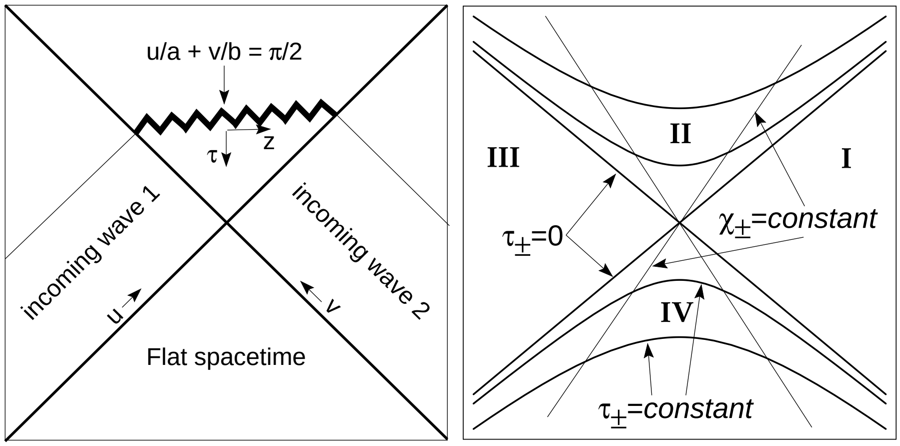

Fig. a   Fig. b

Figure 4.1: Fig.a shows the wave collision in the (u, v) or (τ, z) plane. Fig.b shows how the metric near $u/a + v/b \to \pi/2$ looks in the (τ, χ) plane. The lines $\chi_\pm = const.$ are lines of constant x that cross on the Killing-Cauchy horizons $\tau_\pm = 0$, where $\frac{\partial}{\partial x}$ becomes null.

Changing coordinates by assigning $\chi_\pm = t\, r_0/(r_\pm^2 - |\Upsilon|^2)$, the metric becomes

$$ds^2 \sim -d\tau_\pm^2 + \tau_\pm^2 d\chi_\pm^2 + R(r_\pm)^2 d\Omega. \tag{4.28}$$

In the (τ, χ) plane the metric is the wedge of Rindler spacetime defined in Minkowski coordinates by

$$T^2 - X^2 = \tau_\pm^2, \quad \frac{X}{T} = \tanh \chi_\pm. \tag{4.29}$$

The axion-dilaton colliding plane wave maps to the wedges of Rindler spacetime in the “trapped regions” II and IV and the maximal analytic extension across $\tau_\pm = 0$ gives back the parts of Rindler space that correspond to the non-trapped regions I and III. It is important to remember that χ is proportional to x, and that the spacelike Killing vector $\frac{\partial}{\partial x}$ becomes null on the Killing-Cauchy horizon at $\tau_\pm = 0$. This signals the breakdown of spatial translation invariance in the x-direction just as $\frac{\partial}{\partial t}$ becoming null in regions I and III of figure 4.1 signals the breakdown of time-translation invariance there.

From this point the maximal analytic extension of the axion-dilaton colliding

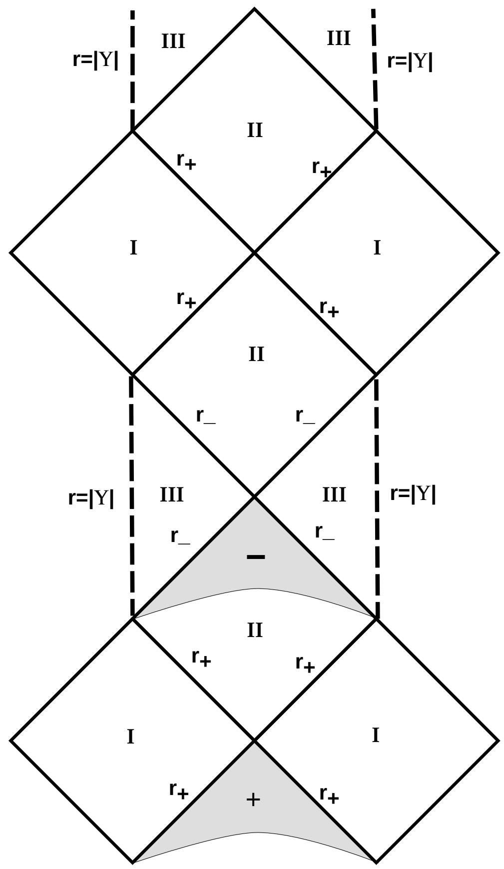

Figure 4.2: The axion-dilaton colliding plane wave metrics analytically extend from the shaded regions of the above diagram into the black hole metric above it.

plane wave metric follows the same steps as for the generic axion-dilaton black hole, which has the same causal structure and maximal analytic extension of a Reissner-Nordstrom black hole, except in the extreme dilaton limit to which we will return later. The Schwarzschild limit was described by Yurtsever in [9].

There is however one problem – we've broken the cyclic boundary conditions on ϕ in the coordinate transformation $\phi \to y/(M^2 - |\Upsilon|^2)^{\frac{1}{2}}$. The cyclic boundary conditions on ϕ, as extended across the surfaces $\tau_\pm = 0$, can be restored by compactifying spacetime in the y-direction for the incoming waves on a circle of radius $\sqrt{M^2 - |\Upsilon|^2}$. If we insist that the maximal extension of the axion- dilaton colliding plane wave spacetime be analytic, compactification of the y-direction is forced on the incoming waves. [9]

The maximal analytic extension of the axion-dilaton colliding plane wave metric has two sandwich waves with translation symmtery in the x and y directions propagating in a universe where the y-coordinate lives on a circle of radius $\sqrt{M^2 - |\Upsilon|^2}$. The waves collide to form either an event horizon r_+ or a Cauchy horizon r_- of an axion-dilaton black hole spacetime, from which the spacetime In figure 4.2, the $(+)$ region is where the $(+)$-branch of the axion-dilaton colliding plane wave metric extends to the black hole spacetime to give an asymptotically flat universe plus an axion-dilaton black hole to the future. The $(-)$ region is where the $(-)$ branch of the colliding wave metric extends from the trapped region II into the axion-dilaton black hole spacetime to the future.

4.5 Comparing string and particle propagation

In sections 3.1.3-4 we compared test particle and string propagation through a single plane wave with a constant profile in $d = 3$. Does this comparison give us any information about test particle vs. string propagation across the Killing-Cauchy horizon of an axion-dilaton black hole? And how is this related to the stringy stretched horizon discussed in section 2.6?

Geodesic focusing in a colliding plane wave background can be examined by calculating the expansion scalar θ. In the axion-dilaton plane wave spacetime (4.9), a null vector $\mathbf{n} = \dot{x}^\mu \partial_\mu$ that is tangent to a null geodesic γ leads to the equation

$$2\, g_{uv}\, \dot{u}\, \dot{v} = -(\frac{p_x^2}{g_{xx}} + \frac{p_y^2}{g_{yy}}) \tag{4.30}$$

along γ, where p_x and p_y are constants of motion along γ.

Null geodesics along which $p_x \neq 0, p_y \neq 0$ do not focus in an axion-dilaton colliding plane wave spacetime, as they would if there were a singularity rather than a Killing-cauchy horizon. The geodesic focusing that defines the focal plane and the Killing-cauchy horizon only occurs in this spacetime for null geodesics that are parallel with $p_x = p_y = 0$. This class of null geodesics delimit the light cone of an event to the infinite past of the plane wave collision. If $p_x = p_y = 0$ then we can choose $\dot{v} = 0$, and $\mathbf{n} = \dot{u}\frac{\partial}{\partial u}$. The geodesic equation $\ddot{u} + ?^{u}_{uu}\dot{u}^2 = 0$ is solved by $\dot{u} = -g^{uv}$, and, using the axion-dilaton colliding plane wave metric (4.9), we get

$$\theta = D_a n^a = \frac{-1}{\sqrt{g}}\frac{\partial}{\partial u}(\sqrt{g}g^{uv}) = \frac{1}{|g_{uv}|\sqrt{g_{xx}g_{yy}}}\frac{\partial}{\partial u}(\sqrt{g_{xx}g_{yy}}) \tag{4.31}$$

$$= \frac{1}{|g_{uv}||\cos(\frac{u}{a}+\frac{v}{b})\cos(\frac{u}{a}-\frac{v}{b})|}\frac{\partial}{\partial u}|\cos(\frac{u}{a}+\frac{v}{b})\cos(\frac{u}{a}-\frac{v}{b})| \tag{4.32}$$

which becomes infinitely negative as $u/a + v/b \to \pi/2$.

The transverse volume element $V_- = \sqrt{g_{xx}g_{yy}} = |\cos(\frac{u}{a}+\frac{v}{b})\cos(\frac{u}{a}-\frac{v}{b})|$ is independent of r_0, so the focusing is controlled by the supersymmetric limit of $r_0 \to 0$, by the Bertotti-Robinson colliding plane wave system. The incoming waves (4.25) when truncated to $d = 3$ are the plane waves we used to compare particle and string geodesic focusing in sections 3.1.3-4. Therefore it seems reasonable to extrapolate those results to the collision region of the axion-dilaton colliding plane wave system.

This focusing of initially parallel light rays defines the Killing Cauchy horizon on the focal plane of the collision region. Parallel light rays delimit causal boundaries of events to the infinite past, so information from the infinite past of the colliding wave spacetime is focused together on the focal plane. This spacetime is on the edge of being singular. Small perturbations of the incoming waves lead to the generic singular solutions. [48] It is interesting to compare this with the infinite buildup of wave fronts from distant perturbations that renders unstable the inner horizon of a charged black hole, as shown by Chandrasekhar and Hartle in [45]. That is also caused by the focusing of information from the infinite past, which in their case represented information from outside the black hole that was being propagated from r_+ through the Cauchy horizon at r_-.

The Killing-Cauchy horizon for the axion-dilaton colliding plane wave metric is mapped to $r = r_\pm$ in the axion-dilaton black hole via the coordinate transformation (4.8). The quantity $V_- = \sqrt{g_{xx}\,g_{yy}}$ becomes $\sqrt{-g_{tt}\,g_{\phi\phi}}$ across the horizon, so the infinite red shift from the black hole point of view $g_{tt} \to 0$ looks from the colliding plane wave point of view like infinite geodesic focusing $V_- \to 0$. From the black hole point of view, the boundary of the trapped region is defined by $\theta_{bh} \to 0$ at $r = r_+$. From the colliding plane wave point of view, the Killing-cauchy horizon is defined by $\theta_{cpw} \to -\infty$ there. So we ought to be able to find a relationship between string propagation through a single plane wave, string propagation through the colliding plane wave focal plane, and string propagation through a black hole event horizon as studied by Susskind and others in [7].

In studying the stringy stretched horizon, Susskind fixed $p = 1$ and looked at $\varepsilon(N)$. In the plots in figure 3.3 we fixed $N = 1$ and varied p instead, with the same effect that as we try to look at the string on a decreasing time scale $\varepsilon = p/N$, the string looks longer and appears to fill more space.

The Killing-Cauchy horizon formed by axion-dilaton colliding plane waves maps to a *past* horizon of an axion-dilaton black hole, as shown in figure 4.2. So the "stringy stretched focal plane" can be viewed as the time-reversed version of the "stringy stretched horizon" described in section 2.5. In other words, suppose we are in the maximally extended colliding plane wave spacetime described in section 4.4, where two waves in a cylindrical universe collide to produce the axion-dilaton black hole spacetime in figure 4.2 at $r = r_+$. A FIDO close to r_+ in region I would see a test string emerging from the collision region at $t = -\infty$ filling the past horizon of the white hole created by the collision and then shrinking rapidly. This is the time-reversed version of what the FIDO at the future horizon sees.

In section 3.3 we illustrated how it should not be possible to focus strings precisely, because of the same stringy causal fuzziness that leads to the stringy stretched horizon and the thermalization of string quantum information around the event horizon of a black hole. In light of this, and in light of the manner in which the inner horizon is relevant to the stringy understanding of black hole thermodynamics [51], one might wonder if the infinite wave front problem at the inner horizon still exists when the spread of information outside of the light cone in string theory is taken into account. It could be possible in string theory that information could diffuse across the inner horizon without leading to infinite wave front density. If this were true it would be very interesting.

Chapter 5 Concluding remarks

In this thesis we examined the relationships between black holes and plane-fronted gravitational waves that are exact conformally invariant backgrounds for various string theories.

The coordinate transformation that relates the trapped region of a Schwarzschild black hole and a colliding plane gravity wave discovered by Ferrari and Ibañez [11, 12] extends naturally to the class of axion-dilaton black holes that are classical solutions to the electric-magnetic duality-invariant action (2.23)

$$S_{eff} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \; (-R + \frac{1}{2}\frac{\partial_\mu \lambda \partial^\mu \bar{\lambda}}{(\mathrm{Im}\lambda)^2} - \sum_{n=1}^{N} F_{\mu\nu}^{(n)\star} \tilde{F}^{(n)\mu\nu}).$$

The local coordinate transformation (4.8)

$$\begin{aligned} r &\rightarrow M \pm r_0 \sin(\frac{u}{a} + \frac{v}{b}), \qquad \theta \rightarrow \frac{\pi}{2} \pm \left(\frac{u}{a} - \frac{v}{b}\right), \\ t &\rightarrow x\, r_0/(M^2 - |\Upsilon|^2)^{\frac{1}{2}}, \qquad \phi \rightarrow 1 + y/(M^2 - |\Upsilon|^2)^{\frac{1}{2}} \end{aligned}$$

transforms an axion-dilaton black hole metric characterized by mass M and complex axion-dilaton charge Υ to the collision region of a colliding axion-dilation plane wave metric (4.9)

$$\begin{aligned} g_{uv} &= \frac{-2\left(\left(M \pm r_0 \sin(\frac{u}{a} + \frac{v}{b})\right)^2 - |\Upsilon|^2\right)}{ab}, \\ g_{xx} &= \frac{(M^2 - |\Upsilon|^2)\cos(\frac{u}{a} + \frac{v}{b})^2}{\left(M \pm r_0 \sin(\frac{u}{a} + \frac{v}{b})\right)^2 - |\Upsilon|^2} \\ g_{yy} &= \cos^2(\frac{u}{a} - \frac{v}{b}) \frac{\left(\left(M \pm r_0 \sin(\frac{u}{a} + \frac{v}{b})\right)^2 - |\Upsilon|^2\right)}{M^2 - |\Upsilon|^2}. \end{aligned}$$

The constants a and b represent the focal lengths of the incoming waves obtained from above through the Khan-Penrose prescription [42] and satisfy the relation $ab = 4(M^2 - |\Upsilon|^2)$. This metric has a Killing-Cauchy horizon at $\frac{u}{a} + \frac{v}{v} = \frac{\pi}{2}$, where the spatial translation Killing vector $\frac{\partial}{\partial x}$ becomes null. The curvature at the Killing-Cauchy horizon is equal to the curvature at $r = r_\pm$ of the corresponding axion-dilaton black hole and so is finite except in the Schwarzschild and extreme electrically or magnetically charged dilaton limits where the curvature at r_- diverges.

The limit $r_0 \to 0$, which for the black hole metrics corresponds to an extreme black hole, takes the axion-dilaton colliding plane wave metric to the Bertotti-Robinson metric (3.39)

$$ds^2 = -du\,dv + \cos^2(\frac{u}{a} + \frac{v}{b})\,dx^2 + \cos^2(\frac{u}{a} - \frac{v}{b})\,dy^2,$$

which has a finite mean focal length $ab = 4(M^2 - |\Upsilon|^2)$ despite the fact that the trapped region of the corresponding black hole has become infinitesimal. The product ab of the non-vanishing parameters describing the colliding waves is related to the entropy of a nonsingular extreme black hole with 1/4 unbroken $N = 4$ supersymmetry through

$$ab = \frac{4S_{extr}}{\pi}. \tag{5.1}$$

An incoming wave obtained from the Bertotti-Robinson collision region can be described in harmonic coordinates as a shock wave of thickness $\Delta U = \pi a/2$, where a is the focal length of that wave, and constant curvature of magnitude $1/a^2 = (\pi/2\Delta U)^2$. If we send $a \to 0$ while keeping the other incoming focal length b finite, then the constraint $ab = 4(M^2 - |\Upsilon|^2)$ says that $M = |\Upsilon|$. The limit $a \to 0$ corresponds to an incoming wave with zero thickness and infinite amplitude, and is singular from the point of view of strings passing through the wave being excited to infinite mass. The black hole corresponding to the $M = |\Upsilon|$ limit has a singular horizon, zero entropy and 1/2 of $N = 4$ supersymmetry unbroken.

The maximal analytic extension of the metric (4.9) across the Killing-Cauchy horizon gives back the non-trapped regions of the corresponding axion-dilaton hole, but requires that the y coordinate live on a circle of radius $\sqrt{M^2 - |\Upsilon|^2}$. The resulting spacetime has two plane-symmetric single waves propagating in a cylindrical universe that collide and form a past horizon of an axion-dilaton black hole, shown in figure 4.2.

The propagation of test particles and test strings in a plane gravitational wave were compared. Geodesic focusing for the axion-dilaton colliding wave system is controlled by the supersymmetric Bertotti-Robinson limit. The single plane waves obtained from this collision region metric therefore make good toy backgrounds to study stringy geodesic focusing. The string equivalent of a massless geodesic equation does not allow for rescaling the affine parameter; consequently light cones as delimited by strings depend on momentum. This introduces a time resolution dependence into string geodesic focusing that is the same time resolution dependence that was analyzed in the stretched black hole horizon by Susskind in [7], suggesting that a "stretched focal plane" is the colliding plane wave analog of a stretched horizon for the black hole. The thermalization of string information at the stringy stretched horizon and the blurring of string geodesic focusing open the possibility that charged black hole inner horizons in the context of string theory might not suffer from the infinite wave front density that renders them unstable in the context of ordinary classical and semi-classical field theory.

Because many $d = 4$ extreme black holes are known to provide nonperturbatively exact backgrounds for selected string theories, we can say now that each of those exact extreme black hole solutions gives us for free a corresponding nonperturbatively exact colliding plane wave background for string propagation. These nonsingular spacetimes are infinitely sensitive to plane-symmetric perturbations in the incoming waves, and the generic solution is singular. So we have a class of exact nonsingular string plane wave scattering backgrounds related by infinitesimal deformations of the incoming states to singularity-forming wave scattering spacetimes for which perturbations around zero string tension fail because the corrections become large.

As for the relevance of this work to more contemporary pressing issues in string theory, in this thesis we have established a relationship between the near-horizon $adS_2 \times S^2$ geometry of the black holes in string theory and that of colliding plane waves. One may attempt to further generalize it for the more general exact backgrounds of string theory with $adS_5 \times S^5$ geometry, which is the near-horizon geometry of D3 branes. It will be a challenge to find the relevant reinterpretation of $adS_5 \times S^5$ spacetime via some analog of colliding waves, which we have studied in this work, and understand their meaning in string theory. Since the black holes we are studying may be understood as an intersection of 4 D3 branes, we may be able to adrress all of these issues directly in ten dimensions, which is the critical dimension for the string theory.

Chapter 6 Appendix

6.1 Details of Colliding Wave Solution

The metric for a spacetime representing the collision of two plane symmetric gravity waves with equal constant, linear polarizations is [1]

$$ds^2 = -2e^{-N}dudv + e^{-U}\sum_{i=1}^{d-2} e^{\lambda_i V}(dx^i)^2, \qquad \sum_{i=1}^{d-2}\lambda_i = 0, \tag{6.1}$$

where N, U, V are functions of u, v obeying the boundary conditions

$$\begin{array}{rcccl} \text{region IV} & u & \leq & 0,\ v \leq 0 & U, N, V = 0 \\ \text{region III} & u & \leq & 0,\ v \geq 0 & U = U(v), \quad V = V(v), \quad N = 0 \\ \text{region II} & u & \geq & 0,\ v \leq 0 & U = U(u), \quad V = V(u), \quad M = 0 \\ \text{region I} & u & \geq & 0,\ v \geq 0 & U = U(u,v), \quad V = V(u,v), \quad N = N(u,v). \end{array} \tag{6.2}$$

The condition $R_{\mu\nu} = 0$ is unamended by higher-derivative terms in regions IV, III, and II, but will not provide an exact solution to the equations of motion for a classical string background in region I, to the future of the scattering event, where the singularity will form. We will first examine the general solution for the Ricci flat case, and then perturb that general solution with the $O(\alpha')$ corrections for the bosonic and heterotic string.

For the above metric, with $D \equiv d - 2$ and the normalization $\sum_{i=1}^{D}\lambda_i^2 = D$, Ricci flatness leads to the equations

$$U_{,uv} - \tfrac{D}{2}U_{,u}U_{,v} = 0 \tag{6.3}$$

[1] The more general case of nonconstant, unequal initial polarizations is not exactly solvable but not immune from the above singularity theorem.

$$V_{,uv} - \tfrac{D}{4}(V_{,u}\,U_{,v} + V_{,v}\,U_{,u}) = 0 \tag{6.4}$$

$$U_{,uu} + N_{,u}\,U_{,u} - \tfrac{1}{2}\,(U_{,u}^{\,2} + V_{,u}^{\,2}) = 0 \tag{6.5}$$

$$U_{,vv} + N_{,v}\,U_{,v} - \tfrac{1}{2}\,(U_{,v}^{\,2} + V_{,v}^{\,2}) = 0 \tag{6.6}$$

$$N_{,uv} + \tfrac{D(D-1)}{4}\,U_{,u}\,U_{,v} - V_{,u}\,V_{,v} = 0. \tag{6.7}$$

This system is integrable. Equations (A3, A4) are constraint equations and will always be satisfied if they are satisfied at the boundary of the interaction region by the incoming waves. To obtain the general solution given boundary conditions on the characteristic surfaces $(u = 0, v)$ and $(u, v = 0)$, we need to make two coordinate transformations. First, define $\alpha \equiv e^{-\frac{D}{2}U(u,v)}$. Because equation (A1) gives $\alpha_{,uv} = 0$, we can then define a coordinate β such that $\beta_{,u} = -\alpha_{,u}$ and $\beta_{,v} = \alpha_{,v}$. Then α and β are timelike and spacelike coordinates, respectively, because

$$-du\,dv = \frac{1}{4\alpha_{,u}\alpha_{,v}}(-d\alpha^2 + d\beta^2). \tag{6.8}$$

Define null coordinates $r \equiv \alpha - \beta$ and $s \equiv \alpha + \beta$. The surface $\{u = 0, v\}$ is the surface $\{r = 1, s\}$ and the surface $\{u, v = 0\}$ is $\{r, s = 1\}$. Equation (A2) becomes

$$V_{,rs} + \frac{1}{2(r+s)}(V_{,r} + V_{,s}) = 0, \tag{6.9}$$

which is a linear hyperbolic second order equation whose solution can be expressed in terms of an integral of the solution to the adjoint equation integrated with the initial data on the null hypersurfaces that bound region I.

The formal solution is

$$\begin{aligned} V(r,s) \;&=\; \int_1^s \left(V_{,s'}(1,s') + \frac{V(1,s')}{2(1+s')} \right) A(1,s';r,s)\,ds' \\ &+\; \int_1^r \left(V_{,r'}(r',1) + \frac{V(r',1)}{2(r'+1)} \right) A(r',1;r,s)\,dr', \end{aligned} \tag{6.10}$$

with $A(r', s'; r, s)$ the kernel of the differential operator adjoint to that acting on

$V(r,s)$. [52] The adjoint equation has the solution

$$A(r',s';r,s) = \left(\frac{r'+s'}{r+s}\right)^{1/2} P_{\perp\frac{1}{2}}(1+2z) \qquad z \equiv \frac{(r'-r)(s'-s)}{(r'+s')(r+s)}. \tag{6.11}$$

The integral has a logarithmic divergence as $r+s = 2\alpha \to 0$, $r' \neq r, s' \neq s$, which corresponds to $z \to \infty$. In this limit, $P_{\perp\frac{1}{2}}(1+2z) \to \frac{1}{\pi}\, z^{-1/2}\, \ln z + \frac{3\ln 2}{\pi}\, z^{-1/2} + O(z^{-3/2})$. The leading behavior of V as $\alpha \to 0$ is $V(\alpha,\beta) = \epsilon(\beta)\ln\alpha + \delta(\beta) + O(\alpha\ln\alpha) + O(\alpha)$. The function $\epsilon(\beta)$ is determined by the incoming amplitudes in the integral

$$\begin{aligned} \epsilon(\beta) &= \frac{1}{\pi}\frac{1}{\sqrt{1+\beta}}\int_{\beta}^{1}[\sqrt{1+s}V(1,s)]_{,s}\left[\frac{1+s}{s-\beta}\right]^{1/2} ds \\ &+ \frac{1}{\pi}\frac{1}{\sqrt{1-\beta}}\int_{-\beta}^{1}[\sqrt{1+r}V(r,1)]_{,r}\left[\frac{r+1}{r-\beta}\right]^{1/2} dr. \end{aligned} \tag{6.12}$$

The equation (A5) integrates to give the relation

$$\frac{e^{-N}}{\alpha_{,u}\alpha_{,v}} \sim e^{-Q/2}\alpha^{\frac{D+1}{D}-2}, \quad Q(\beta) = -\frac{D}{4}\epsilon^2(\beta) + \mu(\beta) + O(\alpha\ln\alpha). \tag{6.13}$$

As $\alpha \to 0$ the metric components

$$\begin{aligned} g_{ii} &\to e^{-\lambda_i\delta}\alpha^{q_i}, \qquad q_i \equiv \tfrac{2}{D} - \lambda_i\epsilon(\beta) \\ g_{\alpha\alpha} = g_{\beta\beta} &\to e^{-\mu(\beta)/2} l_1 l_2\, \alpha^{q-2}, \qquad q \equiv \tfrac{D}{4}\epsilon^2(\beta) + \tfrac{D+1}{D} \end{aligned} \tag{6.14}$$

where $l_1 \equiv 1/2U_{,u}(p)\,, l_2 \equiv 1/2U_{,v}(p)$ are normalization length scales taken at some point p in the interaction region. If we define the timelike coordinate $t = \alpha^{\frac{q}{2}}$ the behavior of the metric as $t \to 0$ is

$$ds^2 = e^{-\mu/2} l_1 l_2(-dt^2 + t^{2p_\beta}\, d\beta^2) + \sum_{i=1}^{D} e^{-\lambda_i\delta(\beta)}\, t^{2p_i}\,(dx^i)^2 \tag{6.15}$$

$$p_\beta \equiv (q-2)/q, \qquad p_i \equiv q_i/q. \tag{6.16}$$

The leading terms of the components of the Riemann tensor are

$$R_{uivi} \sim \frac{t^{2(p_i+p_\beta)-2}}{2D^2}\left(2(D-1)-D(D-4)\lambda_i-\frac{D^2}{2}\epsilon^2(\beta)\right) \quad (6.17)$$

$$R_{uiui},\, R_{vivi} \sim \frac{t^{2(p_i+p_\beta)-2}}{4}\left(\epsilon^2(\beta)\,(1-\lambda_i^2)-\lambda_i\epsilon(\beta)\left(\frac{D}{4}\epsilon^2(\beta)-\frac{1}{D}\right)\right) \quad (6.18)$$

$$R_{ijij} \sim \frac{t^{2(p_i+p_j)-2}}{4D^2}\left(8-4D\epsilon(\beta)(\lambda_i+\lambda_j)+2\lambda_i\lambda_j\epsilon^2(\beta)\right) \quad (6.19)$$

$$R_{uvuv} \sim -\frac{t^{2p_\beta}}{4D}\left(4(D-1)-D^2\epsilon^2(\beta)\right), \quad (6.20)$$

and they agree with the corresponding components for the Kasner metric.

6.2 Properties of Kasner Exponents

The Kasner metric is given in equation (3.45). The Kasner exponents in $\hat{D} = d-1 = D+1$ spatial dimensions, satisfy the relations

$$\sum_{i=1}^{\hat{D}} {p_i}^2 = \sum_{i=1}^{\hat{D}} p_i = 1. \quad (6.21)$$

This is a homogeneous but anistotropic vacuum cosmology with n directions with $p_i > 0$ which expand for increasing time, and m directions with $p_i < 0$ which shrink for increasing time. Because of the above constraints, $1 \leq m \leq \hat{D}-2$ and $2 \leq n \leq \hat{D}-1$. The volume element $\sqrt{-g} = t$ expands as time increases, and shrinks to zero approaching the initial singularity, independent of n, m or $\hat{D}$.

All coordinate invariant quantities must also be invariant under permutations of the Kasner exponents. We will call a permutation invariant product of l different exponents P_l. In $\hat{D}$ spatial dimensions, all P_l will vanish identically for $l > \hat{D}$. These P_l can be expressed as sums of powers of the exponents, (*e.g.*, $P_3 = (\sum {p_i}^3 - 1)/3,$) and vice versa. The dilaton coefficient f_1 is proportional to $P_3 + \frac{1}{2}P_4$. Using identities for P_3 and P_4, it can be shown that

$$P_3 + \alpha P_4 = -\frac{\alpha}{4}\sum_{i=1}^{\hat{D}} p_i^2(p_i-1)(p_i-a), \qquad a = \frac{1}{3}(1+\frac{4}{\alpha}), \quad (6.22)$$

where α is a constant. For f_1, $\alpha = 1/2$ and $a = 3$. Since no exponent can have a value greater than one, f_1 is negative semi-definite, and is zero only in the case where one of the exponents has the value one, and the rest are zero, which is actually flat spacetime.

To determine whether the null convergence condition is satisfied by the $O(\alpha')$ perturbation, we need to calculate the sign of

$$\hat{R}_{\mu\nu}\xi^\mu\xi^\nu = -\frac{4\lambda k_1^2}{t^4}(\frac{t_0}{t})^{2p_1}\left(p_1^2(1-p_1) + (P_3 + 2P_4)\right). \tag{6.23}$$

The first term inside the parentheses is positive. The second term is negative because, from above, if $\alpha = 2$ then $a = 1$, so whether or not the null convergence condition is obeyed at $O(\alpha')$ depends on the Kasner exponent in the spatial direction of the null geodesics in question.

The Kasner exponents live on the intersection of the unit $\hat{D}-1$ sphere $\sum {p_i}^2 = 1$ with the $\hat{D}-1$ hyperplane $\sum p_i = 1$, which is a $\hat{D}-2$ sphere, so for each $\hat{D}$ there is a $(\hat{D}-2)$ - parameter family of possible exponents. To determine the conditions under which $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu$ can be negative, we need only examine the parameter space of exponents near symmetric points with n equal positive exponents p and m equal negative exponents q. The values of p and q are then

$$p = \frac{1}{\hat{D}}\left(1 + \sqrt{(\hat{D}-1)m/n}\right), \quad q = \frac{1}{\hat{D}}\left(1 - \sqrt{(\hat{D}-1)n/m}\right). \tag{6.24}$$

Setting $n = 1, m = \hat{D}-1$ gives $p = 1$, $q = 0$, which is flat spacetime, so the smallest nontrivial value for n is $n = 2$. Explicit calculation shows that

$$\begin{array}{llll} \hat{D} \leq 8 & n \leq 2 \quad \hat{R}_{\mu\nu}\xi^\mu\xi^\nu < 0, & n > 2 & \hat{R}_{\mu\nu}\xi^\mu\xi^\nu > 0 \\ \hat{D} \geq 9 & n \leq 3 \quad \hat{R}_{\mu\nu}\xi^\mu\xi^\nu < 0, & n > 3 & \hat{R}_{\mu\nu}\xi^\mu\xi^\nu > 0 \end{array} \tag{6.25}$$

if the X^1 direction has $p_1 > 0$. However, if $p_1 < 0$, then $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu > 0$ for all $\hat{D}, n,$ and m. Since $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu$ is a smooth function on this parameter space, then we can say that $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu < 0$ in some open neighborhood around those symmetric points where it is negative. If we look near the point ($p_1 = 1$, $p_i = 0\,\forall i \neq 1$), then $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu = 0$ at that point, and decreases away from it in every direction. Near other "flat" solutions, *i.e.* near ($p_1 = 0, p_{i_0} = 1$, $p_i = 0\,\forall i \neq i_0, 1$), variations in the direction of positive p_1 with not more than two other positive exponents lead to $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu < 0$, and all other variations lead to $\hat{R}_{\mu\nu}\xi^\mu\xi^\nu > 0$. So the region of parameter space for Kasner metrics where the null convergence condition is violated in sigma model perturbation theory is some open neighborhood around metrics with either two or three positive Kasner exponents, depending on whether there are eight or more spatial dimensions. (Actually, the number of positive exponents rises again for $\hat{D} \sim 100$.)

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